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Effect of Nuclear Deformability on Reaction Cross Sections*

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We have calculated the effect of nuclear deformability on the cross section for interaction of heavy ions with uranium. Static calculations indicate that these effects should be large; dynamic calculations with liquid-drop parameters indicate that they should be smaller, but measurable with currently available heavy ions (argon incident on uranium). Dynamic calculations with parameters derived from spectroscopic measurements indicate that the effects should be quite small, in agreement with experiment. We have done both classical and quantum-mechanical calculations; the two methods give the same results. The quantum-mechanical calculations also give the probability of Coulomb excitation to the vibrational states of the target nucleus. We discuss the value of several approximations used to estimate total reaction cross sections.

I. INTRODUCTION

Proposed experiments to synthesize the superheavy elements ($Z \approx 114$) will combine very heavy ions as projectiles with heavy element targets. For such experiments to be successful, it is necessary that the projectile and target fuse to form a compound nucleus and that the nucleus so formed dissipate its excitation energy in neutron or γ-ray emission rather than in fission.

The probability for fission is enhanced if several neutrons are emitted, since competition between neutron emission and fission occurs at each stage of evaporation. This probability is also enhanced by the high angular momenta characteristic of heavy-ion-induced reactions. In order to maximize the chance that the nucleus of interest survives fission, it is desirable that the initial compound nucleus have as low an energy and angular momentum as possible. These requirements dictate that the kinetic energy of the incident projectile be as low as possible.

A low energy for the projectile is desirable from another point of view. It is believed that the probability for incomplete fusion reactions increases with increasing kinetic energy. Experimental support for this idea is found in the work of Jodogne, Kowalski, and Miller[1] who investigated the probability for complete fusion as a function of energy for several light element systems. They found not only that the fraction of reactions leading to complete fusion decreases with increasing energy but that the absolute cross section for complete fusion decreases with increasing energy. Measurements by Sikkeland for argon incident on uranium lead to the same conclusion.[2]

The above considerations indicate that experiments aimed at producing superheavy elements will use heavy ions of the lowest possible energy: that is, with energies approximately equal to the Coulomb-barrier energy. For reactions involving heavy ions, it has so far been sufficient to estimate such barriers from the Coulomb energy of two spheres in contact. For some of the reactions contemplated for production of superheavy elements, the Coulomb-barrier energy and the Q for fusion are approximately equal and opposite. Irradiation with heavy ions of the appropriate energy should give complete fusion with good probability and give compound nuclei essentially in the ground state.

It has been pointed out by Beringer,[3] however, that these considerations ignore the deformability of the nuclei. As two charged nuclei approach one another, they tend to flatten into oblate shapes. At any fixed distance between the centers of charge, a lower total energy can be obtained by flattening – the Coulomb-interaction energy decreases more than the deformation energy increases. Because of this flattening the two particles must come closer together than do two spheres before fusing. The Coulomb-interaction energy is higher than for two spheres in contact. At the same time, part of the system energy has been taken from relative motion and put into energy of deformation. The net result is that the "barrier" against nuclear reaction is higher than it would be if the nuclei were rigid spheres. By "barrier" we mean the minimum kinetic energy such that the surfaces of the two nuclei just touch at the classical turn around point. Further development of these ideas has been made by Wong,[4] who considered shell effects on deformability, by Maly and Nix,[5] who have looked into some of the dynamical questions that arise, by Holm, Scheid, and Greiner, who have done dynamical calculations for several nuclei,[6] and by Jensen and Wong, who have also done dynamic calculations for several systems.[7] A closely related problem is that of Coulomb excitation and Coulomb fission, which have been considered by Wilets, Guth, and Tenn[8] and by Beyer and Winther.[9]

Beringer's results[3] indicate that the barrier against nuclear reactions proposed for the production of superheavy elements may be some tens of MeV higher than would be expected from calculations based on spherical nuclei. If this is the case, the bombardments at the barrier calculated for spherical nuclei would produce essentially no nuclear reactions. Bombardments at energies sufficiently high to overcome the barrier would lead to compound nuclei with such high excitation energies that very few would survive fission during the deexcitation process. The implication is that superheavy elements may be more difficult to make than had been anticipated.

Beringer's results also indicate that effects of nuclear flattening should be observable for such systems as neon on uranium, for which there already exist experimental data on reaction cross section as a function of energy. Comparison of such experimental data as are available with the theoretical predictions can be used as a basis for better understanding the feasibility of superheavy element production. We present here such a comparison.

In the following we consider a classical calculation of the equilibrium configuration of two nuclei with quadrupole deformation and a dynamic classical calculation of the interaction between two such nuclei. The results of these calculations are compared with the available experimental data. We then consider a dynamic quantum-mechanical calculation and compare this with the classical result. We consider also the implications of this calculation with respect to fission by Coulomb excitation. Finally, we investigate methods and approximations for calculating nuclear-reaction cross sections in heavy-ion induced reactions.

II. PARAMETERS OF THE PROBLEM

For small deformation, we describe the shape of the nucleus by the expression

$$R = R_0[1 + \beta Y_2^0(\cos\theta)], \qquad (1)$$

where R is the distance from the center of the nucleus to the surface, R_0 is the radius of the undeformed nucleus, β is a deformation parameter, and Y_2^0 is a spherical harmonic. (We note that, with this expression, the volume of the nucleus depends slightly on β.)

The energy V of the two interacting nuclei, relative to the energy of the two spherical nuclei at infinite separation, can be written as

$$V = V_{\rm DEF} + V_{\rm INT},$$

where $V_{\rm DEF}$ is the deformation energy of the two nuclei and $V_{\rm INT}$ is the Coulomb interaction between them. The energy of deformation is taken to be

$$V_{\rm DEF} = \tfrac{1}{2} C \beta^2,$$

where the stiffness constant C is evaluated either from the liquid-drop model or from spectroscopic data. Geilikman[10] has given an expression for the energy of interaction of two charged liquid drops with shapes described by Eq. (1) and a common symmetry axis. His expression was used by Beringer[3] in his calculations, and in corrected form by Wong.[4] The corrected form is

$$V_{\rm INT}=\frac{Z_1Z_2e^2}{r}\left[1+\left(\frac{9}{20\pi}\right)^{1/2}\frac{\sum_i\beta_i R_{i0}^2}{r^2}+\frac{3}{7\pi}\frac{\sum_i\beta_i^2R_{i0}^2}{r^2}+\frac{9}{28\pi}\frac{\sum_i\beta_i^2R_{i0}^4}{r^4}+\frac{27}{10\pi}\frac{\beta_1\beta_2R_{20}^2R_{10}^2}{r^4}+\cdots\right]. \quad (2)$$

Here Z_1 and Z_2 are the atomic numbers of the two nuclei, r is the distance between their centers of charge, $R_{i0}=r_0A_i^{1/3}$, and A_i is the mass number of the ith nucleus. The summations are taken over the two nuclei.

Evaluation of Parameters

For a liquid drop, the deformation energy can also be written as

$$V_{\rm DEF}=(E_s-E_s^0)+(E_C-E_C^0),$$

where the surface energy E_s is given as

$$E_s=E_s^0(1+\beta^2/2\pi+\cdots),$$

and E_C the Coulomb energy is given as

$$E_C=E_C^0(1-\beta^2/4\pi+\cdots).$$

The quantities E_s^0 and E_C^0 are the surface and Coulomb energies of a spherical nucleus. Thus

$$V_{\rm DEF}=\beta^2(2E_s^0-E_C^0)/4\pi,$$

or

$$C=(2E_s^0-E_C^0)/2\pi.$$

TABLE I. Deformability parameters.

Nuclide	C (MeV)
^{238}U	79.4
^{132}Xe	103.2
^{84}Kr	94.5
^{40}Ar	69.8
^{20}Ne	47.3

The quantities $E_s^0=a_sA^{2/3}$ and $E_C=a_CZ^2/A^{1/3}$, can be evaluated from the constants, a_s and a_C, of the semiempirical mass formula. We have used constants evaluated by Green[11] and have obtained the values of C given in Table I for the nuclei of interest to us. In particular, we note that C for uranium is predicted to be about 80 MeV by this model.

Alternatively, we may evaluate the stiffness constant C from spectroscopic data, as Wong has done.[12] This is conveniently done with a simultaneous evaluation of the inertial parameter B, which is necessary for dynamic calculations. For a harmonic oscillator, the spacing, $\hbar\omega$, of the levels is given as

$$\hbar\omega=\hbar\sqrt{C/B}. \quad (3)$$

For ^{238}U, $\hbar\omega$ is about 1 MeV, the spacing between the ground state and the β-vibrational state. Furthermore, for a harmonic oscillator, the mean square value $\overline{\beta^2}$ of the deformation β is given as

$$\overline{\beta^2}=\hbar/2\sqrt{BC}. \quad (4)$$

Thus if we knew both $\overline{\beta^2}$ and $\hbar\omega$ we can determine both B and C.

The quantity $\overline{\beta^2}$ is given in the literature[13] as

$$\mu^2=2\overline{\beta^2}/\bar\beta^2, \quad (5)$$

where $\bar\beta$ is the mean ground-state deformation (0.28 for ^{238}U).[14] Combining Eqs. (3), (4), and (5), we have

$$C=\hbar\omega/\mu^2\bar\beta^2.$$

A value of μ of 0.180 can be obtained from the ground-state rotational band[15] of ^{238}U or of 0.115 from the ratio of $B(E2)$'s for deexcitation to the ground state from the lowest 2+ state in the ground-state band and the lowest 2+ in the first β-vibrational band.[16] Using the second of these two values for μ, which seems more directly related to the properties of the β vibration, we get a value of about 960 MeV for C, or approximately 12 times the liquid-drop value.

This value of C taken together with a value for $\hbar\omega$ of 1 MeV, gives 40 600 amu F^2 for B, or in other units, $B/\hbar^2=972$ MeV^{-1}. It is convenient to parametrize B in terms of the value for irrotational flow,[17]

$$B_{\rm irr}=3AMR_0^2/8\pi, \quad (6)$$

where A is the mass number, M the nucleon mass, and R_0 the radius of the undeformed nucleus. With $R_0=1.5A^{1/3}$ F, $B_{\rm irr}$ is 2440 amu F^2. The value derived from the spectroscopic data is 16.5 times the irrotational flow value; this ratio is consistent with what has been found for other nuclei.[12]

III. STATIC CALCULATIONS

At any separation distance r, there is one equilibrium shape, defined by the solutions to the equations $\partial V/\partial \beta_1 = 0$ and $\partial V/\partial \beta_2 = 0$. For the potential given by Eq. (2), these two equations are linear in β_1 and β_2 and independent. The solution for the equilibrium deformation is therefore straightforward.

The barrier against nuclear reaction by the target and projectile is taken to be the total energy (Coulomb-interaction energy plus deformation energy) when the surfaces of the two nuclei are just touching. For our calculations, each nucleus is considered to have a well-defined surface at a radius given by Eq. (1) with $R_0 = 1.5 A^{1/3}$ F. A radius parameter of 1.5 F is somewhat larger than one expects from nuclear radius measurements; it is, however, appropriate for a model that considers the nucleus to have a well-defined surface. (See the last section of this paper and also a discussion by Thomas.[18])

The calculation of the barrier was done as follows. A value of r, the separation distance, equal to the sum of the spherical radii was chosen as a starting point. Equilibrium values of the shapes were calculated for this distance. Since the equilibrium nuclei are oblate nuclei with a common axis of symmetry their surfaces do not touch at the chosen value of r. A new value of r was picked so that the surfaces would touch. A new equilibrium calculation was done. The cycle was repeated until further iterations produced no significant change in the separation distance or in the values of β_1 and β_2.

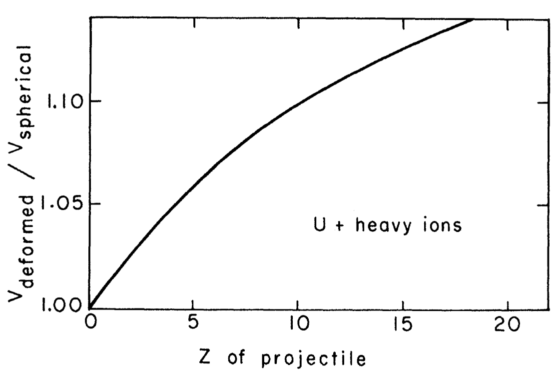

FIG. 1. The ratio of the interaction energy for two touching oblate spheroidal nuclei with equilibrium deformation to the energy of two touching spherical nuclei. One of the two nuclei is uranium; the other has the nuclear charge indicated on the abscissa.

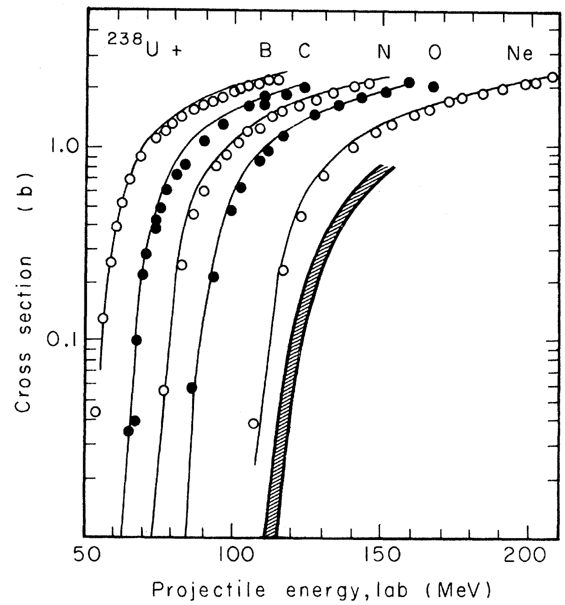

FIG. 2. Experimental and theoretical reaction cross sections for various heavy ions incident on uranium. The experimental data are from Viola and Sikkeland (Ref. 19). The theoretical curves have been calculated using the optical-model parameters given in Table II. The hatched band indicates the estimated cross section for uranium plus neon if the nuclei had their equilibrium deformations.

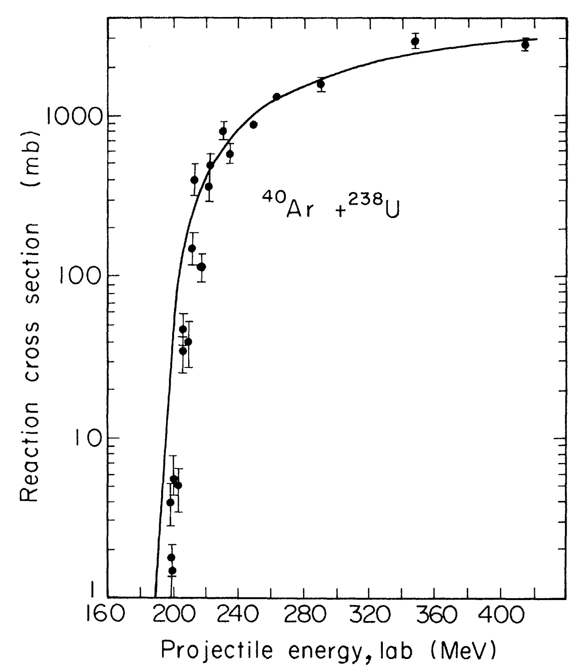

FIG. 3. Experimental and theoretical reaction cross sections for argon incident on uranium. The experimental data are from Sikkeland (Ref. 20). Below about 250 MeV the theoretical curve is based on optical-model calculations with the parameters of Table II. Above 250 MeV the curve was calculated classically.

The results of static calculations for various heavy ions incident on ^{238}U are presented in Fig. 1. For these calculations we have used liquid-drop values of the stiffness constant C. They are in good agreement with results of similar calculations done by Beringer.[3] Here we have plotted against atomic number of the projectile the ratio of the barrier as calculated above to the barrier for two spheres in contact. We note that the calculated barrier is significantly higher than the spherical barrier for the common heavy ions, boron, carbon, nitrogen, oxygen, and neon. In particular, for neon the calculated barrier is about 10% higher than the spherical barrier; for argon the effect is about 15%.

To show that no such effects are observed we have compared in Fig. 2 experimental reaction cross sections for heavy ions incident on uranium (measured by Viola and Sikkeland[19]) with cross sections calculated assuming that the nuclei are rigid spheres of reasonable radius. We also plot, as the cross-hatched band, estimates of the cross section for neon incident on uranium calculated assuming that the potential is given by the sum of Coulomb and deformation energies and that at each separation distance the nuclei have their equilibrium shapes. (See the last section of this paper for the details of this calculation.) We note that the experimental cross sections agree well with those calculated for rigid spherical nuclei and not at all with those for deformable nuclei. A similar comparison for argon ions incident on uranium[20] is shown in Fig. 3. Except at the lowest energies the agreement between experiment and the calculation based on rigid spheres is good.

The spherical nucleus calculations were done with the optical model and a single set of parameters chosen by Auerbach and Porter[21] to reproduce the elastic scattering of heavy ions from gold. (See Table II.) It may be argued that these parameters already compensate for the effects of deformation, since they were chosen to fit experimental data. If this were the case we would expect systematic deviations between experiment and calculation as we consider either light projectiles or light targets. Thomas[22] has shown that these parameters give calculated cross sections that agree (within about 10%) with experiment over a wide range of projectiles, targets, and energies. For instance, the cross sections for such low-Z systems as oxygen and carbon incident on aluminum are satisfactorily predicted with these parameters.

TABLE II. Optical-model parameters used in calculation of reaction cross sections. Both real and imaginary potentials are Woods-Saxon.

$R = 1.26(A_1^{1/3} + A_2^{1/3}) \times 10^{-13}$ cm
$V = 41.8$ MeV
$W = 16.4$ MeV
$a = 0.49 \times 10^{-13}$ cm

The assumption that the interacting nuclei have their equilibrium shapes at any distance is, thus, not in accord with experimental evidence. The measured cross sections at a given energy are substantially larger than the predictions we have made on the basis of this model. If we believe that the deformability parameter C is approximately correct, then we must conclude that the discrepancy arises from our neglect of the dynamics of the process. Only if the relative velocity of target and projectile is very low can the shapes of the nuclei adjust to their equilibrium value. A similar conclusion has been reached by Holm, Scheid, and Greiner,[6] who did dynamic calculations for ^{122}Sn as both target and projectile. They found a 15% increase in the barrier due to dynamic distortion effects compared to increases of up to 35% calculated by the static, or adiabatic, approximations. Jensen and Wong[7] have also shown that the dynamic calculations with spectroscopic parameters give barriers that are only slightly in excess of those for rigid spheres.

IV. DYNAMIC CLASSICAL CALCULATIONS

The equations of motion for two deformable nuclei moving along their line of centers are

$$\mu \ddot{r} = -\partial V(r, \beta_1, \beta_2)/\partial r, \tag{7}$$

$$B_1\ddot{\beta}_1 = -\partial V(r, \beta_1, \beta_2)/\partial\beta_1 + D_1\dot{\beta}_1, \tag{8}$$

$$B_2\ddot{\beta}_2 = -\partial V(r, \beta_1, \beta_2)/\partial\beta_2 + D_2\dot{\beta}_2. \tag{9}$$

The quantity μ is the reduced mass of the system; B_1 and B_2 are the inertial parameters associated with vibration of the two nuclei. The potential V is the sum of Coulomb interaction and deformation energies. The coefficients D_1 and D_2 of the velocity term in Eqs. (8) and (9) represent the viscosity of the two nuclei.

At very large values of r, the Eqs. (8) and (9) become the equations for the damped harmonic oscillator, with the solution

$$\beta = A\exp(-Dt/2B)\exp\left\{\pm it\left[\frac{C}{B}\left(1-\frac{D^2}{4BC}\right)\right]^{1/2}\right\}.$$

The quantity $\sqrt{C/B}$ is the frequency of the undamped oscillator. As the above equation shows, it is convenient to give the viscosity in units of $\sqrt{BC}$. The viscosity represents a radiation damping term, which is small, and the transfer of energy from kinetic energy of collective motion to intrinsic, or nucleonic, excitation. For even-even nuclei there is no intrinsic excitation possible below about 2 MeV. Therefore, unless there is more than this energy in kinetic energy of collective motion, there can be no such transfer; the viscosity is zero.[23] Except for one set of calculations done to show its effect we have assumed the viscosity to be zero.

For a nucleus with a ground-state deformation, such as ^{238}U, there should be an additional equation of motion describing the rotational motion and terms in the potential expression giving the dependence of energy on the angle between the nuclear symmetry axis and the line of centers. Calculations considering the rotational motion alone indicate that this effect can be ignored for our purposes.[24]

For the dynamic calculations we have taken the barrier to be the incident energy that just brings the two nuclear surfaces into contact at the distance of closest approach. The solution was found by numerical integration of Eqs. (7), (8), and (9).

With liquid-drop values for the stiffness constant C, irrotational flow values for the inertial parameter B, and zero viscosity we find for neon incident on uranium that the barrier is about 5% higher than that for rigid spheres and for argon on uranium 10% higher. These are to be compared with the static values of 10 and 15%, respectively. As expected, the dynamic calculations give results that are substantially lower than those from the static calculations. They are still much higher than what we would infer from the experimental results shown in Figs. 2 and 3.

We look now at the effect of viscosity on the system ^{20}Ne incident on ^{238}U. We have used the liquid-drop values for the deformability and the irrotational flow values for the inertial parameter. Figure 4 shows, as the solid line, the ratio of calculated barrier to spherical barrier plotted against viscosity (in units of $\sqrt{BC}$). Only for a rather large viscosity does the calculated barrier fall to approximately the spherical value, in agreement with experiment. Such large viscosities do not seem to be physically reasonable. The maximum kinetic energy of collective motion ($\frac{1}{2}B\,\dot{\beta}^2$) for the system is only 2.9 MeV, and we therefore do not expect much transfer of energy from collective to intrinsic motion.

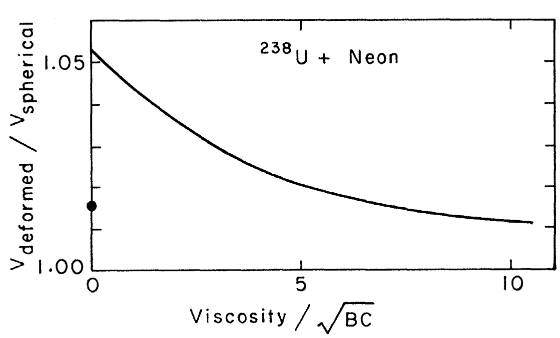

FIG. 4. The ratio of the interaction energy for two touching oblate spheroidal nuclei with dynamic deformation to the energy of two touching spherical nuclei, plotted against nuclear viscosity. The system considered is neon incident on uranium. The line was calculated using an inertial parameter equal to the irrotational flow value. The single point at zero viscosity was calculated with an inertial parameter 10 times as great.

We have seen above that the spectroscopic values of the stiffness constant and the inertial parameter are, respectively, about 12 times the liquid-drop value and 16 times the irrotational flow value. The single point plotted on the vertical axis of Fig. 4 represents the result of a calculation using zero viscosity, the liquid-drop stiffness, and a value of the inertial parameter B equal to 10 B_{irr}. We see that there is now agreement between the calculated barrier and the barrier for rigid spheres (equivalent to the experimental value). Thus, if we use an inertial parameter that is taken from experimental measurements we find that there is essentially no deformation during the interaction, even for zero viscosity.

In Fig. 5 we present results for the system ^{40}Ar incident on ^{238}U. As before the ordinate is the ratio of the barrier for deformable nuclei to the barrier for rigid nuclei. The abscissa is the inertial mass parameter B, in units of the irrotational flow value. For these calculations the viscosity was taken to be zero. For $B=0$, we have the static result mentioned above; for B equal to the irrotational flow value the barrier is about 10% higher than that for rigid spheres. This result indicates that the liquid-drop and irrotational flow parameters are not the correct ones to use. However, if we use B about 10 times the irrotational flow value, the calculated barrier is only slightly in excess of the value for rigid spheres. Since the effect of deformation appears to be small, we expect and find that the reaction cross section is approximately that predicted for rigid spheres.

We must also take into account that the stiffness C is significantly larger than the liquid-drop value. Table III summarizes results for several systems that will be of interest when very heavy ions become available. The systems considered are argon, krypton, xenon, and uranium incident on uranium. Stiffness of 12 times the liquid-drop stiffness and inertial parameters 15 times the irrotational flow value were used. We see that even for uranium incident on uranium the percentage increase in the barrier is very small. The absolute increase (19 MeV) is, however, significant since this energy would have to be dissipated by evaporation of about two neutrons with chances for fission competition.

V. DYNAMIC QUANTUM-MECHANICAL CALCULATIONS

In addition to the classical calculations mentioned above, we have done dynamic quantum-mechanical calculations to determine if a classical model gives a satisfactory picture of the system of the interacting nuclei. In these calculations we have assumed that the motion of the incident ion can be described classically but that the vibrational motion of the target nucleus must be described quantum mechanically. In these calculations, we ignore the vibrational motion of the projectile.

We assume that the wave function describing the oscillation of the projectile can be written as

$$\psi(\beta, t) = \sum_k C_k(t) e^{i\omega_k t} \Phi_k^0(\beta), \tag{10}$$

where the Φ_k^0's are the harmonic-oscillator wave functions, the C_k's are complex coefficients, and $\hbar\omega_k$ is the energy of the kth vibrational state above the ground state. At time zero the nucleus is in its ground state; hence $C_0(0) = 1$ and all other C's are zero. The time dependence of the coefficients is given by the time-dependent Schrödinger equation

$$H_0\psi + V\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial t}, \tag{11}$$

where H_0 is the Hamiltonian of the unperturbed oscillator and V is the perturbing potential due to the presence of a nearby nucleus. V is given by Eq. (2) and is an implicit function of time through

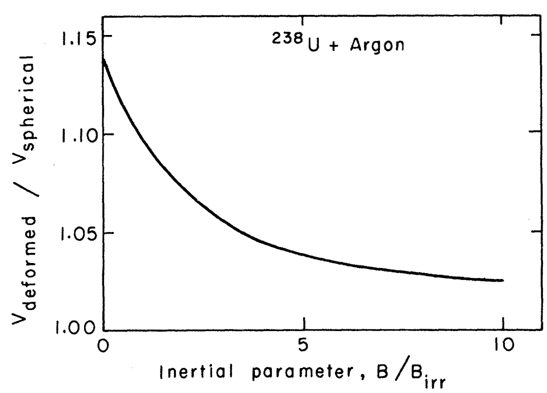

FIG. 5. The ratio of the interaction energy for two touching oblate spheroidal nuclei with dynamic deformation to the energy of two touching spherical nuclei, plotted against inertial parameter. The system considered is argon incident on uranium. The viscosity is taken to be zero.

TABLE III. Barriers (in laboratory system) for various ions incident on uranium calculated classically with spectroscopic parameters.

Ion	Barrier for rigid spheres	Barrier calculated dynamically (MeV) $B = 15B_{irr}$ $C = 12C_{LD}$	Ratio
$_{18}Ar^{40}$	194.09	194.38	1.007
$_{36}Kr^{84}$	406.72	411.54	1.012
$_{54}Xe^{132}$	656.79	668.22	1.017
$_{92}U^{238}$	1311.26	1340.66	1.022

the dependence of r and β on time. The two Eqs. (10) and (11) lead to the relationship

$$\dot{C}_k = \frac{i}{\hbar}\sum_n C_n e^{i(\omega_n - \omega_k)t} V_{kn}, \tag{12}$$

where V_{kn} is the matrix element $\langle\Phi_k|V|\Phi_n\rangle$. We combine Eq. (12) with the radial equation (7) and obtain the solution as a function of time by numerical integration. We have considered only the first eight vibrational states. The value of $\bar{\beta}(= \langle\psi|\beta|\psi\rangle)$ as a function of time agrees very well with the classically calculated value of β as a function of time.

The probability $|C_k|^2$ that the nucleus is in the kth vibrational state is shown as a function of time in Fig. 6. For these calculations, we have considered the system krypton plus uranium, at a laboratory energy for the krypton of 410 MeV. We have used a value of the inertial parameter equal to 9.3 times the irrotational flow value and a value of the stiffness equal to 8.5 times the liquid-

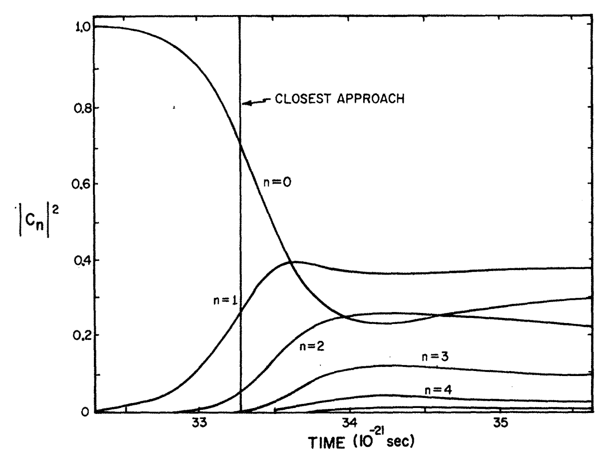

FIG. 6. Probability that ^{238}U nucleus is in the nth excited vibrational state as it is approached by a ^{84}Kr nucleus. Plotted as a function of time. Laboratory energy is 410 MeV. The inertial parameter is 15 times the irrotational flow value; the stiffness is 12 times the liquid-drop value. The vertical line shows the time of closest approach.

drop value. This is the system, and parameters considered by Beyer and Winther[9] in their calculations on Coulomb-induced fission. For comparison with their calculations we can get a crude estimate of the Coulomb-excitation cross section by multiplying the values of $|C_k|^2$ for very long times by the Rutherford scattering cross section at 180°. The results are given in Table IV and are compared there with those of Beyer and Winther. There is fairly good agreement for the cross sections for excitation to the lower vibrational states. This becomes progressively worse for the higher states but is probably about as good as we can expect considering the crudeness of our cross-section calculation and the more realistic oscillator potential used by them.

VI. ESTIMATION OF TOTAL REACTION CROSS SECTION FOR HEAVY-ION-INDUCED REACTIONS

The simplest reactions considered for producing superheavy elements are those in which a heavy ion fuses with some target nucleus to form a compound nucleus. Deexcitation via neutron emission follows. To calculate the expected cross section for such a reaction we must know (1) the cross section for nuclear interaction between target and projectile (reaction cross section), (2) the probability that the interaction will lead to fusion, and (3) the probability that the compound nucleus will deexcite to the desired product. The third of these is outside the scope of this discussion. Experimental evidence on the second has been reviewed by Thomas[22]; see also the work by Sikkeland.[2] We confine our discussion here to the first.

TABLE IV. Probability and estimated cross section for ^{238}U to be left in the nth vibrational state after a head-on encounter with ^{84}Kr at 410 MeV (lab).

n	$C_n{}^2$	$C_n{}^2 \times \sigma_{180}$[a] (mb)	Beyer and Winther[b] (mb)
0	0.206		
1	0.326	34.9	30.
2	0.258	27.4	7.
3	0.133	14.1	1.1
4	0.0525	5.6	0.3
5	0.0203	2.2	0.08
6	0.004 95	0.5	0.02
7	0.000 270	0.02	

[a] σ_{180} is the cross section for Rutherford scattering at 180° and is equal to 10.6 mb for this system.

[b] Estimated from graph in Ref. 9. For these calculations we have used the same parameters that they used.

Spherical Nuclei

The best values of reaction cross sections for heavy-ion-induced reactions have been calculated using a diffuse-well optical model with an appropriate set of well parameters. As has been pointed out by Thomas,[22] one set of such parameters gives calculated cross sections in agreement with experimental values over a wide range of projectiles, targets, and energies. These parameters, from Auerbach and Porter,[21] are given in Table II. Square wells and a parabolic approximation to the barrier shape have been used with some success, but except in special cases, there seems little reason to use these approximations when satisfactory optical-model codes are available.

The principal disadvantage of quantum-mechanical calculations of the reaction cross section is that for energies well above the Coulomb barrier it is necessary to sum partial cross sections for many partial waves – several hundred, for instance, for argon incident on uranium. On the other hand, if many partial waves contribute, we may expect that the classical formula for the reaction cross section will be valid. We investigate here the circumstances under which we can use the classical formula instead of the results of a quantum-mechanical calculation. In addition we will consider a modification of the classical formula suggested by Wong.[25]

The classical formula for reaction cross section σ_R is

$$\sigma_R = \pi R^2 (1 - B/E),$$

where R is the nuclear radius, B is the Coulomb barrier, and E is the kinetic energy in the center-of-mass system. The Coulomb barrier B is given by $Z_1 Z_2 e^2/R$, where $Z_1 e$ and $Z_2 e$ are the nuclear charges of projectile and target. With increasing energy, the cross section calculated classically should asymptotically approach that calculated quantum mechanically, provided that the correct choice of R is made for the classical calculation. Since we are using a model in which the nucleus is considered to have a well-defined surface, we may expect that the required value of R will be somewhat larger than we would expect from the usual nuclear radius measurements.

We can choose a value of R by normalizing the results of the classical calculation to those of the quantum-mechanical calculation at an energy sufficiently high that we expect the two methods to give results that are satisfactorily in agreement. If we have made this choice correctly the two methods should give results in agreement for all energies higher than the one chosen for normalization.

We suggest, arbitrarily, that the normalization be done at an energy for which the quantum-mechanical calculations give a cross section of 1 b. For heavy ions incident on heavy nuclei, this energy is about 20% greater than the classical barrier; at such energies the number of partial waves required by the quantum-mechanical calculations is not excessive.

In Fig. 7 we have plotted a comparison of cross sections calculated quantum mechanically and calculated classically for helium ions and neon ions incident on uranium. The values of R used in the classical calculation were chosen to give agreement between the two methods at a cross section of 1 b. The quantum-mechanical calculations for helium ions on uranium were taken from Huizenga and Igo[26]; those for neon were calculated using the parameters given in Table II.[27] We see that the classical calculation remains within 10 to 20% of the quantum-mechanical one for energies as low as 10% above the classical barrier. This method thus provides a reasonable method for estimating reaction cross sections for energies at which it is impractical to do the quantum-mechanical calculation. We may expect the agreement between the classical and quantum-mechanical calculations to be better than indicated here for heavier ions, since the heavier the ion the more partial waves and the more classical the system. We have used this method to calculate the solid curve shown in Fig. 3 for argon incident on uranium. The low-energy portion of the curve is based on optical-model calculations using the parameters given in Table II. For cross sections greater than 1 b, the curve is calculated classically.

If we take R for the classical calculations to be given as

$$R=r_0(A_1^{1/3}+A_2^{1/3}),$$

where A_1 and A_2 are the mass numbers of target and projectile, then the appropriate values of r_0 are 1.46×10^{-13} cm for neon on uranium and 1.41×10^{-13} cm for argon on uranium. As expected, these values are significantly larger than the value of 1.26×10^{-13} used in the diffuse-well optical-model calculations.

At projectile energies close to the Coulomb barrier, the cross section calculated classically is lower than that calculated quantum mechanically. Wong[25] has suggested that the classical formula be corrected by addition of a term of order

$$d(E-0.5B)/R(E-B),$$

where d is a nuclear diffuseness parameter. The results of a calculation using this correction are shown in Fig. 7. This method has the disadvantage that it predicts infinite cross section at the barrier. At energies well above the barrier it does not appear to provide any advantage over the unmodified classical formula.

Deformable Nuclei

In calculation of reaction cross sections between spherical nuclei, we assume that the interaction energy between the two nuclei is $Z_1Z_2e^2/r$, where r is the distance between centers, and is spherically symmetric. If deformation effects are important, neither of these assumptions is correct. In this case, the interaction energy for nuclei having common symmetry axes will be less than $Z_1Z_2e^2/r$. In addition, the potential will be radially symmetric only if the collision takes place so slowly that the nuclei have their equilibrium deformation at all times.

In Fig. 2 we show an estimate of the cross section for neon incident on uranium, calculated under the assumption that the nuclei do have the equilibrium deformation at all times. Then, as noted, the interaction energy is spherically symmetric. The two curves bounding the shaded area were calculated as follows: Both were calculated using a square-well black-nucleus model. This model with a radius parameter of $r_0=1.5\times10^{-13}$ cm has been shown to give results that are in fairly good agreement with experiment. The lower

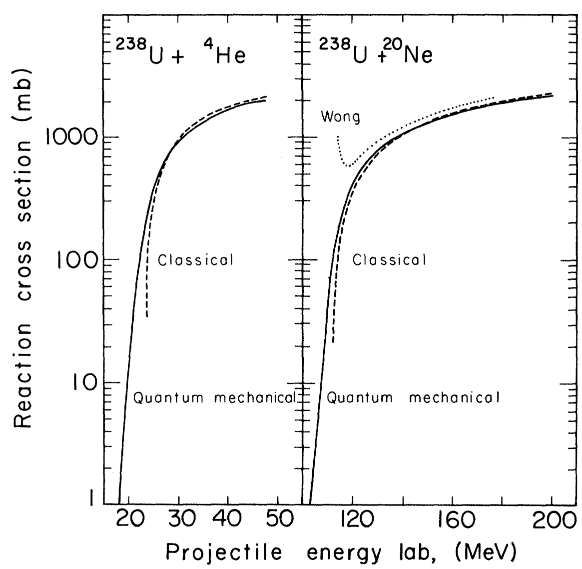

FIG. 7. Reaction cross section for helium ions and neon ions incident on ^{238}U as a function of laboratory energy. Solid curves are based on optical-model calculations; dashed curves are based on classical calculations normalized to the quantum-mechanical result at a cross section of 1 b. The dotted curve is based on an approximation suggested by Wong (Ref. 25).

curve was calculated using a potential energy that varies as $Z_1Z_2e^2/r$ for r greater than the distance between the oblate spheroids that just touch. This energy is everywhere equal to or greater than the interaction energy for the two deformed nuclei; the cross section so determined should therefore be less than the cross section obtained from use of the correct potential. The upper curve was based on the assumption that at the point of contact, the interaction energy is as calculated for the two spheroids, but that it falls off as $1/r$. This potential is everywhere less than the correct one and we will obtain an upper limit for the cross section.

SUMMARY

Static calculations suggest that there should be a substantial decrease in the reaction cross section at a given bombarding energy because of deformation of the two interacting heavy ions. Dynamic calculations with liquid-drop parameters indicate that the effects should be somewhat smaller but still observable for argon incident on uranium. The experimental data are not in agreement with this conclusion. Only by using parameters obtained from spectroscopic parameters do we get agreement between experiment and theory. These conclusions may be somewhat modified by inclusion of shell effects (as has been noted by Wong[4]), by consideration of collisions other than those with zero impact parameter,[6,7] or by taking into account the ground-state deformations of the interacting nuclei.[7]

We have shown that classical dynamic calculations of the deformation give results that are in agreement with the results of quantum-mechanical calculations. We have also shown that for bombarding energies 10 to 20% above the Coulomb barrier a classical calculation of the reaction cross section is in satisfactory agreement with a quantum-mechanical one providing an appropriate adjustment of the radius parameter is made.

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