New Method for Distorted-Wave Analysis of Stripping to Unbound States*

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The distorted-wave Born approximation (DWBA) cross section is discussed for a reaction of the form $A(d,pn)A$ in which n is not observed. We evaluate the slowly convergent oscillating radial integrals by contour integration in the complex radius plane. This avoids the computation of wave functions out to the large distances that the Huby-Mines technique requires. The present technique is also useful for stripping to very weakly bound states. We find that the DWBA angular distribution for stripping to a resonant n state is closely reproduced by using a form factor for the resonant energy of n. The width of the resonance, rather than the spectroscopic factor, is found to be measured by the absolute magnitude of the cross section. The results of calculations with the present method are compared with the results of the Huby-Mines method for the $^{16}O(d,p)^{17}O$ reaction leading to the resonant 5.08-MeV state of ^{17}O.

1. INTRODUCTION

Consider an experiment designed to observe a single-particle transfer reaction of the form $A(d,p)B$. Here d symbolizes a composite particle. It is made up of the particle p (which may be either a nucleon or a composite particle) and a nucleon n that is transferred to the target nucleus and thereby forms a state of the final nucleus B.[1] Bound states of B appear as peaks in the energy spectrum of outgoing p particles. Resonant final states also are seen as peaks, which, however, lie on top of a continuum due to three-body breakup.

When B is a bound state, the reaction can be analyzed in the usual distorted-wave Born approximation (DWBA). Such an analysis frequently gives valuable information on the angular momentum, parity, and spectroscopic factor of the final state. We propose to show that the success of the DWBA analysis can be extended to long-lived resonant states of B that are unstable against emission of n.

The reaction considered is

$$A+d\rightarrow A+p+n-B_d\,, \tag{1}$$

where B_d is the energy for separation of d into n and p. We shall calculate the angular distribution of p, assuming that the direction of n is not observed. The presence of three particles in the final state renders exact calculations prohibitively laborious if realistic potentials are to be used. We therefore restrict ourselves to the DWBA, without necessarily assuming that n and p are sequentially emitted. Rather, the validity of our approximation depends on the *weakness* of inelastic processes (including the breakup itself) in $A+d$ scattering.

The DWBA amplitude evaluated in the present work is the same as that proposed by Huby and Mines.[2] A "convergence factor" $e^{-\alpha r}$ appears in its definition. Huby and Mines evaluate the amplitude for positive values of α and extrapolate numerically to the limit $\alpha\rightarrow 0$. In contrast, *we do not use a convergence factor* in our numerical evaluation of the amplitude. It is of interest to compare our technique with that of Huby and Mines. Alty *et al.*[3] have used the Huby-Mines technique to calculate the proton angular distribution for the reaction $^{16}O(d,p)^{17}O$ leading to the 5.08-MeV, $J^\pi=\frac{3}{2}^+$ state of the final nucleus at 12-MeV incident deuteron energy. We therefore illustrate our method by applying it to the same case.

Other treatments of stripping to resonant states have been given by Bang and Zimanyi[4] and by Bukanov.[5] Bang and Zimanyi choose to compute only the contribution of the $A+n$ resonance pole to the cross section; accordingly they employ a Gamow state for the $A+n$ wave function. They compute the radial integrals by a version of the Huby-Mines method that uses a Gaussian convergence factor; this serves to overcome the exponential increase of the Gamow wave function at large distances. They justify their normalization of the cross section only by appeal to a comparison with experiment. Bukanov, on the other hand, uses $A+n$ wave functions for real energies. He transforms to "prior" form that part of the integral for which the n particle is beyond the range of the nuclear force. In this transformation a surface term arises. He asserts that this is the dominant contribution, and neglects the volume integral on the ground that it is zero if corrections are made for the Coulomb polarization of d. While this may be a good approximation in suitable circumstances, Bukanov's results do not seem to be identically equal to the expressions given by the usual "post" form of the DWBA.

2. THREE-BODY CROSS SECTION

The general expression for the cross section[6]

for the three-body breakup $A+d \rightarrow A+p+n$ is

$$\frac{d\sigma}{d\vec{p}_A d\vec{p}_n d\vec{p}_p} = (2\pi)^{-5}\hbar^{-2} m_d p_d \sum |T|^2 \delta(E_f - E_i)\delta(\vec{P}_f - \vec{P}_i). \tag{2}$$

Here m_d and $\hbar\vec{p}_d$ are the mass and incident momentum of d; E and $\hbar\vec{P}$ are the total energy and momentum of the system, and the subscripts i and f refer to initial and final states; $\hbar\vec{p}_A$, $\hbar\vec{p}_n$, and $\hbar\vec{p}_p$ are the final-state momenta. The transition amplitude T is defined with all scattering wave functions normalized like $e^{i\vec{p}\cdot\vec{r}}$. The summation is over final spins. The result should be suitably averaged over initial spins. Let $\hbar\vec{k}_{An}$ be the relative momentum of A and n, and $\hbar\vec{k}_{Bp}$ the relative momentum of B $(=A+n)$ and p. Then we have

$$d\sigma/d\vec{P}_f d\vec{k}_{An} d\vec{k}_{Bp} = d\sigma/d\vec{p}_A d\vec{p}_n d\vec{p}_p. \tag{3}$$

In the type of experiment considered, only the c.m. angles Ω_p of particle p are observed. The required differential cross section is therefore

$$\frac{d\sigma}{d\Omega_p} = \int d\vec{P}_f \int k_{Bp}^2 dk_{Bp} \int d\vec{k}_{An} \frac{d\sigma}{d\vec{P}_f dk_{An} d\vec{k}_{Bp}}. \tag{4}$$

The final state of the $n+A$ system can be unitarily transformed to a partial-wave representation. Then the integration over the angles of $\vec{k}_{An}$, and the sum over the spin projection of n are replaced by a sum over the angular momentum labels l, j, and m. The P_f and k_{Ap} integrations can be carried out by use of the expression

$$E_f = \tfrac{1}{2}\hbar^2\left(\frac{P_f^2}{m_{\text{total}}} + \frac{k_{An}^2}{\mu_{An}} + \frac{k_{Bp}^2}{\mu_{Bp}}\right), \tag{5}$$

where the quantities μ are reduced masses. The result is

$$\frac{d\sigma}{d\Omega_p} = (2\pi)^{-5}\hbar^{-2} m_d p_d \sum_{ijm} \int dk_{An} \sum |T|^2 k_{Bp}\mu_{Bp}^{-1} k_{Ap}^2 k_{An}^2.$$

If we now define a fictitious cross section

$$\frac{d\sigma_{lj}^F}{d\Omega_p} = (2\pi)^{-2}\hbar^{-4}\mu_{Ad}\mu_{Bp}k_{Bp}k_{Ad}^{-1}\sum_m \sum |T^F|^2$$
$$T^F = (4\pi)^{-1} T, \tag{6}$$

this can be written

$$\frac{d\sigma}{d\Omega_p} = 2\pi^{-1}\hbar^{-2}\mu_{An}\sum_{lj}\int_{E_1}^{E_2} dE_{An} k_{An} \frac{d\sigma_{lj}^F}{d\Omega_p}. \tag{7}$$

Here E_1 and E_2 are the values of $E_{An} = \frac{1}{2}k_{An}^2/\mu_{An}$ that correspond kinematically to the limits of the range of $E_{Bp} = \frac{1}{2}k_{Bp}^2/\mu_{Bp}$ that will be accepted by the detectors (or summed in the data reduction). The fictitious $d\sigma_{lj}^F/d\Omega_p$ has the form of the cross section for stripping to a bound state with angular momenta l,j. Instead of being normalized to unity like a bound state, however, the final n state in T^F has a radial wave function that goes like $(k_{An}r)^{-1} \times \sin(k_{An}r+\gamma)$ at large distances.

Equation (7) shows that different l,j contribute incoherently to the differential cross section. Levin[7] has observed that this facilitates extraction of the contribution of a resonant state of given l,j from the nonresonant background of other l,j. In what follows we assume that the l,j sum in Eq. (7) reduces to a single term (i.e., we suppose that the background part of the observed cross section has been subtracted).

3. DISTORTED-WAVE BORN APPROXIMATION

The three-body breakup has been discussed in a previous paper[8] for the model Hamiltonian

$$H = K + V_n(r_n) + V_p(r_p) + V_{np}(|\vec{r}_n - \vec{r}_p|), \tag{8}$$

with

$$K = \tfrac{1}{2}\hbar^2(p_n^2/m_n + p_p^2/m_p) \tag{9}$$

corresponding to an infinitely massive inert target A. The transition amplitude T was shown to be given by

$$T = \lim_{\alpha\rightarrow 0+} \langle \chi_b^- | e^{-\alpha r_{np}} V_{np} | \psi_a^+ \rangle, \tag{10}$$

where

$$\chi_b^- = \chi_n^-(r_n)\chi_p^-(r_p). \tag{11}$$

We use the notation of Ref. 8, in which χ_n^- and χ_b^- are scattering wave functions for the optical potentials V_n^+ and V_p^+, with incoming boundary conditions. The state ψ_a^+ is the exact scattering eigenfunction of the complete Hamiltonian, with outgoing boundary conditions. The DWBA is obtained on approximating ψ_a^+ by

$$\psi_a^+ \approx \chi_a^+ \equiv \chi_d^+(\vec{R}_d)\beta(|\vec{r}_n - \vec{r}_p|),$$

where

$$\vec{R}_d = (m_n\vec{r}_n + m_p\vec{r}_p)/(m_n+m_p). \tag{12}$$

Here, χ_d^+ is an optical wave function for elastic scattering of d from A, with outgoing boundary conditions, and β is the bound internal wave function of the d particle. Thus

$$T(\mathrm{DWBA}) = \lim_{\alpha \to 0+} \langle \chi_b^- | e^{-\alpha r_{np}} V_{np} | \chi_a^+ \rangle . \tag{13}$$

We shall make the zero-range (ZR) approximation, and later correct for finite-range (FR) effects. In this approximation we have

$$V_{np}\beta(|\vec{r}_n - \vec{r}_p|) \approx D_0 \delta(\vec{r}_n - \vec{r}_p), \quad D_0 = \int d\vec{r}\, V_{np}\beta(r) \tag{14}$$

so that Eq. (13) becomes

$$T(\mathrm{DWBA}) = \lim_{\alpha \to 0+} D_0 \int d\vec{r}\, \chi_n^{-*}(\vec{r}) \chi_p^{-*}(\vec{r}) \chi_d^{+}(\vec{r}) e^{-\alpha r} . \tag{15}$$

Until now, A has been assumed to be infinitely massive. In the ZR approximation, the finite mass of A can be taken into account exactly by replacing the argument of χ_p^- in Eq. (15) by cr, where $c = m_A/m_B$. This substitution will always be made without further comment.

Since a single l,j is assumed to contribute to Eq. (7), only one partial wave of χ_n^- is needed. Then when χ_p^- and χ_d^- are expanded in partial waves, Eq. (15) becomes

$$T^F(\mathrm{DWBA}) = D_0 \sum_{\{\kappa\}} C_{\{\kappa\}}(\vec{k}_{An}, \vec{k}_{Bp}, \vec{k}_{Ad}) f^{lj}_{\{\kappa\}}, \tag{16}$$

$$\{\kappa\} = \{L_p, J_p, L_d, J_d\}.$$

The universal geometrical coefficients $C_{\{\kappa\}}$ relating the radial integrals $f^{lj}_{\{\kappa\}}$ to the DWBA amplitude have been derived by several authors.[9] Satchler[9] gives a complete discussion of these coefficients, although not in the present notation.

The physics of the reaction is entirely determined by the radial integrals

$$f^{lj}_{\{\kappa\}} = 2\pi^{1/2} \frac{M_B}{M_A} \frac{1}{k_{Ad} k_{Bp}} I(L_p, J_p, l, j, L_d, J_d),$$

$$I(L_p, J_p, l, j, L_d, J_d) = \lim_{\alpha \to 0+} \int_0^\infty dr\, \chi_{L_p J_p}(cr) F_{lj}(r) \chi_{L_d J_d}(r) e^{-\alpha r}, \tag{17}$$

where

$$F_{lj}(r) = (k_{An} r)^{-1} \chi_{lj}(r). \tag{18}$$

Beyond the range of the nuclear interaction, the functions χ are all of the form

$$\tfrac{1}{2} i [H_l^-(kr) - \eta H_l^+(kr)] e^{i\sigma_l}, \tag{19}$$

where $H_l^{\pm} = G_l \pm iF_l$ are outgoing and incoming Coulomb functions and η is the partial-wave scattering matrix element. The Coulomb phase shifts are denoted by σ_l.

Standard distorted-wave computer programs can be used to compute $d\sigma^F(\mathrm{DWBA})/d\Omega_p$ from the $f^{lj}_{\{\kappa\}}$. Application of Eq. (7) then gives the desired cross section. The tasks remaining are (a) to calculate the resonant form factor F_{lj} and the optical wave functions χ for d and p, (b) to evaluate the radial integrals $f^{lj}_{\{\kappa\}}$, and (c) to perform the integration over E_{An} indicated in Eq. (7). These tasks are discussed in Secs. 4, 5, and 6, respectively.

4. CALCULATION OF THE FORM FACTOR AND DISTORTED WAVES

The functions $\chi_{lj}, \chi_{L_d J_d}, \chi_{L_p J_p}$ are calculated by solving the partial-wave Schrödinger equation for potentials of the form

$$V = -V_0 U_V - iW_0 U_W - \lambda_\pi^2 \vec{l} \cdot \vec{\sigma} \frac{V_{so}}{r a_{so}} \frac{dU_{so}}{dx_{so}} + V_{\mathrm{Coul}},$$

$$\lambda_\pi^2 = 2.0\ \mathrm{F}^2 \tag{20}$$

with Woods-Saxon shape functions

$$U_\lambda(r) = [1 + e^{x_\lambda}]^{-1}, \quad x_\lambda = (r - R_\lambda)/a_\lambda .$$

The Coulomb potential is taken to be

$$V_{\mathrm{Coul}} = \tfrac{1}{2} e^2 Z_1 Z_2 [3 - (r/R_C)^2]/R_C, \quad (r \le R_C),$$

$$= e^2 Z_1 Z_2 / r, \quad (r > R_C).$$

The geometrical parameters of the potential are

$$R_\lambda = r_{0\lambda} A^{1/3} \text{ and } a_\lambda ,$$

where λ can represent V, W, so, or Coul. The results can subsequently be corrected for nonlocality of the potentials, and for the effects of finite range. As usual, V_d and V_p are chosen to reproduce the elastic-scattering data in the relevant energy range.

For the form factor χ_{lj}, the conventional bound-state procedure is to choose reasonable values for the geometrical parameters (r_{0V}, a_V) of the central well and for the parameters of the spin-orbit potential, and then to vary V_0 to fit the experimentally known separation energy. This procedure can be extended naturally to resonant states. In fact, on physical grounds the procedure may be better justified for resonant states. For bound states, there are really no good guidelines for choosing the parameters of the bound-state well. For resonances, however, scattering measurements can help to determine the potential parameters.

The resonance in the $A+n$ system is character-

ized by an energy E_r, a width $\Gamma = \Gamma_n$, and angular momenta l,j. We assume that no inelastic channels are open near $E_{An} = E_r$; that is, we assume $\Gamma_{\text{total}} = \Gamma_n$. Then V_{An} will be Hermitian. The *single-particle* potential V_{An} corresponds to scattering of n by A *without* virtual excitation of A. To the extent that exchange effects can be neglected, V_{An} must therefore be independent of E_{An}. The energy-dependent exchange effects will be small if the wave function of the scattered nucleon has small overlap with the wave functions of the target nucleons. In particular, this will be true if the wave function of n is small inside A, or if those partial waves of n that are not small inside A have angular momenta that are absent in the shell-model representation of the target state.

For physical scattering of a nucleon n by a nucleus A, there is virtual excitation of A. The corresponding optical potential therefore depends on E_{An}. However, it seems reasonable to neglect such virtual excitation, except at or near resonances, where the scattering wave function is large in the interaction region. We should therefore choose the single-particle potential to reproduce the observed *background* scattering in the energy range of interest. Virtual excitation of A will shift the resonance energy by an amount of the order of the single-particle width. Close to the energy of the physical resonance, the single-particle potential should therefore have a resonance in the l,j partial wave. Although these criteria are not sufficient to determine the single-particle potential uniquely, we believe that they afford a better guide to its choice than is available in the bound-state case.

We use the optical-model code ABACUS[10] to calculate the form factor χ_{lj}. At present, this part of the calculation is not efficient, since the search

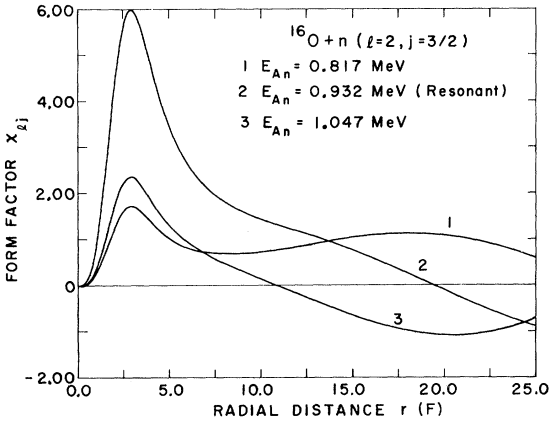

FIG. 1. ^{16}O+n form factors on and off resonance, calculated with the potential parameters given in Table I. The form factors shown have not been corrected for the effects of finite range and nonlocality.

for potential parameters must still be largely by trial and error. However, once the potential is approximately known, ABACUS will automatically make the small parameter changes necessary to fit the resonance energy to any desired degree of accuracy (up to an accuracy of ±1 eV). These small final parameter changes usually do not change the calculated background cross section appreciably. When the potential parameters are known, it is a simple matter to compute the single-particle width Γ_{sp} and the form factor (as a function of E_{An} and r). Figure 1 shows some form factors computed by ABACUS, both on and off resonance, for the 5.08-MeV resonant state of ^{17}O.

This prescription will not (in general) lead to a Γ_{sp} equal to the actual width Γ_n of the physical resonance. This is at it should be. Requiring the width calculated from the single-particle potential to agree with Γ_n would do great violence to the concept of a single-particle potential, and would result in *every* physical resonance having a (spurious) spectroscopic factor of unity.

We correct χ_{lj}, $\chi_{L_pJ_p}$, and $\chi_{L_dJ_d}$ for the effects of nonlocality by using the local-energy approximation of Perey and Saxon.[11] Their prescription gives the wave function for a nonlocal potential equivalent to the local potential V_L as

$$[1 - (\mu\beta_{NL}^2/2\hbar^2)V_L]^{-1/2}$$

times the wave function for V_L. Here β_{NL} is the range of nonlocality, and μ is the reduced mass. We omit the Coulomb potentials in making the nonlocality corrections. Except for the form factor, the nonlocality correction due to the spin-orbit potential is also neglected, since spin averaging should greatly reduce its effects.

We also correct for the finite range of V_{np} by the method of Buttle and Goldfarb.[12] This correction introduces a factor

$$1 - (b^2/a^2)(V_d - V_n - V_p - B_d)/B_d$$

into the radial integrand of Eq. (17). Here V_d, V_n, and V_p are taken to be the optical potentials. The constants a and b specify the ranges of the Hulthén deuteron wave function and of V_{np}, respectively. The Coulomb contributions to this correction cancel almost completely and are neglected. For the same reason as before, we neglect the correction due to spin-orbit potentials acting on d and p.

With the approximations noted, the finite-range and nonlocal corrections (FRNL) can be made by multiplying the form factor by a function that tends to a value near unity at ∞ and becomes small near the origin. Since the correct normalization of the form factor is determined by the general require-

ments of scattering theory, it would be wrong to renormalize it after making the corrections.

5. RADIAL INTEGRALS

The integrand in Eq. (17) at large r is proportional to

$$e^{-\alpha r}\sin(k_{Bp}cr+\gamma_p)\sin(k_{An}r+\gamma_n)\sin(k_{Ad}r+\gamma_d)/r.$$

For a charged particle, the phase γ includes a term proportional to $\ln r$. For positive separation energy B_d of the d particle, the conservation of energy implies

$$k_{Ad}>k_{An}+k_{Bp},$$

so that

$$\Delta k\equiv k_{Ad}-k_{An}-ck_{Bp}>0. \tag{21}$$

Thus the integrand always oscillates. The presence of the factor r^{-1} then ensures convergence even for $\alpha=0$ (though not absolute convergence). The extreme slowness of this convergence causes difficulty with the numerical evaluation of the integral.

For a bound state, the usual procedure is to cut off the integral at a finite upper limit $R\approx 30$ F. This method *may* succeed also for resonant states, at least in those (trivial) cases for which the part of the form factor beyond the barrier region makes a negligible contribution. The danger of using a finite cutoff R in more realistic situations is made clear in Fig. 2, where calculations for two values of R are compared. Some improvement may be gained by averaging[13] over a wide enough range of R values.

Another approach is to replace the resonant state by a very weakly bound state, without altering the energy of the outgoing p particle. This approximation cannot, in general, be relied on, although it may succeed in isolated cases. For the $^{16}O(d,p)$-^{17}O(5.08-MeV) reaction, the bound and resonant form factors are noticeably different (as may be seen from Fig. 7 of Sec. 8). This difference is reflected in the corresponding theoretical cross sections plotted in Fig. 9 of Sec. 7. Even in cases in which the fictitious bound form factor gives the correct shape, there is no reason for the absolute value of the cross section to be correct.

In sum, it seems unwise to rely on the results of calculations with a finite upper cutoff or with a fictitious bound form factor.

Huby and Mines[2] have discussed this problem. They retain the "convergence factor" $e^{-\alpha r}$ and, after evaluating the integral I for several values of α, extrapolate to $\alpha=0$. If one evaluates the integral by a numerical quadrature with weights and abscissas independent of α, the limit as $\alpha\to 0$ will be identically the same as the result of simply omitting the convergence factor. Therefore the Huby-Mines technique cannot improve the convergence unless the quadrature uses weights and abscissas dependent on α. The construction of suitable quadrature formulas is not discussed in Ref. 2. Consider now the conditions for reliability of the numerical extrapolation to $\alpha=0$. If the α^2 term of I is not to exceed say a tenth of the α term, then we must have

$$\alpha/\Delta k<\tfrac{1}{10}.$$

In practice this integral must be cut off at some upper limit R. For adequate accuracy, the exponential should be small at R, say less than $e^{-10}\approx 10^{-4}$. Therefore we must have

$$\alpha R\gtrsim 10.$$

Consequently

$$R\gtrsim 10/\alpha\gtrsim 10(10/\Delta k)=100/\Delta k. \tag{22}$$

For $\Delta k\approx 0.17$ F^{-1}, appropriate to the $^{16}O(d,p)^{17}O$ case treated by Alty *et al.*,[3] Eq. (21) gives

$$R\gtrsim 580 \text{ F}.$$

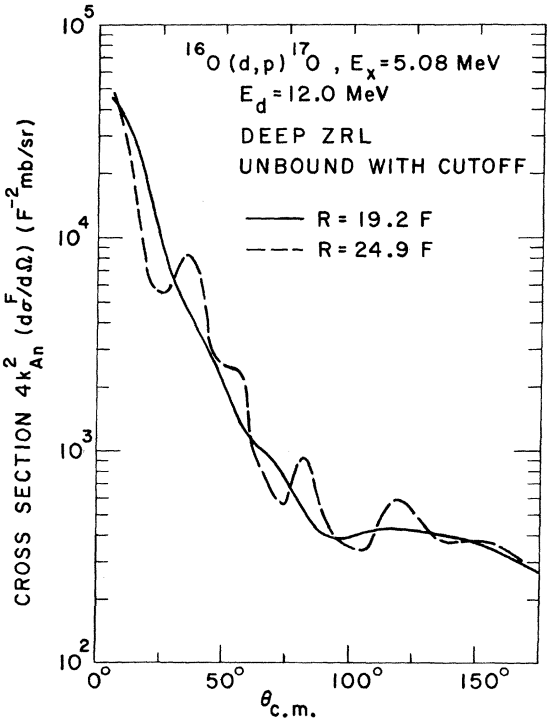

FIG. 2. Cross sections calculated by cutting off the radial integrals at a finite upper limit R. The resonant form factor that was used is curve 2 of Fig. 1.

Therefore, the successful use of the Huby-Mines technique requires evaluation of the three wave functions χ out to very large distances. The integral is the sum of a large number of positive and negative contributions, because the integrand oscillates many times over this distance. The cancellation of positive and negative contributions may reduce the accuracy of the result. Furthermore, the calculation must be done for several values of α. For these reasons we have found the Huby-Mines technique unsuitable.

In the present method, the range of integration is divided into an inner part $0 < r < r_t$, and an outer part $r_t < r < \infty$, with r_t chosen so that the nuclear potentials are negligible in the outer part. The integral over the inner range is evaluated by the standard apparatus of Tamura's program DWMAIN.[14] In the outer range, $\chi_{L_d J_d}$ is split [as in Eq. (19)] into H^+ and H^- parts. Figure 3 shows the complex r plane. The contour for the H^+ part of the integral is deformed into a line V_+ in the positive imaginary direction, completed by an arc Q_+ whose radius tends to infinity. In the same way, the contour for the H^- part becomes $V_- Q_-$.

The function H^+ (H^-) is analytic[15] in r except for a branch point and pole of order l at $r=0$ and a branch line along the negative (positive) imaginary axis. Therefore, these contour deformations do not affect the values of the integrals. For ρ in any sector excluding the branch line, $H_l^{\pm}$ has the asymptotic form

$$H_l^{\pm} \sim e^{\pm i(\rho + \gamma_l)}, \quad |\rho| \to \infty, \tag{23}$$

where γ_l is a phase containing a term proportional to $\ln r$. In conjunction with Eqs. (17)–(19), this asymptotic form implies that the integrands are exponentially small everywhere on the corresponding contours $Q_{\pm}$. The contributions from $Q_{\pm}$ therefore vanish when the radius of the arcs tends to infinity. In the integrals over $V_{\pm}$, the limit $\alpha \to 0$ may be taken under the integral sign. In this way, the convergence factor is altogether eliminated. When the V_+ and V_- contributions are combined, the outer integral becomes

$$\tfrac{1}{2} e^{i\sigma_l} \int_0^{\infty} dy \left[\chi_{L_p J_p}(cr_-) F_{lj}(r_-) H^-_{L_d}(r_-) + \eta_{L_d J_d} \chi_{L_p J_p}(cr_+) F_{lj}(r_+) H^+_{L_d}(r_+) \right]. \tag{24}$$

In this integrand, $r_{\pm} \equiv r_t \pm iy$. The integrand of Eq. (24) is smooth, tends to zero exponentially at large y, and undergoes only a small number of oscillations over the interval in which it is large. It is very suitable for numerical quadrature. Figure 4 illustrates this combined integrand in a typical

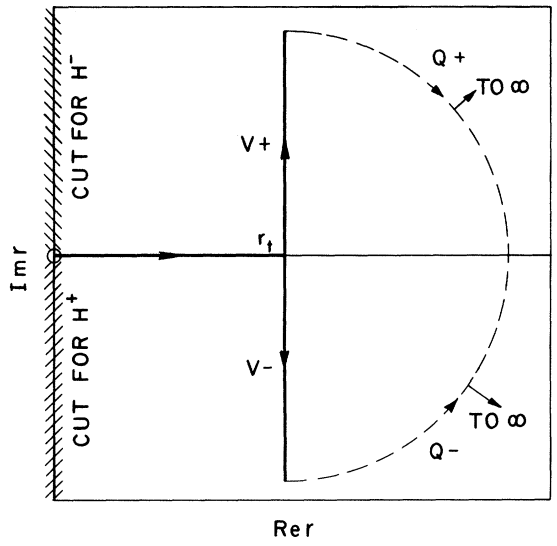

FIG. 3. The complex r plane, showing contours (discussed in the text) for radial integrals.

case. The integrands of Eqs. (24) and (17) do not tend to the same limit as $r \to r_t$. This is to be expected, since Eq. (24) was obtained by combining two integrals over distinct contours running in opposite directions.

In the present method, we truncate the integral in (24) at a suitable value $y_{\max}$. The functions $H^{\pm}(r_{\pm})$ for d are then evaluated by using the asymptotic expansion to start the integration of the radial differential equation for the Coulomb functions. The values and derivatives of F_{lj} and $\chi_{L_p J_p}$ at $r = r_t$ are calculated by DWMAIN, and from these the required values are obtained by integration of the Coulomb

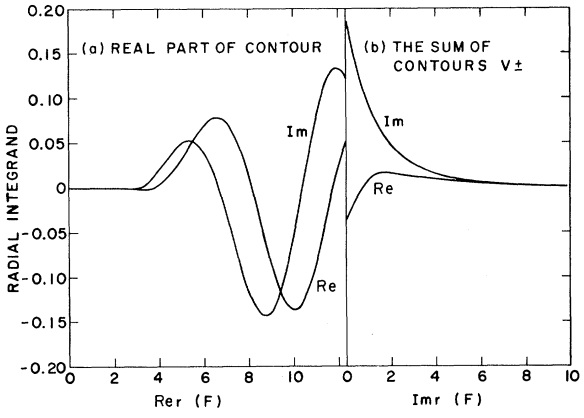

FIG. 4. A typical radial integrand with $L_d=2$, $J_d=1$, $L_p=4$, $J_p=\frac{7}{2}$, $l=2$, $J=\frac{3}{2}$: (a) the oscillating integrand for the original contour; (b) the smooth exponential decay of the sum of the integrands for the contours $V_{\pm}$ that are illustrated in Fig. 3.

differential equation. DWMAIN already furnishes the elastic-scattering matrix element $\eta_{L_d J_d}$. Typically, 30 points are sufficient for the numerical quadrature.[16]

In the separation of $\chi_{L_d J_d}$ in expression (19), numerical errors may arise if the $H^{\pm}_{L_d}$ are much larger than $\chi_{L_d J_d}$ at r_t. Such errors can be avoided if r_t is chosen to be not too far inside the Coulomb-plus-centrifugal barrier for the d particle. Usually, this condition becomes important only at high L_d. In this case the requirement

$$r_t \gtrsim L_d\,(\max)/k_{Ad} \tag{25}$$

ensures good results, where $L_d\,(\max)$ is the highest value of L_d needed to get adequate convergence of the partial-wave expansion. In the case illustrated, the choice $r_t = 12.35$ F was found to be adequate.

The magnitudes of the radial integrals are shown in Fig. 5. The convergence of the partial-wave sum is seen to be as fast as in the bound-state case.

6. INTEGRATION OVER THE ENERGY OF THE UNOBSERVED PARTICLE

The calculation of σ^F_{ij} is laborious even for a single value of E_{An}. It is therefore important that the integration over E_{An} be performed as economically as possible. For the case of a narrow resonance at $E_{An} = E_R$, we substitute

$$E = E_R + \tfrac{1}{2}\Gamma \tan(\tfrac{1}{2}\pi x)\,, \tag{26}$$

where Γ is a constant. Then the integral in Eq. (7) becomes

$$\tfrac{1}{4}\pi\Gamma\int_{x_1}^{x_2} dx\left[1+4\left(\frac{E-E_R}{\Gamma}\right)^2\right] k_{An}\frac{d\sigma^F}{d\Omega_p}. \tag{27}$$

If an approximate Breit-Wigner energy dependence can be assumed for the cross section, i.e., if

$$k_{An}\,(d\sigma^F/d\Omega_p) \propto [(E-E_R)^2 + \tfrac{1}{4}\Gamma^2]^{-1}\,,$$

then the integrand of Eq. (27) becomes a constant. If the energy dependence of the cross section is not too different from a Breit-Wigner dependence, and if Γ is a fairly good estimate of the resonance width, this integral is then suitable for numerical evaluation by Gauss-Legendre quadrature. If the contributions of energies outside the interval (E_1, E_2) are neglected,[17] the quadrature formula gives

$$\frac{d\sigma}{d\Omega_p} \approx \tfrac{1}{2}\hbar^{-2}\Gamma\mu_{An}\sum_{i=i}^{N} W_i \left\{1+4\left[\frac{E(x_i)-E_R}{\Gamma}\right]^2\right\} \times k_{An}(x_i)\frac{d\sigma^F_{ij}(x_i)}{d\Omega_p}\,, \tag{28}$$

where x_i and W_i are the abscissas and weights for the N-point quadrature over the interval $(-1, 1)$.

Figure 6 shows $d\sigma^F/d\Omega_p$ at the values of E_{An} corresponding to the three-point quadrature. The *shape* of the angular distribution is strikingly independent of E_{An}. At first sight, this phenomenon seems hard to explain in view of the strong dependence of the form factors (Fig. 1) on E_{An}. However, the plane-wave Born approximation gives

$$d\sigma^F/d\Omega_p \propto \left|\int_0^\infty dr\, F_l(qr)\chi_{lj}(r)\right|^2 \tag{29}$$

with the momentum transfer defined by

$$q = |\vec{k}_{Ad} - \vec{k}_{Bp}|\,.$$

Here F_l is a regular Coulomb function for zero charge (i.e., a Ricatti-Bessel function). In a suitable one-pole approximation, near a resonance, the wave function χ_{lj} is an energy-dependent linear combination of a free and a resonant wave function. The contribution of the free wave to the right-hand side of Eq. (29) vanishes identically (unless the kinematically impossible condition $q = k_{An}$ is satisfied). If an internal cutoff r_c is introduced, the contribution of the *free* wave is still small if the free wave is small in the interior. Then only the contribution of the resonant wave function is important, and the shape of the angular distribution is approximately independent of E_{An}.

For $N = 1$ our approximation (28) becomes

$$d\sigma/d\Omega_p \approx \hbar^{-2}\Gamma\mu_{An}k_{An}(E_R)[d\sigma^F(E_R)/d\Omega_p]\,. \tag{30}$$

Equation (30) will be a good approximation only if

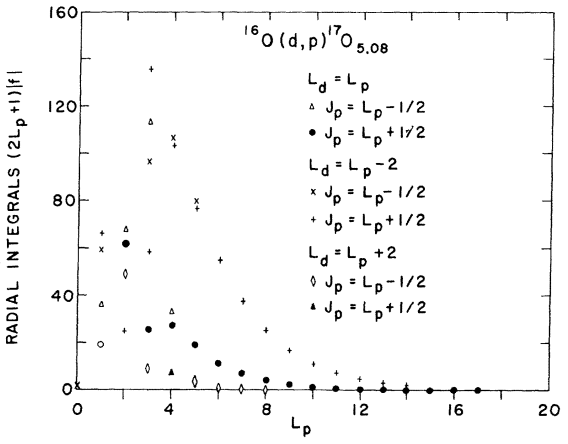

FIG. 5. Magnitudes of the radial integrals as a function of L_p. For $L_p \gtrsim 6$, the points for $J_p = L_p + \frac{1}{2}$ are indistinguishable from those for $J_p = L_p + \frac{1}{2}$. The relative contribution of different L_p values to the forward cross section is roughly proportional to the quantity plotted.

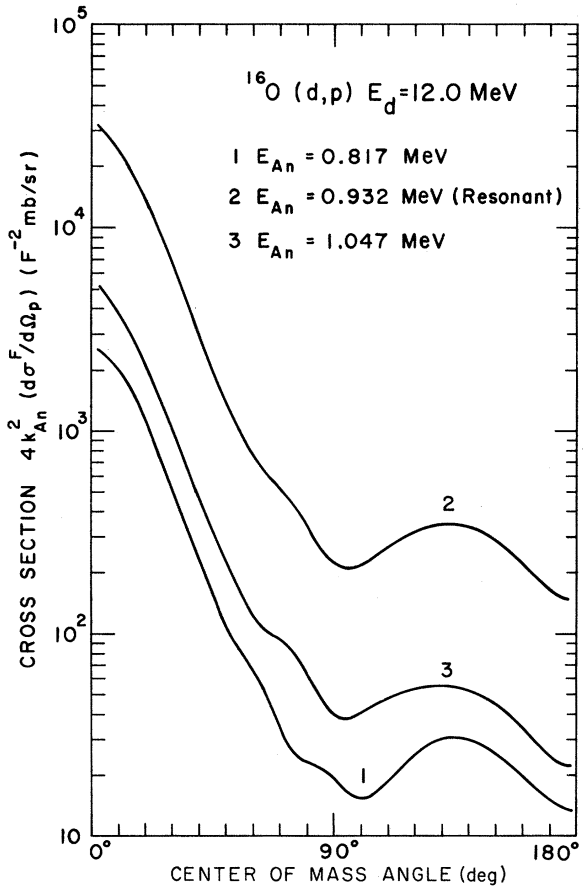

FIG. 6. Typical theoretical cross sections for the $^{16}O(d,p)^{17}O$ reaction to states on and off the $d_{3/2}$ resonance in $^{16}O+n$. Although the corresponding form factors differ considerably in shape, the three angular distributions are remarkably similar.

the resonance is narrow and symmetric and Γ is a good estimate of its width. In the case illustrated, a comparison of Eq. (30) with the result for $N=3$ showed agreement within 1% at all angles.

The accuracy of the $N=1$ result makes it reasonable to analyze data by means of Eq. (30). The shape of the angular distribution can be well determined even if the width Γ is not known.

7. DEVIATIONS FROM THE SINGLE-PARTICLE LIMIT

We have assumed so far that A is inert, and that B can be described as a potential resonance of n and A – i.e., a "single-particle state." The measurement of the deviation from this assumption is a strong motive for DWBA analysis of stripping to bound states. We therefore discuss the possibility of measuring spectroscopic factors also in reactions leading to resonant final states.

For a bound state, the spectroscopic factor S is defined[18] by

$$S=|a|^2, \tag{31}$$

where a is the amplitude of the normalized single-particle state in the expansion of the physical final state. It then follows that

$$S=\left(\frac{d\sigma}{d\Omega_p}\right)_{\rm exp}\Big/\left(\frac{d\sigma}{d\Omega_p}\right)_{\rm sp}. \tag{32}$$

For unbound states, the final state is not normalizable in the usual sense, so that the meaning of the ratio in Eq. (32) is unclear. Indeed, since the basic formula for the cross section [Eq. (2)] is valid only if χ_{lj} is normalized like $\sin(kr+\gamma)$ at large r, the amplitude a seems to have little direct meaning.

What does the ratio (32) actually measure? We expect the conclusion stated in Sec. 6 (that the contribution of the free-wave part of χ_{lj} is small) to extend to the case of charged n. If we assume that the amplitude of the resonant wave function is proportional to $\sin\delta_{lj}$ (where δ_{lj} is the nuclear phase shift), we conclude that

$$d\sigma^F/d\Omega_p=\sin^2\delta_{lj}\,(d\sigma^F/d\Omega_p)_{\rm peak}, \tag{33}$$

where the subscript "peak" refers to the energy at which $\delta_{lj}=\pi/2$. Then by Eq. (7),

$$\frac{d\sigma}{d\Omega_p}=2\pi^{-1}\hbar^{-2}\mu_{An}\left(\frac{d\sigma^F}{d\Omega_p}\right)_{\rm peak}\int_{E_1}^{E_2}dE_{An}\,k_{An}\sin^2\delta_{lj}. \tag{34}$$

Because contributions from outside the nucleus dominate $(d\sigma^F/d\Omega_p)_{\rm peak}$, its experimental value can be estimated from a DWBA calculation even without a detailed knowledge of the physics of the resonant state. *Therefore the experiment measures the integral that appears in Eq.* (34).

Since $\sin^2\delta_{lj}$ can be related to the cross section for elastic scattering of n on A, the integral in Eq. (34) can be measured in two independent ways: by elastic scattering of n on A, and by the reaction $A+d\rightarrow B+p\rightarrow A+n+p$. A comparison of these two methods would be of great interest, since it would check the accuracy of the assumptions of the DWBA analysis. A similar check is not possible for *bound* states of B, because of uncertainties in the magnitude of the single-particle form factors in the external region. In contrast, the magnitude of the *resonant* form factor in the external region is entirely determined by the general requirements of scattering theory.

In the limit of a narrow resonance of width Γ, Eq. (34) reduces to the previously derived Eq. (30). In this approximation *the experiment measures the width Γ of the actual physical resonance in B.*

We regard the determination of S from Γ as a

TABLE I. Optical-model parameters used in analysis of $^{16}O(d,p)^{17}O$.

	V_0 (MeV)	W_0 (MeV)	V_{so} (MeV)	$r_{0V}=r_{0so}$ (F)	r_{0W} (F)	$a_V=a_{so}$ (F)	a_W (F)	r_{0C} (F)
$^{16}O+d$ "deep"	85.29	12.75	0	1.250	0.958	0.606	1.578	1.250
$^{16}O+d$ "shallow"	32.74	4.72	0	1.350	1.929	0.582	1.235	1.350
$^{17}O+p$	57.32	7.694	10.50	1.250	1.207	0.425	0.254	1.250
$^{16}O+n$	51.28[a]	0	9.68	1.325	•••	0.500	•••	•••

[a]Gives a resonance energy E_{An} (c.m.) = 0.933 MeV [E (lab) = 0.992 MeV] and a single-particle width $\Gamma_{c.m.} = 86 \pm 1$ keV.

problem outside the scope of the theory described in this paper. One appropriate method would be that due to Monahan,[19] who treats S as a parameter of an R-matrix model of the resonance, and shows how to choose the value of S that reproduces the experimentally deduced resonance width.

8. COMPARISON WITH EXPERIMENT

We have chosen to compute the cross section for the $^{16}O(d,p)^{17}O$ reaction leading to the $J^\pi = \frac{3}{2}^+$ state at $E_x = 5.08$ MeV in order to compare with the results of calculations[3] using the Huby-Mines[2] technique. The angular distribution for this state has been measured by Alty *et al.*[3] at $E_d = 12.0$ MeV.

The 5.08-MeV state of ^{17}O is unbound against neutron decay and is seen[20] as a resonance in the elastic scattering of neutrons from ^{16}O. The resonance occurs at a neutron energy $E_R = 1.0$ MeV (lab), with a center-of-mass width $\Gamma_n = 90 \pm 5$ keV.[20]

Optical-model parameters giving a reasonable reproduction of the $d_{3/2}$ phase shift for $^{16}O+n$ scattering in the region of this resonance have been reported by Alty *et al.* In order to compare directly with the results of the Huby-Mines technique, this same potential was used in the present case. The parameters are listed in Table I. The single-particle width of the $1d_{3/2}$ resonance, as calculated by ABACUS with this potential, was $\Gamma_{sp} = 86 \pm 1$ keV.

In the calculations[3] with the Huby-Mines technique, two different potentials were used for the incoming deuteron. One of them was characterized as "deep" ($V_d \approx 90$ MeV) and the other as "shallow" ($V_d \approx 40$ MeV). These potentials were used in the present calculation, as was their proton potential.[21] All the potentials used in the present calculation are listed in the table. Calculations with the same parameters were also performed for a fictitious form factor bound by 10 keV (the correct outgoing proton energy was used). In order to obtain this form factor, all the ($^{16}O+n$) parameters except V_0 were held fixed, and V_0 was varied to make the $1d_{3/2}$ state bound. The bound and unbound form factors are shown in Fig. 7.

Calculations for both the deep and shallow deuteron potentials were performed in both the ZRL

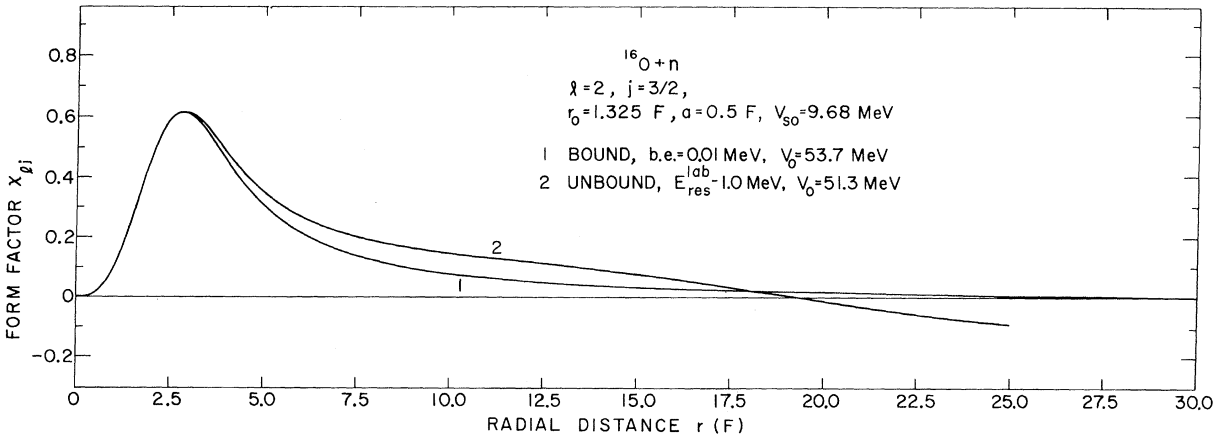

FIG. 7. Bound and resonant single-particle form factors for the $d_{3/2}$ state of $^{16}O+n$ at 0.933-MeV (c.m.) energy. The bound form factor satisfies $\int |\chi|^2 dr = 1$. The resonant form factor has been normalized to have the same maximum value as the bound form factor.

and FRNL approximations. The results of these calculations are shown in Fig. 8. The FRNL parameters used were $a^2/b^2 = 7.0$, $\beta_{NL}(n) = \beta_{NL}(p) = 0.85$ F, $\beta_{NL}(d) = 0.54$ F. It can be seen that the FRNL corrections lead to about a 30% increase in the cross section at 0°. (The form factor was not renormalized after making the FRNL corrections.) In addition, the correction tends to make the predictions of the two potentials more similar. This is an obvious consequence of reducing the contribution of the interior.

Calculations for both the bound and resonant form factors are shown in Fig. 9, along with the data of Alty *et al.* The experimental cross sections have been multiplied by 1.75 in accordance with the corrected target-thickness determination by Naqib and Green.[22] The calculation using the resonant form factor clearly gives a better fit to the slope at forward angles. At backward angles, any conclusions that might be drawn from the fit to the data are nullified by two considerations. Firstly, the cross sections are so small that it is difficult to extract the contribution of the resonant state from the nonresonant background. Secondly, processes other than direct stripping could easily contribute most of the cross section observed at large angles. Thus no significance can be attached to the bound-state calculation being the one that gives the better fit at backward angles.

The bound-state calculation gives a spectroscopic factor of 1.0. For the resonant form factor, the values calculated by Alty *et al.* using the Huby-Mines method[23] also have the same magnitude as the data at forward angles (after the renormalization suggested in Ref. 19). Our calculation, however, gives cross sections that are 1/0.78 times the experimental values at forward angles. We assumed $\Gamma = 86$ keV, and $D_0^2 = 1.48 \times 10^4$ MeV^2 F^3 for the coupling constant[24] in Eq. (14). The FRNL corrections did not appreciably affect the result. Thus comparison of our calculation with the data implies that the width of the state is $\Gamma = 0.78 \times 86 = 67$ keV. This is appreciably smaller than the value $\Gamma = 90 \pm 5$ obtained from neutron scattering experiments.[20]

Serious errors can arise if the contribution of a resonant state to the (d,p) cross section is extracted by relying upon methods appropriate to the case of bound states. For example, suppose that the incoherent background underlying a symmetrical peak is taken as the value of the cross section at the energies $E_R \pm \Delta E$. Here ΔE is assumed to be

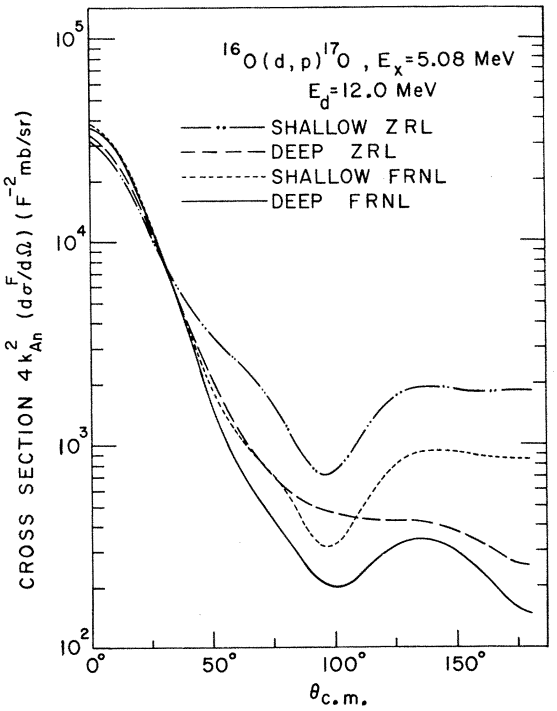

FIG. 8. Comparison of ^{16}O$(d,p)^{17}$O theoretical angular distributions, calculated with "deep" and "shallow" deuteron potentials, with FRNL and without ZRL corrections for finite range and nonlocality. Table I gives the potential parameters.

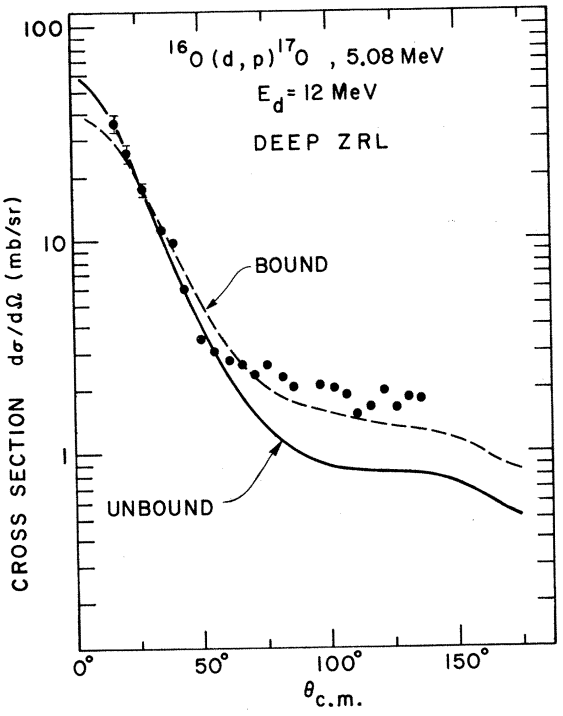

FIG. 9. Data (points) for the ^{16}O$(d,p)^{17}$O reaction leading to the unbound state at $E_x = 5.08$ MeV, together with ZRL DWBA calculations using resonant (solid curve) and bound (dashed curve) form factors, with the potentials listed in Table I. The "deep" deuteron potential was used. The outgoing proton energy was the same in both cases. The theoretical curves have been normalized to give the best fit to the data.

greater than the instrumental resolution width. Then contributions from the resonance wings and under the peak are being neglected. Expressed as a fraction of the total, these contributions amount[25] to about $(2/\pi)(\Gamma/\Delta E)$. For $\Delta E = 2\Gamma$, this is about 30%. It seems not unreasonable to suspect an error of this order in the present case. A possible way to correct for this error would be to solve for Γ in the equation

$$\left(1+\frac{2}{\pi}\frac{\Gamma}{\Delta E}\right)\left(\frac{d\sigma}{d\Omega_p}\right)_{\text{exp}} = \hbar^{-2}\Gamma\mu_{An}k_{An}\left(\frac{d\sigma^F}{d\Omega_p}\right)_{\text{peak}}.$$

In the absence of detailed information on the methods used by Alty *et al.* in extracting the contribution from the 5.08-MeV state, we have not been able to attempt this.

9. CONCLUSIONS

The present method makes the DWBA analysis almost as easy and effective for resonant as for bound final states. In particular, there is no need to use a convergence factor or even to calculate and store the form factor out to very large radii.

For narrow resonances, the shape of the angular distribution can be calculated by using only the on-resonance form factor. This possibility can greatly facilitate the practical analysis of experiments. The absolute magnitude of the cross section in essence determines the width of the final state. The spectroscopic factor S can be inferred from this width.

In our analysis of the $^{16}O(d,p)^{17}O$ reaction to the 5.08-MeV state, we find agreement with the angular distribution calculated by Alty *et al.* The data indicate that the 5.08-MeV state is a good $d_{3/2}$ single-particle state. The width of the resonance deduced from the (d,p) cross section is somewhat smaller than that deduced from (n,n) measurements. This discrepancy may be due to neglect of the intrinsic width of the state in reducing the data.

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[1]Such reactions as (d,p), (α,t), and $(^3\text{He},d)$ are common examples. For the sake of definiteness, the reader may like to keep in mind the (d,p) reaction. Until Sec. 8, the symbols d, n, and p will nevertheless retain their generalized meanings.

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[16]This method is also useful for very weakly-bound final states. It is therefore feasible to investigate the transition through zero binding of $A+n$.

[17]This approximation has the advantage that the midpoint of the interval of integration corresponds to E_R. For odd N, E_r is then one of the energies used in the quadrature formula. Because of the way in which the data are usually reduced, this treatment is adequate.

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