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Faddeev Equations for Realistic Three-Nucleon Systems. I. Complete Angular Momentum Reduction and Antisymmetrization of States*

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A complete angular momentum reduction of the Faddeev equations is carried out for the case of realistic, nonrelativistic three-nucleon systems with (local and/or nonlocal) interactions having general spin, isospin, and velocity dependence. Antisymmetrization of states with respect to particle exchange is properly accounted for by using properties of the permutation group and the isospin formalism. Expressions for the Faddeev equations, in the form of coupled two-variable integral equations, are obtained in two different coupling schemes; J-j [coupling of the (relative orbital plus total spin) angular momentum of a nucleon pair with the total angular momentum of the third nucleon (in the c.m. system) to give the total angular momentum (in the c.m. system)]; and $\mathcal{L}$-$\mathcal{S}$ [coupling of the total orbital angular momentum (relative orbital angular momentum of a nucleon pair plus the orbital angular momentum of the third nucleon) in the c.m. system with the total spin angular momentum (total spin angular momentum of a nucleon pair plus the spin angular momentum of the third nucleon) to give the total angular momentum in the c.m. system].

I. INTRODUCTION

Since the work of Faddeev[1] on the nonrelativistic quantum theory of three-particle systems, many applications of the Faddeev formalism have been investigated.[2] In the case of two-body local interactions, one encounters formidable practical difficulty associated with the problem of obtaining numerical solutions for a set of coupled integral equations in two continuous variables. However, with the recent development of several numerical techniques,[3,4] it now appears feasible to solve two-variable integral equations with presently available computer facilities.

In applying the Faddeev formalism to three-nucleon systems, we must first make a complete angular momentum reduction of the dynamical equations which takes proper account of the intrinsic spins, isospins, and statistics of the nucleons.

The original Faddeev equations involve the nine independent components of the particle momenta. The dependence of the wave function on the components of the total linear momentum is easily accounted for by separating out the c.m. motion.

For the six remaining independent variables, Omnès[5] chose the c.m. energies of the particles and the Euler angles of orientation of the closed triangle formed by the c.m. momenta. By expanding in eigenstates of the total angular momentum squared and the components of the total angular momentum along a space-fixed and "body"-fixed axis in the plane of the momentum triangle, the Faddeev equations were reduced to a set of coupled integral equations in three continuous variables. Osborn and Noyes[6] expanded the two-body transition amplitudes in relative angular momentum components and further reduced Omnès's equations to a set of coupled integral equations in two continuous variables. Osborn[7] has applied these equations to the calculation of the binding energy of a simple system of three spinless identical particles interacting pairwise through an s-wave Yukawa interaction.

Another method of angular momentum reduction was proposed by Ahmadzadeh and Tjon.[8] After separating out the c.m. motion, they expanded three-particle states in simultaneous eigenstates of $(\vec{L})^2$, L_z, $(\vec{l})^2$, and l_z, where $\vec{L}$ and $\vec{l}$ are, respectively, the relative angular momentum of a pair of particles and the angular momentum of the third particle in the c.m. system. The coupled two-variable equations obtained by this method have been used to determine the binding energy of a system of three identical spinless bosons interacting via two-body local Yukawa interactions,[9,10] and the binding energy of C^{12} on the basis of a three α-particle model.[11] They have also been used to determine the bound-state energy and wave function of the triton for local nucleon-nucleon interactions containing soft-core repulsion and tensor-coupling terms.[12]

El-Baz *et al.*[13] have generalized the Ahmadzadeh-Tjon method for the case of particles with intrinsic spins. They use the well-known separable expansion formula[14] for a spherical harmonic whose argument is a vector sum of two vectors, and use a graphical method[15] for handling the angular momentum algebra.

The Omnès and Ahmadzadeh-Tjon formulations of the Faddeev equations are simply related by a unitary transformation, as is expected. This has been explicitly shown by Balian and Brezin.[16] Although the Ahmadzadeh-Tjon equations are unsymmetrical in the variables introduced, they appear to be more convenient for physical applications.

Other reductions of the Faddeev equations include that of Doolen,[17] which is based on the Jacob-Wick[18] helicity formalism, and that of Lee,[19] which is based on the *SU*(3) representation of three-particle states.

In this paper, we generalize the method of El-Baz *et al.* to the case of two-body tensor interactions and antisymmetrization in terms of the isospin formalism for the three-nucleon system. Extensive use is made of the graphical techniques of Yutsis, Levinson, and Vanagas (YLV)[20] for angular momentum algebra involved in obtaining the final expressions.

In Sec. II, we give a brief summary of the Faddeev equations, the definitions of kinematic variables, and the normalization, orthogonality, and transformation properties of free-particle linear and orbital angular momentum eigenstates. Section III contains a general discussion of the antisymmetrization procedure used in classifying our three-particle states. In Sec. IV, the Faddeev equations are derived for the *J-j* coupling scheme. In this scheme, the (relative orbital plus total spin) angular momentum of a nucleon pair is coupled with the total angular momentum of the third nucleon (in the c.m. system) to give the total angular momentum (in the c.m. system). The specific details of antisymmetrization for this coupling scheme are also discussed. In Sec. V, a similar presentation is given for the $\mathcal{L}$-$\mathcal{S}$ coupling scheme. In this case, the total orbital angular momentum (relative orbital angular momentum of a nucleon pair plus the orbital angular momentum of the third nucleon) in the three-particle c.m. system is coupled to the total spin angular momentum (total spin angular momentum of a nucleon pair plus the spin of the third nucleon) to give the total angular momentum in the c.m. system. Section VI contains a discussion and a brief summary of the results of this paper. A graphical derivation of results in Secs. IV and V is presented in Appendix A. In Appendix B, we give the properties of the isospin eigenstates of the three-nucleon system which are used in this paper.

II. FADDEEV EQUATIONS, KINEMATIC VARIABLES, AND LINEAR MOMENTUM AND ANGULAR MOMENTUM BASES

In order to clarify the notation and conventions involved in this paper, we present here a summary of the Faddeev equations and associated kinematics.

The nonrelativistic three-particle scattering matrix T for particles of mass m_1, m_2, and m_3 can be decomposed as[1,8]

$$T = T^{(1)} + T^{(2)} + T^{(3)} . \tag{2.1}$$

The $T^{(i)}$'s satisfy the Faddeev equations

$$T^{(i)}(s) = T_i(s) - \sum_{j \neq i} T_i(s) G_0(s) T^{(j)}(s), \quad i = 1, 2, 3 . \tag{2.2}$$

$G_0(s)$ is the three-particle Green's function

$$G_0(s) = (H_0 - s)^{-1}, \tag{2.3}$$

(H_0 being the three-particle kinetic energy operator), s is the total energy of the three-particle system, the T_i's are the off-shell two-body T-matrices which satisfy the Lippmann-Schwinger equations

$$T_i(s) = V_i - V_i G_0(s) T_i(s), \tag{2.4}$$

where V_i is the interaction between the pair of particles j and k ($i \neq j \neq k$).

Following the conventions of Refs. 8 and 9, we define linear momentum combinations

$$\vec{p}_1 = \frac{m_3 \vec{k}_2 - m_2 \vec{k}_3}{[2m_2 m_3(m_2 + m_3)]^{1/2}} ,$$
$$\vec{q}_1 = \frac{m_1(\vec{k}_2 + \vec{k}_3) - (m_2 + m_3)\vec{k}_1}{[2m_1(m_2+m_3)(m_1+m_2+m_3)]^{1/2}} , \tag{2.5}$$

with definitions for ($\vec{p}_2$, $\vec{q}_2$) and ($\vec{p}_3$, $\vec{q}_3$) following from (2.5) by cyclic permutation of the indices 1, 2, and 3. $\vec{k}_i$ represents the momentum of particle i in the space-fixed coordinate system. The total-momentum combination

$$\vec{P} = \frac{\vec{k}_1 + \vec{k}_2 + \vec{k}_3}{[2(m_1 + m_2 + m_3)]^{1/2}} , \tag{2.6}$$

together with $\vec{p}_i$ and $\vec{q}_i$, satisfies the relation

$$E = \sum_i \frac{(\vec{k}_i)^2}{2m_i} = (\vec{p}_j)^2 + (\vec{q}_j)^2 + (\vec{P})^2 . \tag{2.7}$$

The above choice of momentum variables gives unity for the Jacobian of the transformation relating ($\vec{p}_i, \vec{q}_i, \vec{P}$) and ($\vec{p}_j, \vec{q}_j, \vec{P}$), $i, j = 1, 2, 3$.

In terms of the mass factors

$$\alpha_{ij} = \left[\frac{m_i m_j}{(m_i + m_k)(m_j + m_k)}\right]^{1/2} = \alpha_{ji} ,$$
$$\beta_{ij} = (1 - \alpha_{ij}{}^2)^{1/2} = -\beta_{ji} \quad (ijk \text{ cyclic}), \tag{2.8}$$

introduced by Ball and Wong,[9] the linear relations between ($\vec{p}_i, \vec{q}_i$) and ($\vec{p}_j, \vec{q}_j$) are

$$\vec{p}_i = -\alpha_{ij}\vec{p}_j - \beta_{ij}\vec{q}_j ,$$
$$\vec{q}_i = \beta_{ij}\vec{p}_j - \alpha_{ij}\vec{q}_j \quad (ijk \text{ cyclic}). \tag{2.9}$$

For three identical particles, we have $\alpha_{ij}=1/2$ and $\beta_{ij}=\sqrt{3}/2$.

In the c.m. system ($\vec{P}=0$), three-particle eigenstates of linear momentum

$$|\vec{k}_1\vec{k}_2\vec{k}_3\rangle = |\vec{p}_1,\vec{q}_1\rangle_1 = |\vec{p}_2,\vec{q}_2\rangle_2 = |\vec{p}_3,\vec{q}_3\rangle_3, \quad (2.10)$$

are assumed to have the orthonormality property

$$\langle\vec{p}',\vec{q}'|\vec{p},\vec{q}\rangle = \delta^{(3)}(\vec{p}'-\vec{p})\delta^{(3)}(\vec{q}'-\vec{q}). \quad (2.11)$$

The partial-wave expansion of $|\vec{p},\vec{q}\rangle$ is

$$|\vec{p},\vec{q}\rangle = \sum_{\substack{L,M\\ l,m}} Y^*_{LM}(\hat{p})Y^*_{lm}(\hat{q})|pLM;qlm\rangle, \quad (2.12)$$

where

$$|pLM;qlm\rangle = \int d\hat{p}\,d\hat{q}\,Y_{LM}(\hat{p})Y_{lm}(\hat{q})|\vec{p},\vec{q}\rangle. \quad (2.13)$$

In (2.13), $d\hat{p}$ and $d\hat{q}$ denote solid-angle differentials. The normalization (2.11) requires that

$$\langle p'L'M';q'l'm'|pLM;qlm\rangle = \frac{\delta(p'-p)}{p^2}\,\frac{\delta(q'-q)}{q^2}\,\delta_{LL'}\delta_{MM'}\delta_{ll'}\delta_{mm'}. \quad (2.14)$$

III. ANTISYMMETRIZATION OF STATES

If the isospin formalism is used, a composite of nucleons can be treated as a system of identical fermions with intrinsic spin $\frac{1}{2}$ and isospin $\frac{1}{2}$. The generalized Pauli principle requires that the total state function of the nucleon system be antisymmetric under simultaneous exchanges of the space, spin, and isospin coordinates of any pair of nucleons.

There are a number of works which give procedures for generally constructing the antisymmetric states of a three-nucleon system. Blatt and Derrick,[21] in particular, have given an elegant group-theoretical discussion. These procedures, however, give states that are much more suitable for variational calculations of binding energies than they are for scattering and bound-state calculations based on the Faddeev equations.

A general classification of antisymmetric states of three nucleons, which is suitable for use in the Faddeev equations, can be obtained very simply after performing a complete angular momentum reduction of these equations. The technical details of the reduction are given in Secs. IV and V. In this section, we schematically indicate the general features of our antisymmetrization scheme.

We will work with three-nucleon states $|(\psi)\rangle_i$ which are constructed to be antisymmetric under the exchange of the dynamical coordinates of nucleons j and k (with ijk a cyclic ordering). Fully antisymmetric three-nucleon states can be easily generated via particle exchange operations on the $|(\psi)\rangle_i$.

Let P_{ij} be the ij particle-exchange operator, i.e.,

$$P_{ij} = P_r(ij)P_\sigma(ij)P_\tau(ij), \quad (3.1)$$

where P_r, P_σ, and P_τ are, respectively, the space, spin, and isospin exchange operators. The three P_{ij} operators are odd permutation operators and elements of the permutation group S_3.[22] The three remaining elements of the group are the even permutation operators e, P_{123}, and P_{132}, where e is the identity operator, and P_{123} and P_{132} are cyclic permutation operators. From group-element multiplications, it easily follows that

$$|\psi\rangle_A = (e+P_{123}+P_{132})|(\psi)\rangle_i \quad (3.2)$$

is completely antisymmetric with respect to particle exchange.

Now let $|\alpha(i,jk)\rangle_i$ be a free-particle state of three nucleons which is antisymmetric with respect to jk exchange (ijk cyclic), and is an eigenstate of a complete set of commuting operators (including the free-particle kinetic energy H_0) which is appropriate for this symmetry.

In the J-j coupling scheme, $\alpha(i,jk)$ denotes the quantum numbers associated with the operators: H_0, $(\vec{L}_i)^2$, $(\vec{S}_i)^2=(\vec{s}_j+\vec{s}_k)^2$, $(\vec{s}_j)^2$, $(\vec{s}_k)^2$, $(\vec{J}_i)^2=(\vec{L}_i+\vec{S}_i)^2$, $(\vec{l}_i)^2$, $(\vec{s}_i)^2$, $(\vec{j}_i)^2=(\vec{l}_i+\vec{s}_i)^2$, $(\vec{\mathcal{J}})^2=(\vec{J}_i+\vec{j}_i)^2$, $\mathcal{J}_z$, $(\vec{t}_j)^2$, $(\vec{t}_k)^2$, $(\vec{T}_i)^2=(\vec{t}_j+\vec{t}_k)^2$, $(\vec{t}_i)^2$, $(\vec{\mathcal{T}})^2=(\vec{T}_i+\vec{t}_i)^2$, $\mathcal{T}_z$. Here $\vec{L}_i$ is the relative orbital angular momentum of the jk pair (ijk cyclic); $\vec{l}_i$ is the orbital angular momentum of nucleon i in the c.m. system, $\vec{s}_i$ is the spin angular momentum of nucleon i, and $\vec{t}_i$ is the spin isospin of nucleon i. In the $\mathcal{L}$-$\mathcal{S}$ coupling scheme, the corresponding set of operators is H_0, $(\vec{L}_i)^2$, $(\vec{l}_i)^2$, $(\vec{\mathcal{L}})^2=(\vec{L}_i+\vec{l}_i)^2$, $(\vec{S}_i)^2$, $(\vec{s}_j)^2$, $(\vec{s}_k)^2$, $(\vec{s}_i)^2$, $(\vec{\mathcal{S}})^2=(\vec{S}_i+\vec{s}_i)^2$, $(\vec{\mathcal{J}})^2=(\vec{\mathcal{L}}+\vec{\mathcal{S}})^2$, $\mathcal{J}_z$, and the same isospin operators as in the J-j scheme. Since the intrinsic spin and isospin quantum numbers are equal to $\frac{1}{2}$ for all nucleons, they will not usually be explicitly specified from here on.

The completely antisymmetrized state

$$|\alpha(1,23)\rangle_A = (e+P_{123}+P_{132})|\alpha(1,23)\rangle_1 \quad (3.3)$$

is an eigenstate of H_0, $(\vec{\mathcal{J}})^2$, and $\mathcal{J}_z$.

If H_0 and $O_s(1,2,3)$, $s=1,2,\ldots,16$, denote the operators associated with the quantum numbers $\alpha(1,23)$, then it is easily shown that the state

$$P_{123}|\alpha(1,23)\rangle_1 = |P_{123}\alpha(1,23)\rangle_2 = |\alpha(2,31)\rangle_2 \quad (3.4)$$

is a simultaneous eigenstate of the operators H_0 and $P_{123}O_s(1,2,3)P_{123}^{-1}\equiv O_s(2,3,1)$, $s=1,2,\ldots,16$, with the same quantum numbers originally associated with H_0 and $O_s(1,2,3)$, $s=1,2,\ldots,16$.

Since the $T^{(i)}$ operators in the Faddeev equations (2.2) satisfy

$$P_{12}T^{(1)}P_{12}^{-1}=T^{(2)}\,,\qquad P_{123}T^{(1)}P_{123}^{-1}=T^{(2)}\,,\quad \text{etc.,} \tag{3.5}$$

and

$$P_{12}|\psi\rangle_A=-|\psi\rangle_A\,,\quad \text{etc.,}\qquad P_{123}|\psi\rangle_A=|\psi\rangle_A\,,\quad \text{etc.,} \tag{3.6}$$

where $|\psi\rangle_A$ is an arbitrary (completely) antisymmetrized state, we have the relations

$$\begin{aligned}{}_1\langle\alpha(1,23)|T^{(2)}(s)|\psi\rangle_A &= {}_3\langle P_{132}\alpha(1,23)|P_{132}T^{(2)}(s)|\psi\rangle_A \\ &= {}_3\langle\alpha(3,12)|T^{(1)}(s)|\psi\rangle_A\,, \\ {}_1\langle\alpha(1,23)|T^{(3)}(s)|\psi\rangle_A &= {}_2\langle P_{123}\alpha(1,23)|P_{123}T^{(3)}(s)|\psi\rangle_A \\ &= {}_2\langle\alpha(2,31)|T^{(1)}(s)|\psi\rangle_A\,,\quad \text{etc.}\end{aligned} \tag{3.7}$$

Note that ${}_1\langle\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A$ would vanish if $|\alpha(1,23)\rangle_1$ were symmetric under 23 exchange.

The states $|\alpha(1,23)\rangle_1$ are assumed to satisfy the orthonormality relations

$${}_1\langle\alpha'(1,23)|\alpha(1,23)\rangle_1=\frac{\delta(p_1'-p_1)}{p_1^2}\,\frac{\delta(q_1'-q_1)}{q_1^2}\,\delta_{\alpha'\alpha}\,, \tag{3.8}$$

where $\delta_{\alpha'\alpha}$ denotes a product of Kronecker δ functions associated with the discrete quantum numbers. The effective closure relation

$$\begin{aligned}1&=\sum_\alpha\int_0^\infty p_1^2dp_1\int_0^\infty q_1^2dq_1\,|\alpha(1,23)\rangle_1\,{}_1\langle\alpha(1,23)| \\ &= \mathop{\mathrm{S}}_{\alpha}|\alpha(1,23)\rangle_1\,{}_1\langle\alpha(1,23)|\end{aligned} \tag{3.9}$$

may be inserted adjacent to a state which is antisymmetric under 23 exchange.

Thus the Faddeev equations for ${}_1\langle\alpha(1,23)|T^{(i)}|\psi\rangle_A$, $i=1,2,3$, with $|\psi\rangle_A$ fixed, reduce to a single integral equation for the matrix element, $\langle\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A$.

$$\begin{aligned}{}_1\langle\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A &= {}_1\langle\alpha(1,23)|T_1(s)|\psi\rangle_A-\mathop{\mathrm{S}}_{\alpha'}\,{}_1\langle\alpha(1,23)|T_1(s)|\alpha'(1,23)\rangle_1\frac{1}{E_{\alpha'}-s} \\ &\quad\times\left({}_1\langle\alpha'(1,23)|T^{(2)}(s)|\psi\rangle_A+{}_1\langle\alpha'(1,23)|T^{(3)}(s)|\psi\rangle_A\right) \\ &= {}_1\langle\alpha(1,23)|T_1(s)|\psi\rangle_A-\mathop{\mathrm{S}}_{\alpha'}\langle\alpha(1,23)|T_1(s)|\alpha'(1,23)\rangle_1\frac{1}{E_{\alpha'}-s} \\ &\quad\times\mathop{\mathrm{S}}_{\alpha''}\left({}_3\langle P_{132}\alpha'(1,23)|\alpha''(1,23)\rangle_1+{}_2\langle P_{123}\alpha'(1,23)|\alpha''(1,23)\rangle_1\right) \\ &\quad\times{}_1\langle\alpha''(1,23)|T^{(1)}(s)|\psi\rangle_A\,.\end{aligned} \tag{3.10}$$

Equation (3.10) may be further simplified if we note that

$${}_3\langle P_{132}\alpha'(1,23)|\alpha''(1,23)\rangle_1={}_2\langle P_{123}\alpha'(1,23)|\alpha''(1,23)\rangle_1\,. \tag{3.11}$$

(3.11) follows from the fact that

$$(P_{132}-P_{123})|\alpha'(1,23)\rangle_1 \tag{3.12}$$

is *symmetric* with respect to 23 exchange and is consequently orthogonal to $|\alpha''(1,23)\rangle_1$. Thus

$$\begin{aligned}{}_1\langle\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A &= {}_1\langle\alpha(1,23)|T_1(s)|\psi\rangle_A-2\mathop{\mathrm{S}}_{\alpha'}\,{}_1\langle\alpha(1,23)|T_1(s)|\alpha'(1,23)\rangle_1\frac{1}{E_{\alpha'}-s} \\ &\quad\times\mathop{\mathrm{S}}_{\alpha''}\,{}_3\langle P_{132}\alpha'(1,23)|\alpha''(1,23)\rangle_1\,{}_1\langle\alpha''(1,23)|T^{(1)}(s)|\psi\rangle_A\,.\end{aligned} \tag{3.13}$$

An alternative form of (3.13), which will be used later on, is

$$\begin{aligned}{}_1\langle\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A &= {}_1\langle\alpha(1,23)|T_1(s)|\psi\rangle_A-2\mathop{\mathrm{S}}_{\alpha'}\,{}_1\langle\alpha(1,23)|T_1(s)|P_{123}\alpha'(1,23)\rangle_2\frac{1}{E_{\alpha'}-s} \\ &\quad\times{}_1\langle\alpha'(1,23)|T^{(1)}(s)|\psi\rangle_A\,.\end{aligned} \tag{3.14}$$

We conclude this section by considering the exchange symmetry of three-particle bound-state wave functions which are extracted from the three-particle T matrix.[23] Let $|\psi_B\rangle$ be a normalized three-particle bound state:

$$(E_B - H)|\psi_B\rangle = 0, \quad \langle\psi_B|\psi_B\rangle = 1$$
$$(H = H_0 + V_1 + V_2 + V_3 = H_0 + V). \tag{3.15}$$

From the formal expression for $T(s)$

$$T(s) = V - V\frac{1}{H-s}V, \tag{3.16}$$

it follows that

$${}_1\langle\alpha(1,23)|T(s)|\psi\rangle_A \underset{s\approx E_B}{\approx} -{}_1\langle\alpha(1,23)|V|\psi_B\rangle\frac{1}{E_B - s}\langle\psi_B|V|\psi\rangle_A. \tag{3.17}$$

The residue of ${}_1\langle\alpha(1,23)|T(s)|\psi_B\rangle$ at the bound-state pole, which may be easily extracted from the numerical solution of Eqs. (3.13) or (3.14), is thus proportional to

$${}_1\langle\alpha(1,23)|V|\psi_B\rangle = (E_B - p_1{}^2 - q_1{}^2)\,{}_1\langle\alpha(1,23)|\psi_B\rangle \tag{3.18}$$

in the c.m. system. The equality (3.18) is derived by representing V as $H - H_0$ and using (3.15) and

$$(p_1{}^2 + q_1{}^2 - H_0)|\alpha(1,23)\rangle_1 = 0. \tag{3.19}$$

Now

$$\begin{aligned}{}_1\langle\alpha(1,23)|T(s)|\psi\rangle_A &= {}_1\langle\alpha(1,23)|T^{(1)}(s) + T^{(2)}(s) + T^{(3)}(s)|\psi\rangle_A\\ &= \langle(e + P_{132} + P_{123})\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A\\ &= \langle(e + P_{132} + P_{123})\tfrac{1}{2}(1 - P_{23})\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A,\end{aligned} \tag{3.20}$$

and consequently

$$\begin{aligned}{}_2\langle P_{12}\alpha(1,23)|T(s)|\psi\rangle_A &= \langle(e + P_{132} + P_{123})P_{12}\tfrac{1}{2}(1 - P_{23})\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A\\ &= -\langle(e + P_{132} + P_{123})\tfrac{1}{2}(1 - P_{23})\alpha(1,23)|T^{(1)}(s)|\psi\rangle_A\\ &= -{}_1\langle\alpha(1,23)|T(s)|\psi\rangle_A, \quad \text{etc.}\end{aligned} \tag{3.21}$$

The expression for ${}_1\langle\alpha(1,23)|T(s)|\psi\rangle_A$ given by (3.20) yields, according to (3.17) and (3.18), components of $|\psi_B\rangle$,

$${}_1\langle\alpha(1,23)|\psi_B\rangle,$$

which satisfy

$${}_2\langle P_{12}\alpha(1,23)|\psi_B\rangle = -{}_1\langle\alpha(1,23)|\psi_B\rangle, \quad \text{etc.} \tag{3.22}$$

These components are thus consistent with the complete antisymmetry of $|\psi_B\rangle$.

IV. J-j COUPLING SCHEME

In the J-j coupling scheme, we form (c.m. system) eigenstates of total angular momentum $\mathcal{J}$ and projection of total angular momentum on the z axis of a space-fixed coordinate system $\mathcal{J}_z$ from a direct product of the eigenstates of $\vec{J}$ (relative orbital plus total spin angular momentum of a nucleon pair) and $\vec{j}$ (total angular momentum of the third nucleon in the c.m. system). The isospin components of these states have total isospin $\mathcal{T}$, and z component of total isospin $\mathcal{T}_z$. They are formed from a direct product of the eigenstates of $\vec{T}$ (total isospin of a nucleon pair) and $\vec{t}$ (isospin of the third nucleon). The complete list of commuting operators which characterize the J-j coupling scheme was given in Sec. III.

The explicit construction of the complete set of J-j basis states is given by

$$\begin{aligned}|p,q,\alpha\rangle_i = |p,q,\alpha(i,jk)\rangle_i &= |[p(LS)J,q(ls)j]\mathcal{J}\mathcal{J}_z;(Tt)\mathcal{T}\mathcal{T}_z\rangle_i\\ &= \sum_{m_J,m_j}\langle Jm_Jjm_j|\mathcal{J}\mathcal{J}_z\rangle\,|p(LS)Jm_J;q(ls)jm_j\rangle_i\,|(Tt)\mathcal{T}\mathcal{T}_z\rangle_i\end{aligned} \tag{4.1}$$

with

$$|p(LS)Jm_J;\, q(ls)jm_j\rangle_i = \sum_{m_L,m_S}\sum_{m_l,m_s} \langle Lm_L Sm_S|Jm_J\rangle \langle lm_l\, sm_s|jm_j\rangle |pLm_L;\, qlm_l\rangle |Sm_S\rangle_i |sm_s\rangle \tag{4.2}$$

and

$$|Sm_S\rangle_i = |(s_j s_k)Sm_S\rangle_i = \sum_{m_{s_j},m_{s_k}} \langle s_j\, m_{s_j} s_k m_{s_k}|Sm_S\rangle\, |s_j m_{s_j}\rangle\, |s_k m_{s_k}\rangle \quad (ijk \text{ cyclic}), \tag{4.3}$$

where $|s_i m_{s_i}\rangle$ are the orthonormal spin eigenstates for nucleon i and $|(Tt)\mathcal{T}\mathcal{T}_z\rangle_i$ are the three-nucleon isospin states defined in Appendix B. The symbols $\langle a\alpha b\beta|c\gamma\rangle$ are Clebsch-Gordan coefficients. (We will use the Condon-Shortley phase convention throughout this work.) For convenience, the same symbol is used for an operator and its eigenvalues. The quantum numbers s and t are, of course, always $\frac{1}{2}$ for nucleons. Antisymmetry with respect to jk exchange requires that

$$(-1)^{L+S+T} = -1\,.$$

We may use (2.12) to expand the states (4.1) in eigenstates of linear momentum:

$$|[p(LS)J,\, q(ls)j]\mathcal{J}\mathcal{J}_z;\, (Tt)\mathcal{T}\mathcal{T}_z\rangle_i = \sum_{m_J,m_j} \langle Jm_J jm_j|\mathcal{J}\mathcal{J}_z\rangle \sum_{m_L,m_S}\sum_{m_l,m_s} \langle Lm_L Sm_S|Jm_J\rangle \langle lm_l sm_s|jm_j\rangle$$
$$\times \int d\hat{p}\int d\hat{q}\, Y_{Lm_L}(\hat{p}) Y_{lm_l}(\hat{q})\,|\vec{p},\vec{q};\, Sm_S,\, sm_s;\, (Tt)\mathcal{T}\mathcal{T}_z\rangle_i\,. \tag{4.4}$$

The orthonormality relation for the states given by (4.1) is

$${}_i\langle [p'(L'S')J', q'(l's)j']\mathcal{J}'\mathcal{J}'_z;\, (T't)\mathcal{T}'\mathcal{T}'_z\,||[p(LS)J,q(ls)j]\mathcal{J}\mathcal{J}_z;\, (Tt)\mathcal{T}\mathcal{T}_z\rangle_i$$
$$= \frac{\delta(p'-p)}{p^2}\,\frac{\delta(q'-q)}{q^2}\,\delta_{L'L}\delta_{S'S}\delta_{J'J}\delta_{l'l}\delta_{j'j}\delta_{\mathcal{J}'\mathcal{J}}\,\delta_{\mathcal{J}'_z\mathcal{J}_z}\,\delta_{T'T}\delta_{\mathcal{T}'\mathcal{T}}\delta_{\mathcal{T}'_z\mathcal{T}_z}\,. \tag{4.5}$$

The Faddeev equation, in the J-j coupling scheme, becomes

$$\psi_s^{(1)}(p,q,\alpha) = \varphi_s^{(1)}(p,q,\alpha) - 2\sum_{\alpha_2}\int_0^\infty p_2^2 dp_2 \int_0^\infty q_2^2 dq_2 \frac{{}^{(1)}K_2}{p_2^2+q_2^2-s}\,\psi_s^{(1)}(p_2,q_2,\alpha_2), \tag{4.6}$$

where

$$\psi_s^{(1)}(p,q,\alpha) = {}_1\langle p,q,\alpha|T^{(1)}(s)|\psi\rangle_A, \tag{4.7}$$

$$\psi_s^{(1)}(p_2,q_2,\alpha_2) = {}_1\langle p_2,q_2,\alpha_2|T^{(1)}(s)|\psi\rangle_A, \tag{4.7a}$$

$$\varphi_s^{(1)}(p,q,\alpha) = {}_1\langle p,q,\alpha|T_1(s)|\psi\rangle_A, \tag{4.8}$$

and

$${}^{(1)}K_2 = {}_1\langle p,q,\alpha|T_1(s)|p_2,q_2,\alpha_2\rangle_2\,. \tag{4.9}$$

Using (4.4) in (4.9), we obtain

$${}^{(1)}K_2 = \sum_{\substack{\text{(all magnetic numbers}\\ \text{except } \mathcal{J}_z,\, \mathcal{J}_{2z})}} \langle \mathcal{J}\mathcal{J}_z|Jm_J jm_j\rangle \langle Jm_J|Lm_L Sm_S\rangle \langle jm_j|lm_l\, sm_s\rangle \langle J_2 m_{J_2} j_2 m_{j_2}|\mathcal{J}_2\mathcal{J}_{2z}\rangle$$
$$\times \langle L_2 m_{L_2} S_2 m_{S_2}|J_2 m_{J_2}\rangle \langle l_2 m_{l_2} s_2 m_{s_2}|j_2 m_{j_2}\rangle \int d\hat{p}\int d\hat{q}\int d\hat{p}_2\int d\hat{q}_2\, Y^*_{Lm_L}(\hat{p}) Y^*_{lm_l}(\hat{q}) Y_{L_2,m_{L_2}}(\hat{p}_2) Y_{l_2 m_{l_2}}(\hat{q}_2)$$
$$\times {}_1\langle\vec{p},\vec{q};Sm_S,sm_s;\, (Tt)\mathcal{T}\mathcal{T}_z\,|\,T_1(s)\,|\,\vec{p}_2,\vec{q}_2;\, S_2m_{S_2},\, s_2m_{s_2};(T_2t_2)\mathcal{T}_2\mathcal{T}_{2z}\rangle_2\,. \tag{4.10}$$

The spin and isospin eigenstates, $|S_2m_{S_2}, s_2m_{s_2}\rangle_2$ and $|(T_2t_2)\mathcal{T}_2\mathcal{T}_{2z}\rangle$, may be expressed in terms of $|\,\rangle_1$-type states by using (B5) of Appendix B, and the relation

$$|S_i m_{S_i},\, s_i m_{s_i}\rangle_i = |(s_j s_k)S_i m_{S_i};\, s_i m_{s_i}\rangle_i$$
$$= \sum_{m_{s_j},m_{s_k}}\sum_{S_k,m_{S_k}} \langle s_j m_{s_j} s_k m_{s_k}|S_i m_{S_i}\rangle \langle S_k m_{S_k}|s_i m_{s_i} s_j m_{s_j}\rangle\, |(s_i s_j)S_k m_{S_k};\, s_k m_{s_k}\rangle_k \quad (ijk \text{ cyclic}). \tag{4.11}$$

After setting $|\vec{p}_2,\vec{q}_2\rangle_2 = |\vec{p}_1,\vec{q}_1\rangle_1$, the matrix element of $T_1(s)$ in Eq. (4.10) reduces to

$$
\begin{aligned}
{}_1\langle \vec{p}_1,\vec{q}_1; Sm_S, sm_s; (Tt)\mathcal{T}\mathcal{T}_z \,|\, T_1(s) \,|\, \vec{p}_2,\vec{q}_2; S_2m_{S_2}, s_2m_{s_2}; (T_2t_2)\mathcal{T}_2\mathcal{T}_{2z}\rangle_2 \\
&= \sum_{m_{s_3},m_{s_1}} \sum_{S_1,m_{S_1}} \langle s_3m_{s_3}s_1m_{s_1} | S_2m_{S_2}\rangle \langle S_1m_{S_1} | s_2m_{s_2}s_3m_{s_3}\rangle \delta_{s\,s_1}\delta_{m_s\,m_{s_1}} \\
&\quad\times \sum_{\substack{T_z,t_z,\\ T_1,T_{1z},t_{1z}}} \langle TT_z tt_z | \mathcal{T}\mathcal{T}_z\rangle (-1)^{t_2-T_2-\mathcal{T}_2} \hat{T}_1\hat{T}_2 W(t_1t_3\mathcal{T}_2t_z; T_2T_1) \\
&\quad\times \langle T_1T_{1z}t_1t_{1z} | \mathcal{T}_2\mathcal{T}_{2z}\rangle \delta_{t\,t_1}\delta_{t_z\,t_{1z}}\, {}_1\langle \vec{p},\vec{q}; Sm_S; TT_z \,|\, T_1(s) \,|\, \vec{p}_1,\vec{q}_1; S_1m_{S_1}; T_1T_{1z}\rangle_1 ,
\end{aligned}
\tag{4.12}
$$

where $\hat{T}_1$ denotes $(2T_1+1)^{1/2}$ and W is the usual Racah coefficient. Charge conservation requires $\mathcal{T}_{2z} = \mathcal{T}_z$ for a nonvanishing matrix element.

The two-nucleon interaction is assumed to be invariant under the usual space-time translations, inversions, and Galilean boosts, as well as general rotations in ordinary space and rotations about the z axis in isospin space. The appropriate partial-wave expansion of (4.12) is then[24]

$$
\begin{aligned}
{}_1\langle \vec{p},\vec{q}; Sm_S; TT_z \,|\, T_1(s) \,|\, \vec{p}_1,\vec{q}_1; S_1m_{S_1}; T_1T_{1z}\rangle_1 \\
&= \delta^{(3)}(\vec{q}-\vec{q}_1)\delta_{SS_1}\delta_{TT_1}\delta_{T_zT_{1z}} \sum_{J',J_1}\sum_{L',m_{L'}}\sum_{L_1,m_{L_1}} Y_{L'm_{L'}}(\hat{p}) Y^*_{L_1m_{L_1}}(\hat{p}_1) \langle L'm_{L'}Sm_S | J'm_{j'} = m_{L'}+m_S\rangle \\
&\quad\times \langle L_1m_{L_1}S_1m_{S_1} | J_1m_{J_1} = m_{L_1}+m_{S_1}\rangle \delta_{J'J_1}\delta_{m_{L_1},m_{L'}+m_S-m_{S_1}} \bar{\delta}_{L_1,L'} \tau^{J_1STT_z}_{L_1,L'}(p^2,p_1^2; s-q^2) ,
\end{aligned}
\tag{4.13}
$$

where

$$\bar{\delta}_{L_1,L'} = \delta_{L_1,L'} - \delta_{|L_1-L'|,2} ,$$

with L', L_1 taking on values from $|J_1-S|$ to $|J_1+S|$. Space-reflection invariance restricts $|L_1-L'|$ to 0 or 2 in the case of tensor forces. For $S=1$, $J\neq 0$, and $(-1)^{T+J_1}=-1$,

$$-i\pi p\,\tau^{J1TT_z}_{L_1,L}(p^2,p^2; s=p^2) + \delta_{L_1,L}$$

is a two by two unitary and symmetric matrix with respect to the L_1,L indices.

Following a suggestion made by El-Baz *et al.*,[13] we use (2.9) and the relation[14]

$$
r^l Y^*_{lm}(\hat{r}) = \sum_{\lambda=0}^{l} \sum_{m_\lambda=-\lambda}^{\lambda} \frac{\sqrt{4\pi}}{\hat{\lambda}} \binom{2l+1}{2\lambda}^{1/2} (sr_a)^\lambda (tr_b)^{\lambda-l} \langle \lambda\, m_\lambda\, l-\lambda\, m-m_\lambda | l\, m\rangle Y^*_{\lambda m_\lambda}(\hat{r}_a) Y^*_{l-\lambda,\,m-m_\lambda}(\hat{r}_b) ,
\tag{4.14}
$$

where $\vec{r} = s\vec{r}_a + t\vec{r}_b$, $\hat{\lambda} = (2\lambda+1)^{1/2}$, and $\binom{2l+1}{2\lambda}$ is the binomial coefficient, to obtain

$$
\begin{aligned}
q_1^l p_1^{L_1} Y^*_{lm}(\hat{q}_1) Y_{L_1,m_{L_1}}(\hat{p}_1) = \sum_{\lambda,m_\lambda}\sum_{\Lambda,m_\Lambda} \frac{4\pi}{\hat{\lambda}\hat{\Lambda}} \binom{2l+1}{2\lambda}^{1/2} \binom{2L_1+1}{2\Lambda}^{1/2} (\beta_{12}p_2)^\lambda (-\alpha_{12}q_2)^{l-\lambda} (-\alpha_{12}p_2)^\Lambda (-\beta_{12}q_2)^{L_1-\Lambda} \\
\times \langle \lambda\, m_\lambda\, l-\lambda\, m-m_\lambda | l\, m\rangle \langle \Lambda\, m_\Lambda\, L_1-\Lambda\, m_{L_1}-m_\Lambda | L_1 m_{L_1}\rangle \\
\times Y^*_{\lambda m_\lambda}(\hat{p}_2) Y^*_{l-\lambda,\,m-m_\lambda}(\hat{q}_2) Y^*_{\Lambda m_\Lambda}(\hat{p}_2) Y^*_{L_1-\Lambda,\,m_{L_1}-m_\Lambda}(\hat{q}_2) .
\end{aligned}
\tag{4.15}
$$

After substituting (4.12) and (4.13) into (4.10) and integrating over $\hat{p}$ and $\hat{q}$, we use (4.15) and the partial-wave expansion

$$
T^{J_1STT_z}_{L_1,L,l}(p_1,q_1) = \sum_{r,m_r} T^{J_1STT_z}_{L_1,L,l,r}(p_2,q_2) Y^*_{rm_r}(\hat{p}_2) Y_{rm_r}(\hat{q}_2)
\tag{4.16}
$$

with

$$T^{J_1STT_z}_{L_1,L,l}(p_1,q_1)=\frac{2}{q}\,\delta(q^2-q_1^2)\,\frac{\tau^{J_1STT_z}_{L_1,L}(p^2,p_1^2;s-q^2)}{p_1^{L_1}q_1^{l}} \tag{4.17}$$

to obtain

$$\begin{aligned}{}^{(1)}K_2={}&\frac{1}{4\pi}\sum_{S_1,J_1}\delta_{SS_1}\delta_{JJ_1}\delta_{\mathcal{T}_z\mathcal{T}_{2z}}(-1)^{t_2-T_2-\mathcal{T}_2}\hat{T}\hat{T}_2W(t_1t_3\mathcal{T}_2t_2;T_2T)\sum_{T_z,t_z}\langle TT_zt t_z|\mathcal{T}\mathcal{T}_z\rangle\langle TT_zt t_z|\mathcal{T}_2\mathcal{T}_{2z}\rangle\\
&\times\sum_{L_1}\bar{\delta}_{L_1,L}\sum_{\lambda,\Lambda,r,r_1,r_2}\binom{2l+1}{2\lambda}^{1/2}\binom{2L_1+1}{2\Lambda}^{1/2}(\beta_{12}p_2)^{\lambda}(-\alpha_{12}q_2)^{l-\lambda}(-\alpha_{12}p_2)^{\Lambda}(-\beta_{12}q_2)^{L_1-\Lambda}\\
&\times T^{J_1STT_z}_{L_1,L,l,r}(p_2,q_2)(2r+1)(2L_2+1)^{1/2}(2l_2+1)^{1/2}(2r_1+1)(2r_2+1)[2(L_1-\Lambda)+1]^{1/2}[2(l-\lambda)+1]^{1/2}\\
&\times\begin{pmatrix}L_2&r&r_1\\0&0&0\end{pmatrix}\begin{pmatrix}\Lambda&\lambda&r_1\\0&0&0\end{pmatrix}\begin{pmatrix}r&l_2&r_2\\0&0&0\end{pmatrix}\begin{pmatrix}L_1-\Lambda&l-\lambda&r_2\\0&0&0\end{pmatrix}G_{J-j}\,,\end{aligned} \tag{4.18}$$

where G_{J-j} is the geometrical factor given by

$$\begin{aligned}G_{J-j}={}&\frac{1}{2\mathcal{J}+1}\sum_{\substack{\text{all } m\text{'s},\\ J_z}}(-1)^{m_r}\delta_{\mathcal{J}\mathcal{J}_2}\delta_{\mathcal{J}_z\mathcal{J}_{2z}}\delta_{m_{L_1}+m_{S_1},\,m_L+m_S}\delta_{m_\Lambda+m_\lambda,\,m_{L_2}-m_r}\delta_{m_r+m_{l_2},\,m_{L_1}+m_l-m_\Lambda-m_\lambda}\\
&\times\langle\mathcal{J}\mathcal{J}_z|Jm_Jjm_j\rangle\langle Jm_J|Lm_LSm_S\rangle\langle jm_j|lm_lsm_s\rangle\langle J_2m_{J_2}j_2m_{j_2}|\mathcal{J}_2\mathcal{J}_{2z}\rangle\langle L_2m_{L_2}S_2m_{S_2}|J_2m_{J_2}\rangle\\
&\times\langle l_2m_{l_2}s_2m_{s_2}|j_2m_{j_2}\rangle\langle Lm_LSm_S|J_1m_L+m_S\rangle\langle L_1m_{L_1}S_1m_{S_1}|J_1m_{L_1}+m_{S_1}\rangle\langle s_3m_{s_3}s_1m_{s_1}|S_2m_{S_2}\rangle\\
&\times\langle s_2m_{s_2}s_3m_{s_3}|S_1m_{S_1}\rangle\langle\lambda m_\lambda l-\lambda\, m_l-m_\lambda|lm\rangle\langle\Lambda m_\Lambda L_1-\Lambda\, m_{L_1}-m_\Lambda|L_1m_{L_1}\rangle\\
&\times\begin{pmatrix}L_2&r&r_1\\m_{L_2}&m_r&m_r-m_{L_2}\end{pmatrix}\begin{pmatrix}\Lambda&\lambda&r_1\\m_\Lambda&m_\lambda&-m_\lambda-m_\Lambda\end{pmatrix}\begin{pmatrix}r&l_2&r_2\\m_r&m_{l_2}&-m_r-m_{l_2}\end{pmatrix}\begin{pmatrix}L_1-\Lambda&l-\lambda&r_2\\m_{L_1}-m_\Lambda&m_l-m_\lambda&m_\Lambda+m_\lambda-m_{L_1}-m_l\end{pmatrix}.\end{aligned} \tag{4.19}$$

The quantities in the brackets in (4.19) are Wigner 3-j symbols which are related to the Clebsch-Gordan coefficients by

$$\langle a\alpha b\beta|c\gamma\rangle=(-1)^{a-b+\gamma}(2c+1)^{1/2}\begin{pmatrix}a&b&c\\\alpha&\beta&-\gamma\end{pmatrix}. \tag{4.20}$$

In deriving (4.18) and (4.19), we have used the relations

$$\begin{aligned}\int d\hat{p}_2Y_{L_2m_{L_2}}(\hat{p}_2)Y^*_{rm_r}(\hat{p}_2)Y^*_{\lambda m_\lambda}(\hat{p}_2)Y^*_{\Lambda m_\Lambda}(\hat{p}_2)={}&\frac{1}{4\pi}(-1)^{m_r}\delta_{m_\Lambda+m_\lambda,\,m_{L_2}-m_r}\hat{L}_2\hat{r}\hat{\Lambda}\hat{\lambda}\\
&\times\sum_{r_1}\hat{r}_1^{\,2}\begin{pmatrix}L_2&r&r_1\\0&0&0\end{pmatrix}\begin{pmatrix}L_2&r&r_1\\m_{L_2}&-m_r&m_r-m_{L_2}\end{pmatrix}\begin{pmatrix}\Lambda&\lambda&r_1\\0&0&0\end{pmatrix}\begin{pmatrix}\Lambda&\lambda&r_1\\m_\Lambda&m_\lambda&-m_\Lambda-m_\lambda\end{pmatrix}\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\int d\hat{q}_2Y_{rm_r}(\hat{q}_2)Y_{l_2m_{l_2}}(\hat{q}_2)Y^*_{L_1-\Lambda,\,m_{L_1}-m_\Lambda}(\hat{q}_2)Y^*_{l-\lambda,\,m_l-m_\lambda}(\hat{q}_2)={}&\frac{1}{4\pi}\delta_{m_r+m_{l_2},\,m_{L_1}+m_l-m_\Lambda-m_\lambda}\hat{r}\hat{l}_2\\
&\times[2(L_1-\Lambda)+1]^{1/2}[2(l-\lambda)+1]^{1/2}\sum_{r_2}\hat{r}_2^{\,2}\begin{pmatrix}r&l_2&r_2\\0&0&0\end{pmatrix}\begin{pmatrix}r&l_2&r_2\\m_r&m_{l_2}&-m_r-m_{l_2}\end{pmatrix}\begin{pmatrix}L_1-\Lambda&l-\lambda&r_2\\0&0&0\end{pmatrix}\\
&\times\begin{pmatrix}L_1-\Lambda&l-\lambda&r_2\\m_{L_1}-m_\Lambda&m_l-m_\lambda&m_\Lambda+m_\lambda-m_{L_1}-m_l\end{pmatrix},\end{aligned} \tag{4.22}$$

which are obtained by repeated applications of the addition theorem for the spherical harmonics.

The geometrical factor G_{J-j} can be simplified by angular momentum algebra (see Appendix A):

$$G_{J-j}=(-1)^{\mathcal{J}-s+l_1+S_1+J_2+r}\{LSJ\}\hat{J}\hat{j}\hat{J}_2\hat{j}_2\hat{S}_1\hat{S}_2\hat{l}\hat{L}_1\sum_y(-1)^y(2y+1)\,{}^1\begin{Bmatrix} r & r_2 & L_1-\Lambda & L_1 \\ l_2 & l-\lambda & \Lambda & L_2 \\ y & l & \lambda & r_1 \end{Bmatrix}\,{}^5\begin{Bmatrix} s & L_1 & j_2 & \mathcal{J} & J_2 \\ L_2 & j & s_2 & s_3 & S_1 \\ J_1 & S_2 & l_2 & y & l \end{Bmatrix}, \tag{4.23}$$

where $\{LSJ\}$ indicates the triangular relation among L, S, and J. The two quantities in the brackets in the y summation in (4.23) are the 12-j symbol of the first kind and the 15-j symbol of the fifth kind, which can be written in terms of 6-j symbols as[20]

$${}^1\begin{Bmatrix} r & r_2 & L_1-\Lambda & L_1 \\ l_2 & l-\lambda & \Lambda & L_2 \\ y & l & \lambda & r_1 \end{Bmatrix}=\sum_x(-1)^{R_4-x}(2x+1)\begin{Bmatrix} r & y & x \\ l & r_2 & l_2 \end{Bmatrix}\begin{Bmatrix} r_2 & l & x \\ \lambda & L_1-\Lambda & l-\lambda \end{Bmatrix}\begin{Bmatrix} L_1-\Lambda & \lambda & x \\ r_1 & L_1 & \Lambda \end{Bmatrix}\begin{Bmatrix} L_1 & r_1 & x \\ r & y & L_2 \end{Bmatrix}, \tag{4.24}$$

with $R_4=y+l_2+L_2+r_1+r_2+r$, and

$${}^5\begin{Bmatrix} s & L_1 & j_2 & \mathcal{J} & J_2 \\ L_2 & j & s_2 & s_3 & S_1 \\ J_1 & S_2 & l_2 & y & l \end{Bmatrix}$$

$$=\sum_{x_1,x_2}(2x_1+1)(2x_2+1)(-1)^{j_2+s_2+s+L_1-L_2+j-S_1+y+x_2}\begin{Bmatrix} s_3 & l & x_1 \\ j & S_2 & s \end{Bmatrix}\begin{Bmatrix} S_1 & y & x_2 \\ L_2 & J_1 & L_1 \end{Bmatrix}\begin{Bmatrix} s_3 & l & x_1 \\ S_1 & y & x_2 \\ s_2 & l_2 & j_2 \end{Bmatrix}\begin{Bmatrix} j & S_2 & x_1 \\ J_1 & L_2 & x_2 \\ \mathcal{J} & J_2 & j_2 \end{Bmatrix}, \tag{4.25}$$

where the last two brackets are the usual 9-j symbols.

From (4.16), we have

$$T^{J_1STT_z}_{L_1,L,l,r}(p_2,q_2)=2\pi\int_{-1}^{+1}\frac{2}{q}\delta(q^2-q_1^2)\frac{T^{J_1STT_z}_{L_1,L}(p^2,p_1^2;s-q^2)}{p_1^{L_1}q_1^{l}}P_r(\cos\theta)d(\cos\theta)\ , \tag{4.26}$$

where θ is the angle between $\hat{p}_2$ and $\hat{q}_2$. Equation (4.26) reduces to[13]

$$T^{J_1STT_z}_{L_1,L,l,r}(p_2,q_2)=\frac{2\pi}{\alpha_{12}\beta_{12}p_2q_2}\frac{T^{J_1STT_z}_{L_1,L}(p^2,p_1^2;s-q^2)}{(p_2^2+q_2^2-q^2)^{L_1/2}q^{l+1}}P_r(\cos\theta')[H(\cos\theta'+1)-H(\cos\theta'-1)]\ , \tag{4.27}$$

where

$$\cos\theta'=\frac{\beta_{12}^2p_2^2+\alpha_{12}^2q_2^2-q^2}{2\alpha_{12}\beta_{12}p_2q_2} \tag{4.27a}$$

and $H(x)$ is the Heaviside unit function.

The final result for the integral part of the Faddeev equation (4.6) is

$$\int_0^\infty p_2^2dp_2\int_0^\infty q_2^2dq_2\frac{{}^{(1)}K_2}{p_2^2+q_2^2-s}\,\psi_s^{(1)}(p_2,q_2,\alpha_2)$$

$$=\tfrac{1}{2}\sum_{S_1,J_1}\delta_{JJ_1}\delta_{SS_1}\delta_{\mathcal{T}_z\mathcal{T}_{2z}}(-1)^{t_2-T_2-\mathcal{T}_2}\hat{T}\hat{T}_2W(t_1t_3\mathcal{T}_2t_2;\,T_2T)\sum_{T_z,t_z}\langle TT_ztt_z|\mathcal{T}\mathcal{T}_z\rangle\langle TT_ztt_z|\mathcal{T}_2\mathcal{T}_{2z}\rangle$$

$$\times\sum_{L_1}\bar{\delta}_{L_1,L}\sum_{\lambda\Lambda rr_1r_2}\binom{2l+1}{2\lambda}^{1/2}\binom{2L_1+1}{2\Lambda}^{1/2}(\alpha_{12})^{l-\lambda+\Lambda-1}(\beta_{12})^{\lambda+L_1-\Lambda-1}(-1)^{L_1+l-\lambda}(2L_2+1)^{1/2}(2l_2+1)^{1/2}$$

$$\times(2r_1+1)(2r_2+1)[2(L_1-\Lambda)+1]^{1/2}[2(l-\lambda)+1]^{1/2}\hat{r}^2\begin{pmatrix} L_2 & r & r_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \Lambda & \lambda & r_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r & l_2 & r_2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L_1-\Lambda & l-\lambda & r_2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times G_{J-j}\,\frac{1}{q^{l+1}}\int_0^\infty dq_2\,q_2^{L_1-\Lambda+l-\lambda+1}\int_{|\alpha_{12}q_2-q|/\beta_{12}}^{(\alpha_{12}q_2+q)/\beta_{12}} dp_2\,p_2^{\Lambda+\lambda+1}\,\frac{\tau_{L_1,L}^{J_1STT_z}(p^2,p_2^2+q_2^2-q^2;s-q^2)}{(p_2^2+q_2^2-s)(p_2^2+q_2^2-q^2)^{L_1/2}}$$

$$\times P_r\left(\frac{\beta_{12}^2p_2^2+\alpha_{12}^2q_2^2-q^2}{2\alpha_{12}\beta_{12}p_2q_2}\right)\psi_s^{(1)}(p_2,q_2,\alpha_2)\,. \tag{4.28}$$

The τ function is related to the two-body t matrix for $S=0$ and $T=0$ or 1 by the relation

$$\tau_{L,L}^{L0TT_z}(p^2,p_1^2;s-q^2)=-\frac{2}{\pi}\,t_L(p,p_1;(s-q^2)^{1/2})\,, \tag{4.29}$$

where t_L is normalized so that $t_L(k,k;k)=e^{i\delta_L}\sin\delta_L/k$ with the Lth partial-wave phase shift δ_L. For $T_z=-1$ (proton-proton system), we must include the Coulomb interaction in the two-body t matrix. The complete off-shell two-body t matrix can be calculated for the general case of tensor and singular core interactions by a straightforward integral-equation approach.[24,25]

For the scattering problem, we need to specify the inhomogeneous term $\varphi_s^{(i)}(p,q,\alpha)$ in Eq. (4.6). This term contains the matrix element ${}^{(i)}K_i$ in addition to ${}^{(i)}K_j$ given in Eq. (4.18). The calculation of ${}^{(i)}K_i$ is much simpler than that of ${}^{(i)}K_j$. We will only present the result for ${}^{(1)}K_1$:

$${}^{(1)}K_1=\frac{2}{q}\delta(q^2-q_1^2)\delta_{jj_1}\delta_{ll_1}\delta_{ss_1}\delta_{JJ_1}\delta_{SS_1}\delta_{\mathcal{J}\mathcal{J}_1}\delta_{\mathcal{J}_z\mathcal{J}_{1z}}\delta_{\mathcal{T}_z\mathcal{T}_{1z}}\sum_{T_z,t_z}\langle TT_ztt_z|\mathcal{T}\mathcal{T}_z\rangle\langle TT_ztt_z|\mathcal{T}_1\mathcal{T}_{1z}\rangle$$

$$\times\bar{\delta}_{L_1,L}\{lsj\}\{Jj\mathcal{J}\}\{LSJ\}\{L_1S_1J_1\}\,\tau_{L_1,L}^{J_1STT_z}(p^2,p_1^2;s-q^2)\,, \tag{4.30}$$

where subscript 1 indicates the initial-state quantum numbers.

For the special case of spinless identical particles ($\vec{s}_1=\vec{s}_2=\vec{s}_3=0$ and $\vec{t}_1=\vec{t}_2=\vec{t}_3=0$), G_{J-j} as defined in Eq. (4.23) reduces to

$$G_{J-j}=(-1)^{\mathcal{L}+r_1+r_2+r}\,(\hat{l}\hat{L})\begin{Bmatrix} r_1 & r_2 & \mathcal{L}\\ l_2 & L_2 & r\end{Bmatrix}\begin{Bmatrix} L_1 & \Lambda & L_1-\Lambda\\ l & \lambda & l-\lambda\\ \mathcal{L} & r_1 & r_2\end{Bmatrix}. \tag{4.31}$$

Substituting (4.31) into (4.18), we obtain ${}^{(1)}K_2$ for the special case of spinless identical particles

$${}^{(1)}K_2=\frac{1}{4\pi}\sum_{\lambda\Lambda r_1r_2r}(-1)^{\mathcal{L}+r_1+r_2+r}\binom{2l+1}{2\lambda}^{1/2}\binom{2L+1}{2\Lambda}^{1/2}(\beta_{12}p_{12})^{\lambda}(-\alpha_{12}q_2)^{l-\lambda}(-\alpha_{12}p_2)^{\Lambda}(-\beta_{12}q_2)^{L-\Lambda}$$

$$\times\hat{r}_1^2\hat{r}_2^2\hat{r}^2\hat{l}\hat{L}\hat{l}_2\hat{L}_2[2(L-\Lambda)+1]^{1/2}[2(l-\lambda)+1]^{1/2}\begin{pmatrix} L_2 & r & r_1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} \Lambda & \lambda & r_1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} r & l_2 & r_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} L-\Lambda & l-\lambda & r_2\\ 0 & 0 & 0\end{pmatrix}T_{L,L,l}^{L000}(p_2,q_2)\,, \tag{4.32}$$

which agrees with the result of El-Baz *et al.*[13] when their phase factor $(-1)^{\mathcal{L}}$ is multiplied by the correction factor $(-1)^{\mathcal{L}_1+\mathcal{L}_2+J}$.

V. $\mathcal{L}$-$\mathcal{S}$ COUPLING SCHEME

In the $\mathcal{L}$-$\mathcal{S}$ coupling scheme, we form (c.m. system) eigenstates of total angular momentum $\mathcal{J}$ and projection of total angular momentum on the z axis of a space-fixed coordinate system $\mathcal{J}_z$ from a direct product of the eigenstates of $\mathcal{L}$ (relative orbital angular momentum of a nucleon pair plus angular momentum of the third nucleon in the c.m. system) and $\mathcal{S}$ (total spin angular momentum of a nucleon pair plus spin angular momentum of the third nucleon). The isospin parts of these states are the same as those in the J-j coupling scheme. The complete list of commuting operators which characterize the $\mathcal{L}$-$\mathcal{S}$ coupling scheme was given in Sec. III.

The explicit construction of the complete set of $\mathcal{L}$-$\mathcal{S}$ basis states is given by

$$\begin{aligned}|p,q;\alpha\rangle_i &= |p,q;\alpha(i,jk)\rangle_i\\ &= |[pq(Ll)\mathcal{L},(Ss)\mathcal{S}]\mathcal{J}\mathcal{J}_z;(Tt)\mathcal{T}\mathcal{T}_z\rangle_i\\ &= \sum_{m_{\mathcal{L}},m_{\mathcal{S}}}\langle\mathcal{L}m_{\mathcal{L}}\mathcal{S}m_{\mathcal{S}}|\mathcal{J}\mathcal{J}_z\rangle|pq(Ll)\mathcal{L}m_{\mathcal{L}}\rangle_i\,|(Ss)\mathcal{S}m_{\mathcal{S}}\rangle_i\,|(Tt)\mathcal{T}\mathcal{T}_z\rangle_i\,,\end{aligned} \tag{5.1}$$

with

$$|pq(Ll)\mathcal{L}m_{\mathcal{L}}\rangle_i = \sum_{m_L, m_l} \langle Lm_L lm_l | \mathcal{L}m_{\mathcal{L}}\rangle | pLm_L; qlm_l\rangle_i \tag{5.2}$$

and

$$|(Ss)\mathcal{S}m_{\mathcal{S}}\rangle_i = \sum_{m_S, m_s} \langle Sm_S sm_s | \mathcal{S}m_{\mathcal{S}}\rangle | Sm_S\rangle_i | sm_s\rangle_i \,. \tag{5.3}$$

The eigenstates given by (5.1)–(5.3) are related to those of the J-j coupling scheme (4.1) by a unitary transformation:

$$|[pq(Ll)\mathcal{L},(Ss)\mathcal{S}]\mathcal{J}\mathcal{J}_z; (Tt)\mathcal{T}\mathcal{T}_z\rangle_i = \sum_{J,j} \hat{J}\hat{j}\,\hat{\mathcal{L}}\hat{\mathcal{S}} \begin{Bmatrix} L & l & \mathcal{L} \\ S & s & \mathcal{S} \\ J & j & \mathcal{J} \end{Bmatrix} |[p(LS)J, q(ls)j]\mathcal{J}\mathcal{J}_z; (Tt)\mathcal{T}\mathcal{T}_z\rangle_i \,. \tag{5.4}$$

The two-particle spin eigenstate $|Sm_S\rangle_i$ is the same as that defined in (4.3). As in the J-j scheme, antisymmetrization with respect to jk exchange requires that

$$(-1)^{L+S+T} = -1 \,.$$

The orthonormality relation and the expansion of the states (5.1) in eigenstates of linear momentum are similar to those given in (4.5) and (4.4), respectively.

The Faddeev equation, in the $\mathcal{L}$-$\mathcal{S}$ coupling scheme, is the same as (4.6) with $|p,q,\alpha\rangle_1$ denoting $\mathcal{L}$-$\mathcal{S}$ basis states and the kernel ${}^{(1)}K_2$ given by

$$\begin{aligned} {}^{(1)}K_2 = \sum_{\substack{\text{(all magnetic quantum} \\ \text{numbers except } \mathcal{J}_z, \mathcal{J}_{2z})}} & \langle \mathcal{J}\mathcal{J}_z | \mathcal{L}m_{\mathcal{L}}\mathcal{S}m_{\mathcal{S}}\rangle \langle \mathcal{L}m_{\mathcal{L}} | Lm_L lm_l\rangle \langle \mathcal{S}m_{\mathcal{S}} | Sm_S sm_s\rangle \langle \mathcal{L}_2 m_{\mathcal{L}_2} \mathcal{S}_2 m_{\mathcal{S}_2} | \mathcal{J}_2\mathcal{J}_{2z}\rangle \\ & \times \langle L_2 m_{L_2} l_2 m_{l_2} | \mathcal{L}_2 m_{\mathcal{L}_2}\rangle \langle S_2 m_{S_2} s_2 m_{s_2} | \mathcal{S}_2 m_{\mathcal{S}_2}\rangle \int d\hat{p}_2 \int d\hat{q}_2 \int d\hat{p} \int d\hat{q}\, Y^*_{Lm_L}(\hat{p}) Y^*_{lm_l}(\hat{q}) Y_{L_2 m_{L_2}}(\hat{p}_2) Y_{l_2 m_{l_2}}(\hat{q}_2) \\ & \times {}_1\langle \vec{p},\vec{q}; Sm_S sm_s; (Tt)\mathcal{T}\mathcal{T}_z | T_1(s) | \vec{p}_2,\vec{q}_2; S_2 m_{S_2}, s_2 m_{s_2}; (T_2 t_2)\mathcal{T}_2\mathcal{T}_{2z}\rangle \,. \end{aligned} \tag{5.5}$$

(5.5) differs from (4.10) only in the first six Clebsch-Gordan coefficients, and hence the angular momentum reduction for ${}^{(1)}K_2$ in the $\mathcal{L}$-$\mathcal{S}$ coupling scheme is identical with that given by (4.18) except for the replacement of $G_{J-j}\,\delta_{JJ_1}$ by $G_{\mathcal{L}-\mathcal{S}}$, where

$$\begin{aligned} G_{\mathcal{L}-\mathcal{S}} = \frac{1}{2\mathcal{J}+1} \sum_{\substack{\text{(all magnetic} \\ \text{quantum numbers)}}} & (-1)^{m_r} \delta_{\mathcal{J}\mathcal{J}_2} \delta_{\mathcal{J}_z, \mathcal{J}_{2z}} \delta_{m_{L_1}+m_{S_1}, m_L+m_S} \delta_{m_\Lambda+m_\lambda, m_{L_2}-m_r} \delta_{m_r+m_{l_2}, m_{L_1}+m_l-m_\Lambda-m_\lambda} \\ & \times \langle \mathcal{J}\mathcal{J}_z | \mathcal{L}m_{\mathcal{L}}\mathcal{S}m_{\mathcal{S}}\rangle \langle \mathcal{L}m_{\mathcal{L}} | Lm_L lm_l\rangle \langle \mathcal{S}m_{\mathcal{S}} | Sm_S sm_s\rangle \langle \mathcal{L}_2 m_{\mathcal{L}_2} \mathcal{S}_2 m_{\mathcal{S}_2} | \mathcal{J}_2\mathcal{J}_{2z}\rangle \langle L_2 m_{L_2} l_2 m_{l_2} | \mathcal{L}_2 m_{\mathcal{L}_2}\rangle \\ & \times \langle S_2 m_{S_2} s_2 m_{s_2} | \mathcal{S}_2 m_{\mathcal{S}_2}\rangle \langle L m_L S m_S | J_1 m_L + m_S\rangle \langle L_1 m_{L_1} S_1 m_{S_1} | J_1 m_{L_1} + m_{S_1}\rangle \\ & \times \langle s_3 m_{s_3} s_1 m_{s_1} | S_2 m_{S_2}\rangle \langle s_2 m_{s_2} s_3 m_{s_3} | S_1 m_{S_1}\rangle \langle \lambda m_\lambda\, l-\lambda\, m-m_\lambda | lm\rangle \langle \Lambda m_\Lambda\, L_1-\Lambda\, m_{L_1}-m_\Lambda | L_1 m_{L_1}\rangle \\ & \times \begin{pmatrix} L_2 & r & r_1 \\ m_2 & m_r & m_r - m_{L_2} \end{pmatrix} \begin{pmatrix} \Lambda & \lambda & r_1 \\ m_\Lambda & m_\lambda & -m_\lambda - m_\Lambda \end{pmatrix} \begin{pmatrix} r & l_2 & r_2 \\ m_r & m_{l_2} & -m_r - m_{l_2} \end{pmatrix} \begin{pmatrix} L_1-\Lambda & l-\lambda & r_2 \\ m_{L_1}-m_\Lambda & m_l-m_\lambda & m_\Lambda+m_\lambda-m_L-m_l \end{pmatrix} . \end{aligned} \tag{5.6}$$

(5.6) reduces to (see Appendix A)

$$G_{\mathcal{L}-\mathcal{S}} = (-1)^{r_1+r_2+r+2s+\mathcal{S}+\mathcal{S}_2+S_2+\mathcal{L}_2}\, \hat{\mathcal{S}}\hat{\mathcal{L}}\hat{\mathcal{S}}_2\hat{\mathcal{L}}_2\hat{J}_1^2\hat{S}_2\hat{S}_1\hat{l}\hat{L}_1 \begin{Bmatrix} s_2 & s_3 & S_1 \\ s & \mathcal{S}_2 & S_2 \end{Bmatrix} \begin{Bmatrix} r_1 & r_2 & \mathcal{L}_2 \\ l_2 & L_2 & r \end{Bmatrix} \begin{Bmatrix} L_1 & \Lambda & L_1-\Lambda \\ l & \lambda & l-\lambda \\ \mathcal{L}_2 & r_1 & r_2 \end{Bmatrix} {}^2\!\begin{bmatrix} J_1 & l & s & \mathcal{J} \\ S_1 & S & \mathcal{L}_2 & \mathcal{L} \\ L_1 & \mathcal{S}_2 & L & \mathcal{S} \end{bmatrix} , \tag{5.7}$$

where the square bracket is the 12-j symbol of the second kind[20]:

$$\begin{bmatrix} J_1 & l & s & \mathcal{J} \\ S_1 & S & \mathcal{L}_2 & \mathcal{L} \\ L_1 & \mathcal{S}_2 & L & \mathcal{S} \end{bmatrix} = (-1)^{S_1-s-\mathcal{L}_2+\mathcal{L}} \sum_x (2x+1) \begin{Bmatrix} L_1 & \mathcal{S}_2 & x \\ s & J_1 & S_1 \end{Bmatrix} \begin{Bmatrix} L & \mathcal{S} & x \\ s & J_1 & S \end{Bmatrix} \begin{Bmatrix} L_1 & \mathcal{S}_2 & x \\ \mathcal{J} & l & \mathcal{L}_2 \end{Bmatrix} \begin{Bmatrix} L & \mathcal{S} & x \\ \mathcal{J} & l & \mathcal{L} \end{Bmatrix} . \tag{5.8}$$

The final expression for the integral part of (4.6) in the $\mathcal{L}$-$\mathcal{S}$ coupling scheme is the same as (4.28) with $G_{\mathcal{L}-\mathcal{S}}$ replacing $G_{J-j}\delta_{JJ_1}$.

For the special case of spinless identical particles ($\vec{s}_1=\vec{s}_2=\vec{s}_3=0$, $\vec{t}_1=\vec{t}_2=\vec{t}_3=0$), $G_{\mathcal{L}-\mathcal{S}}$ and ${}^{(1)}K_2$ reduce to (4.31) and (4.32), respectively.

The matrix element ${}^{(i)}K_i$ required for the inhomogeneous term $\varphi_s^{(i)}$ can be obtained by a similar method. The final expression for ${}^{(1)}K_1$ in the $\mathcal{L}$-$\mathcal{S}$ coupling scheme is

$$
\begin{aligned}
{}^{(1)}K_1 = & \sum_{J,J_1} \frac{2}{q}\delta(q^2-q_1^2)(-1)^{L+S+\mathcal{L}+\mathcal{S}+L_1+S_1+\mathcal{L}_1+\mathcal{S}_1+2s}\delta_{ll_1}\delta_{ss_1}\delta_{JJ_1}\delta_{SS_1}\delta_{TT_1}\delta_{T_zT_{1z}}\delta_{\mathcal{T}_z\mathcal{T}_{1z}} \\
& \times\hat{\mathcal{L}}\,\hat{\mathcal{S}}\,\hat{\mathcal{L}}_1\,\hat{\mathcal{S}}_1\hat{J}_1^2\sum_{T_z}\langle TT_z t t_z|\mathcal{T}\mathcal{T}_z\rangle\langle TT_z t t_z|\mathcal{T}_1\mathcal{T}_{1z}\rangle\bar{\delta}_{L_1,L} \\
& \times\sum_x(2x+1)\begin{Bmatrix}\mathcal{S}_1 & L_1 & x\\ l & \mathcal{J} & \mathcal{L}_1\end{Bmatrix}\begin{Bmatrix}\mathcal{S} & L & x\\ l & \mathcal{J} & \mathcal{L}\end{Bmatrix}\begin{Bmatrix}\mathcal{S}_1 & L_1 & x\\ J_1 & s & S_1\end{Bmatrix}\begin{Bmatrix}\mathcal{S} & L & x\\ J_1 & s & S\end{Bmatrix}\tau_{L_1,L}^{J_1 STT_z}(p^2,p_1^2;s-q^2),
\end{aligned}
\qquad (5.9)
$$

where subscript 1 indicates the initial-state quantum numbers.

VI. SUMMARY AND DISCUSSION

We have obtained complete angular momentum reductions of the Faddeev equations for three-nucleon systems in two different coupling schemes (J-j and $\mathcal{L}$-$\mathcal{S}$). The two-nucleon interaction is assumed to have general space, spin, isospin, and velocity dependence consistent with invariance under the usual space-time translations and inversions, Galilean boosts, rotations in ordinary space, and rotations about the z axis in isospin space. Complete antisymmetrization of states with respect to particle exchange is easily accomplished by using the properties of the permutation group and the isospin formalism. The extraction of a properly antisymmetrized wave function for the three-nucleon system from the solution of the Faddeev equation (3.14) was briefly discussed in Sec. III.

For the special case of separable two-nucleon interactions, the results of this paper can be used to reduce the Faddeev equation (3.14) to a set of coupled integral equations in one continuous variable.

In future publications, we will give a detailed theoretical and numerical analysis of three-nucleon bound-state wave functions, electromagnetic form factors, and low-energy scattering parameters, based on the formalism of this paper and that of Refs. 3 and 25.

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APPENDIX A. GRAPHICAL REDUCTIONS OF GEOMETRICAL FACTORS G_{J-j} AND $G_{\mathcal{L}-\mathcal{S}}$

In this Appendix, we present graphical reductions of G_{J-j} and $G_{\mathcal{L}-\mathcal{S}}$, defined in (4.19) and (5.6) by the graphical method of Yutsis, Levinson, and Vanagas (YLV).[20]

We first consider G_{J-j}, which can be reduced to

$$G_{J-j}=(-1)^{3\mathcal{J}-s_3-r_1-r_2-r}\hat{J}\hat{j}\,\hat{J}_2\hat{j}_2\hat{S}_1\hat{S}_2\hat{l}\hat{L}_1\{LSJ\}H_{J-j}, \qquad (A1)$$

where $\{LSJ\}$ indicates the triangular relation among L, S, and J, and H_{J-j} is graphically given in Fig. 1. In drawing the graphs in figures, we have made two minor modifications of the YLV method. We have dropped the directional arrows for lines corresponding to integer spins and have retained the directional arrows for lines corresponding to half-integer spins. Also, when the sum of the upper three arguments of the 3-j symbol representing a node is an even integer, the node is enclosed by a circle, and the positive or negative sign omitted because, in this case, the orientation of the node is irrelevant. Orientation of the node is positive or negative depending upon whether the labeling is counterclockwise or clockwise, respectively. By cutting through the lines l, l_2, L_1, and L_2, H_{J-j} can be decomposed into a product of H_2 and H_3, i.e.,

$$H_{J-j}=\sum_y(2y+1)H_2H_3, \qquad (A2)$$

where H_2 and H_3 are diagrammatically represented in Fig. 1. H_2 reduces to the 12-j symbol of the first kind:

$$H_2={}^{1}\begin{Bmatrix} r & r_2 & L_1-\Lambda & L_1\\ l_2 & l-\lambda & \Lambda & L_2\\ y & l & \lambda & r_1\end{Bmatrix}, \qquad (A3)$$

which can be reduced to products of the 6-j symbols as given in Eq. (4.24). Similarly, the diagram H_3 reduces to

$$H_3=(-1)^{L_1+S_1+J_2+y+2s_2}\;{}^{5}\begin{Bmatrix} s & L_1 & j_2 & \mathcal{J} & J_2\\ L_2 & j & s_2 & s_3 & S_1\\ J_1 & S_2 & l_2 & y & l\end{Bmatrix}, \qquad (A4)$$

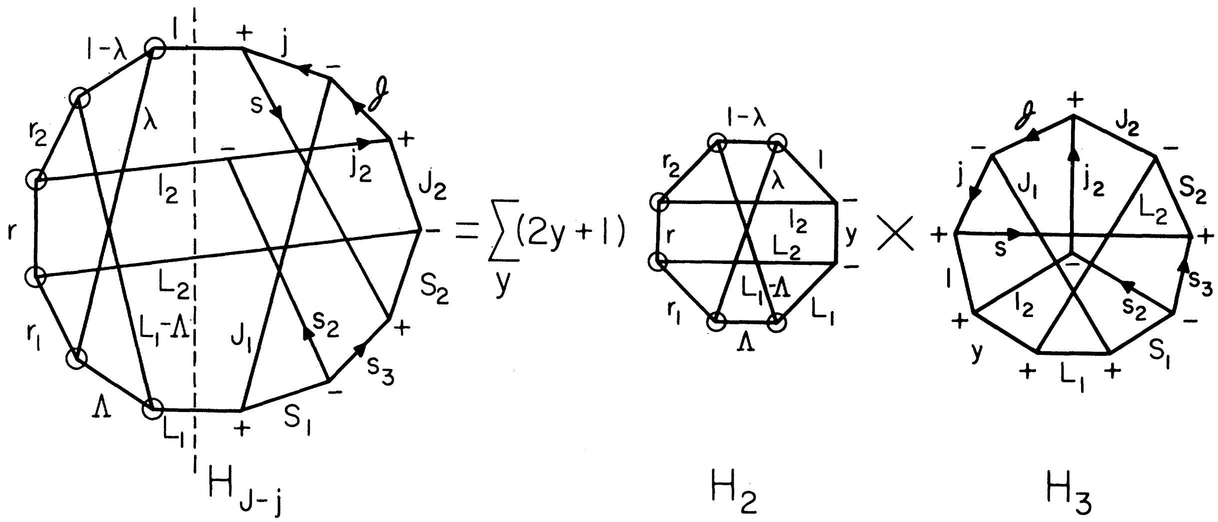

FIG. 1. Graphic representation of H_{J-j} and its reduction in terms of H_2 and H_3. H_{J-j} is related to G_{J-j} according to (A1). The dashed line indicates where the cut is made.

where the last factor is the 15-j symbol of the fifth kind, which reduces to Eq. (4.25). Combining (A1), (A2), (A3), and (A4), we obtain the final result G_{J-j} given in Eq. (4.23).

In the case of the $\mathcal{L}$-$\mathcal{S}$ coupling scheme, we can reduce $G_{\mathcal{L}-\mathcal{S}}$ to

$$G_{\mathcal{L}-\mathcal{S}} = (-1)^{\mathcal{J}+s-r_1-r_2-r}\, \hat{\mathcal{S}}\hat{\mathcal{L}}\,\hat{s}_2\hat{\mathcal{L}}_2\hat{J}_1^{\,2}\hat{S}_1\hat{S}_2\hat{l}\hat{L}_1 H_{\mathcal{L}-\mathcal{S}}\,, \tag{A5}$$

where $H_{\mathcal{L}-\mathcal{S}}$ is given graphically in Fig. 2.

Again cutting the lines l, l_2, L_1, and L_2 in $H_{\mathcal{L}-\mathcal{S}}$, we reduce $H_{\mathcal{L}-\mathcal{S}}$ to a product of H_2 and H_4:

$$H_{\mathcal{L}-\mathcal{S}} = \sum_y (2y+1) H_2 H_4\,, \tag{A6}$$

where H_2 is graphically given in Fig. 1, and defined in Eq. (A3), and H_4 is graphically given in Fig. 2. H_4 can be further reduced by making two cuts indicated in Fig. 2:

$$H_4 = h_1 h_2 h_3\,. \tag{A7}$$

The graphical representations of h_1, h_2, and h_3 are given in Fig. 3. The h_1, h_2, and h_3 factors reduce to

$$h_1 = (-1)^{l+L_1+\mathcal{L}_2} \begin{Bmatrix} l & L_1 & \mathcal{L}_2 \\ L_2 & l_2 & y \end{Bmatrix}\,, \tag{A8}$$

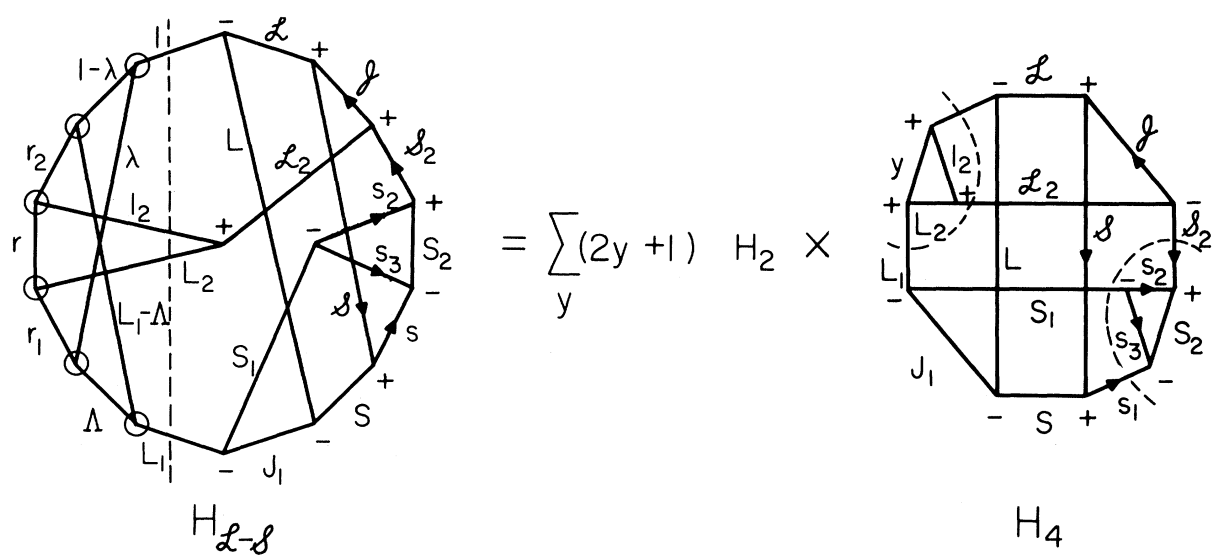

FIG. 2. Graphic representation of $H_{\mathcal{L}-\mathcal{S}}$ and its reduction in terms of H_2 and H_4. $H_{\mathcal{L}-\mathcal{S}}$ is related to $G_{\mathcal{L}-\mathcal{S}}$ according to (A5). The cuts in $H_{\mathcal{L}-\mathcal{S}}$ and H_4 are indicated by the dashed lines. The graphic representation of H_2 is given in Fig. 1.

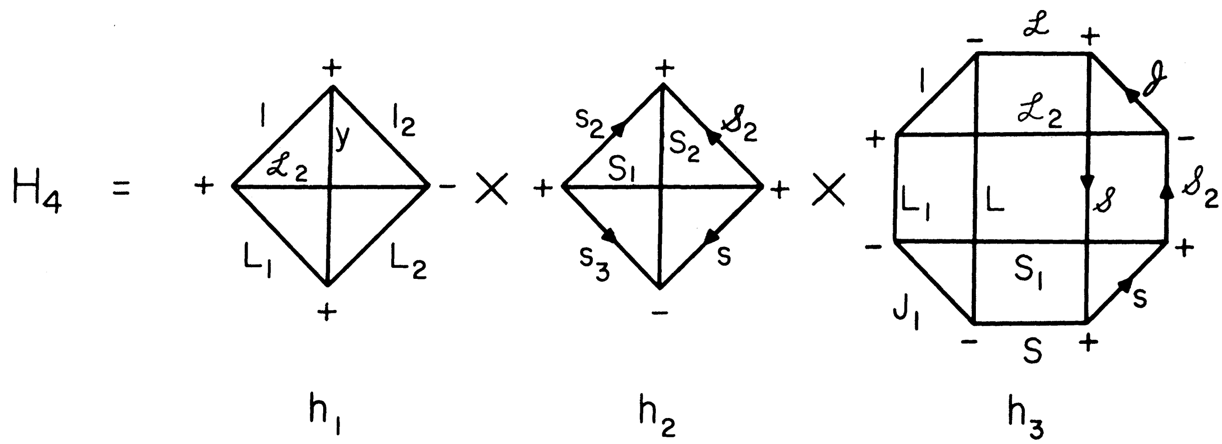

FIG. 3. Graphic reduction of $H_4 = h_1 h_2 h_3$. The graphic representation of H_4 is given in Fig. 2.

$$h_2 = (-1)^{s_2+S_2+3\mathcal{S}_2} \begin{Bmatrix} s_2 & s_3 & S_1 \\ s & \mathcal{S}_2 & S_2 \end{Bmatrix}, \tag{A9}$$

and

$$h_3 = (-1)^{L+l+\mathcal{S}+\mathcal{J}} \, {}^{2}\!\begin{bmatrix} J_1 & l & s & \mathcal{J} \\ S_1 & S & \mathcal{L}_2 & \mathcal{L} \\ L_1 & \mathcal{S}_2 & L & \mathcal{S} \end{bmatrix} . \tag{A10}$$

The last bracket factor is the 12-j symbol of the second kind, which reduces to Eq. (5.8). Using the easily verified relation,[20]

$$\sum_y (2y+1) \, {}^{1}\!\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ y & k_2 & k_3 & k_4 \end{Bmatrix} \begin{Bmatrix} l_1 & k_2 & y \\ j_4 & l_4 & k_1 \end{Bmatrix}$$

$$= (-1)^{l_2+l_3+j_3+k_3-k_1} \begin{Bmatrix} j_2 & k_4 & k_1 \\ j_3 & l_3 & j_4 \\ l_2 & k_3 & k_2 \end{Bmatrix} \begin{Bmatrix} j_2 & k_4 & k_1 \\ l_4 & l_1 & j_1 \end{Bmatrix}, \tag{A11}$$

we can sum over y in (A6) to obtain

$$\sum_y (2y+1) H_2 h_1 = (-1)^{r_1+r_2+\mathcal{L}_2} \times \begin{Bmatrix} L_1 & \Lambda & L_1-\Lambda \\ l & \lambda & l-\lambda \\ \mathcal{L}_2 & r_1 & r_2 \end{Bmatrix} \begin{Bmatrix} r_1 & r_2 & \mathcal{L}_2 \\ l_2 & L_2 & r \end{Bmatrix}. \tag{A12}$$

Combination of (A5) through (A12) yields the result for $G_{\mathcal{L}-\mathcal{S}}$ given in (5.7).

APPENDIX B. ISOSPIN EIGENSTATES

The proton and neutron isospin states are

$$|p\rangle = |t, t_z = -\tfrac{1}{2}\rangle ,$$

and (B1)

$$|n\rangle = |t, t_z = +\tfrac{1}{2}\rangle ,$$

respectively.

For the three-nucleon system, we couple the isospins of the kl pair to form the total isospin $\vec{\mathrm{T}}_i$ of the pair, and then obtain the total isospin $\vec{\mathcal{T}}_i$ by coupling $\vec{\mathrm{T}}_i$ and the isospin of the third particle $\vec{\mathrm{t}}_i$, i.e., $\vec{\mathcal{T}}_i = \vec{\mathrm{T}}_i + \vec{\mathrm{t}}_i$. In this coupling scheme, the eigenstates of total isospin are

$$\begin{aligned} |\mathcal{T}\mathcal{T}_z\rangle_i &= |(T_i t_i)\mathcal{T}_i \mathcal{T}_{iz}\rangle_i , \\ &= |[(t_k t_l)T_i t_i]\mathcal{T}_i \mathcal{T}_{iz}\rangle_i , \\ &= \sum_{T_{iz}, t_{iz}} \langle T_i T_{iz} t_i t_{iz} | \mathcal{T}_i \mathcal{T}_{iz}\rangle |(t_k t_l)T_i T_{iz}\rangle |t_i t_{iz}\rangle \end{aligned}$$

$$(ikl \text{ cyclic}), \tag{B2}$$

where

$$|(t_k t_l)T_i T_{iz}\rangle = \sum_{t_{kz}, t_{lz}} \langle t_k t_{kz} t_l t_{lz} | T_i T_{iz}\rangle \times |t_k t_{kz}\rangle |t_l t_{lz}\rangle . \tag{B3}$$

From these definitions, it is straightforward to show that

$$|(T_i t_i)\mathcal{T}_i \mathcal{T}_{iz}\rangle_i = \sum_{T_k'} (-1)^{t_k+T_k'-T_i}[(2T_i+1)(2T_k'+1)]^{1/2} \times W(t_k t_l \mathcal{T}_i t_i; T_i T_k')|(T_k' t_k)\mathcal{T}_i \mathcal{T}_{iz}\rangle_k \tag{B4}$$

and

$$|(T_i t_i)\mathcal{T}_i \mathcal{T}_{iz}\rangle_i = \sum_{T_l'} (-1)^{t_i-T_i-\mathcal{T}_i}[(2T_i+1)(2T_l'+1)]^{1/2} \times W(t_l t_k \mathcal{T}_i t_i; T_i T_l')|(T_l' t_l)\mathcal{T}_i \mathcal{T}_{iz}\rangle_l$$

$$(ikl \text{ cyclic}), \tag{B5}$$

where W is the usual Racah coefficient.

The projection quantum numbers for various three-nucleon states are:

$\mathcal{T}_z = \frac{3}{2}, \quad T_z = 1, \quad t_z = \frac{1}{2}$ for the trineutron system;
$\mathcal{T}_z = \frac{1}{2}, \quad T_z = 1, \quad t_z = -\frac{1}{2}$
or
$\mathcal{T}_z = \frac{1}{2}, \quad T_z = 0, \quad t_z = \frac{1}{2}$ for the triton;
$\mathcal{T}_z = -\frac{1}{2}, \quad T_z = 0, \quad t_z = -\frac{1}{2}$
or
$\mathcal{T}_z = -\frac{1}{2}, \quad T_z = -1, \quad t_z = \frac{1}{2}$ for He^3;
$\mathcal{T}_z = -\frac{3}{2}, \quad T_z = -1, \quad t_z = -\frac{1}{2}$ for the triproton system. (B6)

The two-nucleon Coulomb interaction is

$$V_C(r_{12}) = \frac{e^2}{r_{12}} \, \tfrac{1}{2}(T_z{}^2 - T_z)$$

for point nucleons. Thus the two-nucleon t matrix $\tau_{L_1,L}^{J_1 S T T_z}$ will have Coulomb contributions for $T_z = -1$, but not for $T_z = 0$ or 1.

In the presence of Coulomb interactions, total isospin is not conserved, so that $\mathcal{T}$ is not necessarily equal to $\mathcal{T}_2$ in (4.18), i.e., isospin mixing occurs. In the absence of Coulomb interaction, the two-nucleon t matrix is independent of T_z so that we can sum over T_z in (4.18) to obtain

$$\sum_{T_z, t_z} \langle T T_z t t_z | \mathcal{T} \mathcal{T}_z \rangle \langle T T_z t t_z | \mathcal{T}_2 \mathcal{T}_z \rangle = \delta_{\mathcal{T}\mathcal{T}_2}. \qquad \text{(B7)}$$

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