[10]Usually wave functions are tabulated according to amplitudes $C$ appearing in $|\alpha JM\rangle = \sum_{a\le b} C_{ab}^{\alpha J}(1+\delta_{ab})^{-1/2} \times A_{JM}^{\dagger}(ab)|\omega\rangle$ with $\sum_{a\le b}|C|^2=1$. However, for formal manipulations the unrestricted form (5.5) is often more convenient. The connection between the amplitudes is $\eta_{ab}^{\alpha J} = (-)^{j_a-j_b+J}\eta_{ba}^{\alpha J} = C_{ab}^{\alpha J}(1+\delta_{ab})^{1/2}$.

[11]The first three results were given, aside from the configuration amplitude, by S. Yoshida, Nucl. Phys. 33, 685 (1962).

[12]cf. N. K. Glendenning, Ann. Rev. Nucl. Sci. 13, 191 (1963).

[13]J. C. Hardy and I. S. Towner, Phys. Letters 25B, 98 (1967); D. G. Fleming, J. Cerny, and N. K. Glendenning, Phys. Rev. 165, 1153 (1968).

PHYSICAL REVIEW C VOLUME 2, NUMBER 2 AUGUST 1970

# Comment on the Number of Degrees of Freedom in Fluctuation Analysis*

H. L. Harney†
*Max-Planck-Institut für Kernphysik, Heidelberg, Germany*

and

A. Richter‡
*Argonne National Laboratory, Argonne, Illinois 60439*

(Received 24 October 1969; revised manuscript received 2 February 1970)

Different methods of calculating the number of degrees of freedom $N_{\rm eff}$ in statistical cross-section fluctuations are compared. The underlying physical assumptions of the methods are discussed critically, and numerical examples for $N_{\rm eff}$ are given.

## I. INTRODUCTION

The number of degrees of freedom $N_{\rm eff}$ is a fundamental quantity in analyses of cross-section fluctuations, since its knowledge allows one to estimate the relative values of the direct and compound-nucleus cross sections. In Sec. II we investigate the "basic cross-section method" for the calculation of $N_{\rm eff}$ and the modifications of it due to Gibbs. We show that the lack of statistical independence of the basic cross sections raises difficulties when one solves for $N_{\rm eff}$ within the Gibbs model. In Sec. III, we present calculations of $N_{\rm eff}$ for recently published experiments on $^{27}$Al$(\alpha, p)^{30}$Si and $^{31}$P$(d, \alpha)^{29}$Si in order to demonstrate the differences between the approaches discussed in Sec. II.

In addition, it is pointed out that a realistic calculation of the fluctuation damping coefficient does not require the knowledge of the capture cross sections of all open compound-nucleus decay channels.

## II. COMPARISON OF DIFFERENT FORMALISMS

In an article on the "Limitation of the Number of Degrees of Freedom in Fluctuation Analysis," Gibbs[1] discusses the applicability of an expression for this number given by Bondorf and Leachman.[2] These authors decompose the differential cross section $\sigma(\theta)$ into a number $N$ of "basic cross sections" $\sigma_\mu$ and write

$$\sigma = \sum_\mu \sigma_\mu , \tag{1}$$

where the $\mu$ are the four spin projections of the colliding and outgoing particles. The variance of the fluctuations of each $\sigma_\mu$ is

$$(\langle \sigma_\mu{}^2\rangle - \langle\sigma_\mu\rangle^2)/\langle\sigma_\mu\rangle^2 = 1 , \tag{2}$$

where the angular brackets represent energy averages. In Ref. 2, the $\sigma_\mu$ are assumed to be statistically independent; and hence the number $N_{\rm eff}$ of degrees of freedom, which is defined as

$$N_{\rm eff}(\theta) = \langle\sigma(\theta)\rangle^2/[\langle\sigma^2(\theta)\rangle - \langle\sigma(\theta)\rangle^2] , \tag{3}$$

reduces to

$$N_{\rm eff}^{\rm all}(\theta) = [\sum_{\mu=1}^{N}\langle\sigma_\mu(\theta)\rangle]^2/[\sum_{\mu=1}^{N}\langle\sigma_\mu(\theta)\rangle^2] . \tag{4}$$

For computational purposes the averaged basic cross sections $\langle\sigma_\mu\rangle$ may be identified with Hauser-Feshbach expressions. Thus the analysis of several fluctuation experiments[3-5] has been based on Eq. (4). However, the $\sigma_\mu$ need not be statistically independent. Indeed, if their number $N$ is larger than the number $\Lambda$ of statistically independent scattering matrix elements $U_{lj l'j'}^{J\pi}$ responsible for the reaction cross section, the assumption of independence of the $\sigma_\mu$ must fail. The purpose of the paper by Gibbs[1] is to give an appropriate correction to $N_{\rm eff}^{\rm all}$ in this case. From a schematic model, in which all basic cross sections are assumed to be equal and each of them is constructed out of $\Lambda$ independent amplitudes with equal weight, he finds

$$N_{\rm eff}(\theta)=N^{\rm all}_{\rm eff}(\theta)\Lambda[\Lambda+N^{\rm all}_{\rm eff}(\theta)-1]^{-1}. \tag{5}$$

In order to estimate $\Lambda$, the collision matrix element is factored into a product

$$U^{J\pi}_{ljl'j'}=\alpha^{J\pi}_{ljl'j'}x^{J\pi}(E) \tag{6}$$

such that the factor $\alpha$ varies slowly with the energy $E$ and describes the single-particle features of the cross section, while $x$ contains the rapid energy variation due to the compound-nucleus states.[1] The statistical assumption

$$\langle x^{J\pi}(E)x^{\bar{J}\bar{\pi}*}(E)\rangle=\delta_{J\bar{J}}\delta_{\pi\bar{\pi}} \tag{7}$$

is introduced. It follows immediately from the indexing of $\alpha$ and $x$ in Eq. (6) that $\Lambda$ is equal to the number of parity states times the number of $J$ values.

It is clear that Eq. (6) leads to much less statistically independent amplitudes than does the random-phase approximation used by Ericson[6] and by Brink, Stephen, and Tanner.[7] These authors assume that every two $U$ matrix elements with different sets of indices $\{cc'\}=\{ljJ\pi, l'j'J\pi\}$ are statistically independent, i.e., that

$$\langle U_{cc'}U_{\bar{c}\bar{c}'}{}^*\rangle=\delta_{c\bar{c}}\delta_{c'\bar{c}'}|\alpha_{cc'}|^2. \tag{8}$$

Hence, the statistical assumption (7) implies much more important corrections to Eq. (4) than does the assumption (8). Though most fluctuation experiments (see, e.g., the review by Ericson and Mayer-Kuckuk[8]) have been successfully interpreted in terms of the statistical model defined by Eq. (8), there are some experiments[3,4] that seem to indicate a breakdown of the Ericson model, since they revealed an unexpectedly small number of degrees of freedom. This is the reason for the introduction of the model of Ref. 1.

We want to point out that this model may, however, not be treated numerically in a satisfying way: Ref. 1 presents a "realistic calculation of $N_{\rm eff}$" which starts from dropping the very schematic assumptions of equal weight of all basic cross sections and all independent amplitudes. This model is defined by Eqs. (6) and (7) and by

$$\langle|x^{J\pi}(E)|^4\rangle=2, \tag{9a}$$

$$\langle[x^{J\pi}(E)]^2\rangle=0. \tag{9b}$$

The basic cross sections may be written as

$$\sigma_\mu(E)=|\sum_{J\pi}a^{J\pi}_\mu x^{J\pi}(E)|^2, \tag{10}$$

where the coefficients $a^{J\pi}_\mu$ are defined in terms of the quantities $\alpha$ and geometrical coefficients $b$ by

$$a^{J\pi}_\mu=\sum_{ljl'j'}b^{\mu J\pi}_{ljl'j'}\alpha^{J\pi}_{ljl'j'}. \tag{11}$$

One derives

$$\langle\sigma^2\rangle-\langle\sigma\rangle^2=\sum_\mu(\sum_{J\pi}\sigma^{J\pi}_\mu)^2+\sum_{\mu\neq\nu}\sum_{J\pi}\sigma^{J\pi}_\mu\sigma^{J\pi}_\nu+A, \tag{12}$$

where

$$\sigma^{J\pi}_\mu=|a^{J\pi}_\mu|^2, \tag{13}$$

and

$$A=\sum_{\mu\neq\nu}\sum_{J\pi\neq J'\pi'}a^{J\pi}_\mu a^{J\pi*}_\nu a^{J'\pi'*}_\mu a^{J'\pi'}_\nu. \tag{14}$$

In order to evaluate Eq. (12), two assumptions are made. (i) The $a^{J\pi}_\mu$ are considered to be random numbers with the expectation value zero. The expectation value of $A$ is then also zero, and the term $A$ is neglected. (ii) The $\sigma^{J\pi}_\mu$ are calculated by employing usual Hauser-Feshbach methods. The first approximation implies an error of unknown importance. The second one implies an inconsistency, as can be seen as follows. In Gibbs's model we have

$$\sigma^{J\pi}_\mu=|\sum_{ljl'j'}b^{J\pi}_{ljl'j'}\alpha^{J\pi}_{ljl'j'}|^2, \tag{15}$$

while in the Hauser-Feshbach theory,[9] which makes the assumption (8), we have

$$\sigma^{J\pi}_\mu(\text{H.F.})=\sum_{ljl'j'}|b^{J\pi}_{ljl'j'}|^2|\alpha^{J\pi}_{ljl'j'}|^2. \tag{16}$$

In fact, every statistical model in which the strong Hauser-Feshbach assumptions are weakened in the sense of Ref. 1 leads to this difficulty: partial cross sections of the type of $\sigma^{J\pi}_\mu$ that cannot be calculated, since they contain unknown interference terms. We notice that in Ref. 1 the results of Eq. (12) may very well be reproduced by the schematic model of Eq. (5), if an appropriate $\Lambda$ is introduced. Hence, though the "realistic calculation of $N_{\rm eff}$" in Ref. 1 is mathematically not satisfying, the Hauser-Feshbach calculation of $\sigma^{J\pi}_\mu$ provides a way of estimating $\Lambda$. This way may, however, be somewhat complicated and not quite transparent. Equation (12) has usually been evaluated in the applications of Gibbs's model.[4,5]

No interference terms appear if the Hauser-Feshbach assumptions (8) are introduced. We complement Eq. (8) by

$$\langle|U_{cc'}(E)|^4\rangle=2|\alpha_{cc'}|^4 \tag{17a}$$

and

$$\langle[U_{cc'}(E)]^2\rangle=0 \tag{17b}$$

to obtain

$$\langle\sigma^2\rangle-\langle\sigma\rangle^2=\sum_\mu(\sum_{cc'}|b^\mu_{cc'}\alpha_{cc'}|^2)^2+\sum_{\mu\neq\nu}\sum_{cc'}|b^\mu_{cc'}b^\nu_{cc'}|^2|\alpha_{cc'}|^4+A', \tag{18}$$

where the term $A'$ is defined by

$$A' = \sum_{\mu \neq \nu} \sum_{cc' \neq \bar{c}\bar{c}'} b^{\mu}_{cc'} b^{\nu *}_{cc'} b^{\mu *}_{\bar{c}\bar{c}'} b^{\nu}_{\bar{c}\bar{c}'} |\alpha_{cc'}|^2 |\alpha_{\bar{c}\bar{c}'}|^2 . \tag{19}$$

The $|\alpha^{J\pi}_{ljl'j'}|^2$ may be directly expressed by transmission coefficients, and the $b^{\mu J\pi}_{ljl'j'}$ are geometrical quantities. So even the term $A'$ can be calculated exactly. Equations (18) and (19) are equivalent to the expression (12) given[10] in a paper by von Witsch *et al.*[11]

However, the last equations cannot account for the large cross-section variances deduced in Refs. 3, 4. These experiments thus seem to be cases in which the strong assumption of Eq. (8) fails. The degree to which the $U$ matrix elements are statistically independent may then be expressed by the parameter $\Lambda$ which should be defined by Eq. (5). This procedure leaves open the question where the correlations between $U$ matrix elements arise: from Eqs. (6), (7) or any other condition which creates correlations. Indeed, any restriction of the statistical independence leading to the same number $\Lambda$ yields the same effective number of degrees of freedom.

### III. NUMERICAL CALCULATIONS

We have calculated $N_{\text{eff}}(\theta)$ for $^{27}\text{Al}(\alpha, p_0)^{30}\text{Si}$ at the energy $E_\alpha^{\text{c.m.}} = 6.3$ MeV (Ref. 3) under the assumptions (8) and (17) and with the aid of formula (12) of Ref. 11. In Fig. 1, $N_{\text{eff}}(\theta)$ is compared with two curves from Ref. 1, namely, $N^{\text{all}}_{\text{eff}}(\theta)$ and $N^{J\pi}_{\text{eff}}(\theta)$. The latter is calculated by use of Eq. (5).

At most angles, $N^{\text{all}}_{\text{eff}}$ and $N_{\text{eff}}$ are rather close together while $N_{\text{eff}}$ and $N^{J\pi}_{\text{eff}}$ differ widely. This result is expected from Eq. (5), since our calculation of $N_{\text{eff}}$ implied the value $\Lambda = 27$ because of Eq. (8), while the calculation of $N^{J\pi}_{\text{eff}}$ was done with $\Lambda = 7.2$ deduced[1] from Eqs. (6) and (7). The differences between $N^{\text{all}}_{\text{eff}}$ and $N_{\text{eff}}$ indicate that even in the Hauser-Feshbach-Ericson model the basic cross sections $\sigma_\mu$ are not statistically independent. The case[5] of $^{31}\text{P}(d, \alpha_0)^{29}\text{Si}$ described in Fig. 2 is similar to that shown in Fig. 1.

In addition, Gibbs's model was very recently discussed in fluctuation studies with $(d, \alpha)$ reactions on nuclei in the mass range $A = 24-41$ and was found to give satisfactory agreement with experiment only if one invokes very large direct-interaction contributions to the cross section.[12] This turned out to be inconsistent with results from straightforward Hauser-Feshbach analyses.

The method of calculating cross sections of the type $\sigma_\mu$ or $\sigma^{J\pi}_\mu$ in Refs. 3–5 differs from that used in our calculations. These cross sections contain the sum over the decay widths of the compound nucleus with respect to all open channels, as indicated in Eq. (4) of Ref. 3. In the analysis[3] of $^{27}\text{Al}(\alpha, p)$, this was performed by calculating the capture cross sections for 42 exit channels explicitly; in Ref. 5, as many as 3040 channels were considered. In the formalism of Ref. 11, this procedure is simplified by the introduction of a level-density formula. It turns out[13] that only the spin-distribution parameter $\sigma_{\text{res}}^2$ of the residual nucleus, to which the compound nucleus mainly decays, enters into the final result for $N_{\text{eff}}$. The quantity $\sigma_{\text{res}}^2$ was taken from the work of Bormann *et al.*[14] This result is even independent of $\sigma_{\text{res}}^2$ whenever

$$4\sigma_{\text{res}}^2 >> J_{\text{max}} , \tag{20}$$

where $J_{\text{max}}$ is the maximum angular momentum produced with a sizable probability in the compound nucleus. The condition (20) is often fulfilled. Indeed, we have verified[14] that for the range 4.75

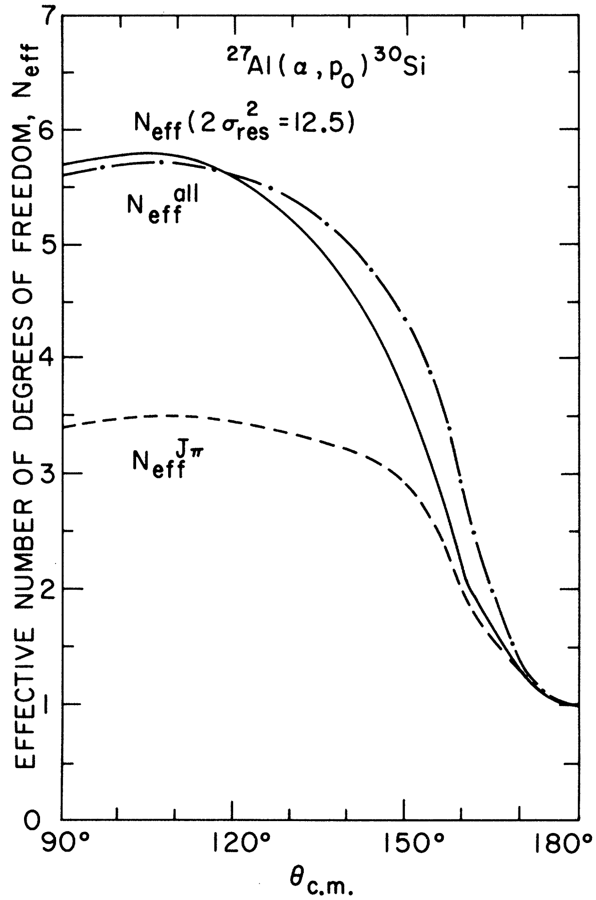


FIG. 1. Effective number of degrees of freedom calculated from different approaches for the reaction $^{27}\text{Al}(\alpha, p_0)^{30}\text{Si}$ at $E_\alpha = 6.3$ MeV (Ref. 3). The dashed curves $N^{\text{all}}_{\text{eff}}(\theta)$ and $N^{J\pi}_{\text{eff}}(\theta)$ are taken from Ref. 1. The full curve $N_{\text{eff}}(\theta)$ has been calculated by use of formula (12) of Ref. 10 with $2\sigma_{\text{res}}^2 = 12.5$.

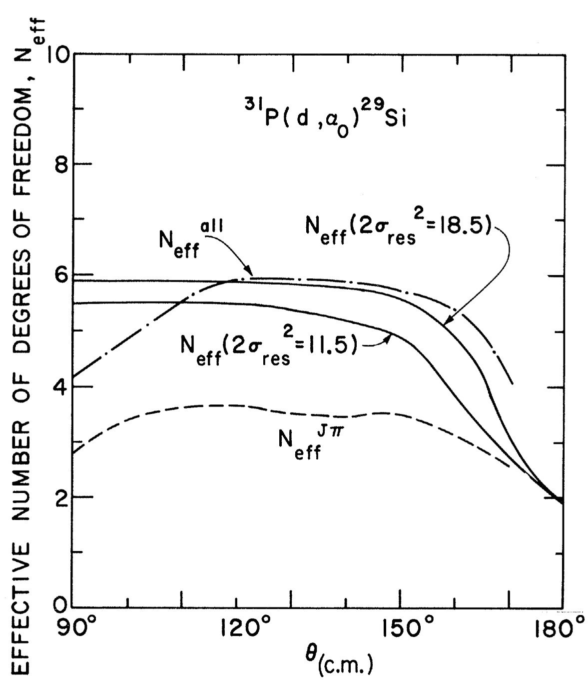


FIG. 2. Effective number of degrees of freedom for $^{31}P(d, \alpha_0)^{29}Si$ at $E_\alpha = 9.5$ MeV. The dashed curves $N_{eff}^{all}(\theta)$ and $N_{eff}^{J\pi}(\theta)$ are taken from Ref. 5. The full curves $N_{eff}(\theta)$ are calculated according to Ref. 10 with $2\sigma_{res}^2 = 11.5$ and $2\sigma_{res}^2 = 18.5$.

$\le \sigma_{res}^2 \le 7.75$, the quantity $N_{eff}(\theta)$ for the reaction $^{27}Al(\alpha, p_0)$ is practically independent of $\sigma_{res}^2$. For the case[5] of $^{31}P(d, \alpha_0)^{29}Si$ at $E_d = 9.5$ MeV, the function $N_{eff}(\theta)$ is as given in Fig. 2 for the parameters $2\sigma_{res}^2 = 11.5$ and $2\sigma_{res}^2 = 18.5$. The difference between these curves is again small, as in the case of $^{27}Al(\alpha, p)$, and hence $N_{eff}(\theta)$ may be calculated fairly well without computing the decay widths of several thousand exit channels explicitly. Since the quantum numbers of most of the exit channels are not known anyhow, the explicit summations carried out in Refs. 3–5 suffer from the same ambiguity as the method of Ref. 11, which thus is as accurate and very much simpler.

We notice that these results are rather insensitive to the details of the transmission coefficients. In Figs. 1 and 2 we have used the optical-model transmission coefficients from Refs. 3 and 5, respectively. Figure 3, however, illustrates that one can as well use "sharp cutoff" transmission coefficients. The resulting differences are in general unimportant compared to the finite range of data errors of the experimental results.

## IV. CONCLUSION

The methods proposed by Gibbs[1] and by Bondorf and Leachman[2] to calculate the number $N_{eff}$ of degrees of freedom in fluctuation analysis have been

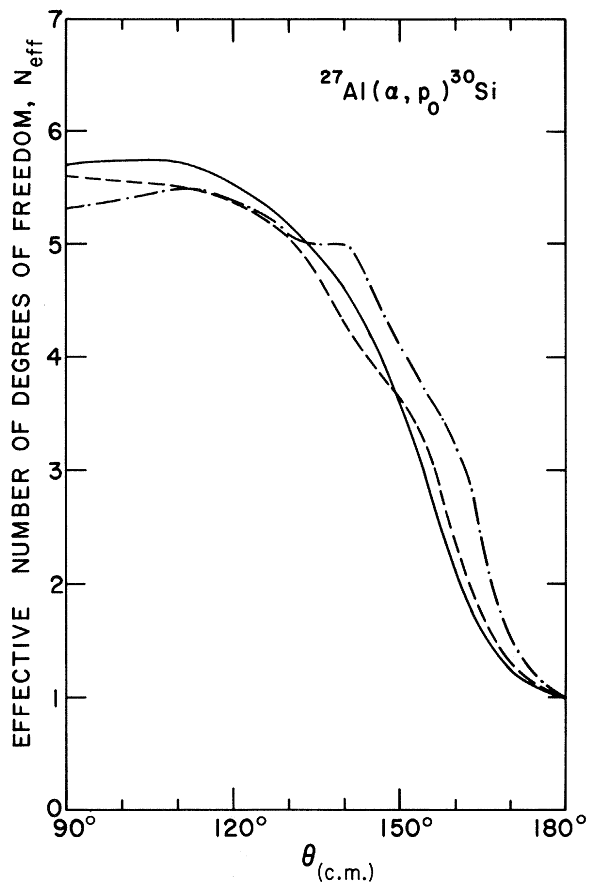


FIG. 3. Dependence of $N_{eff}(\theta)$ for $^{27}Al(\alpha, p_0)^{30}Si$ on the transmission coefficients $T_l$. The spin-distribution parameter is always such that $2\sigma_{res}^2 = 12.5$. The full curve is calculated with the $T_l$ coefficients given in Ref. 3. It is identical to the full curve of Fig. 1. The dashed curve is calculated with $T_l = 1$ if $l \le 5$, $T_l = 0$ if $l > 5$ for the entrance channel $^{27}Al + \alpha$, and $T_{l'} = 1$ if $l' \le 3$, $T_{l'} = 0$ if $l' > 3$ for the exit channel $^{30}Si + p$. The dash-dot curve was obtained with $T_l = 1$ if $l \le 7$, $T_l = 0$ if $l > 7$ for the entrance channel, and $T_{l'} = 1$ if $l' \le 4$, $T_{l'} = 0$ if $l' > 4$ for the exit channel.

compared with the method of von Witsch *et al.*[11] The results from Refs. 2 and 11 – both of which are based on the Hauser-Feshbach-Ericson model – are rather similar. The method of Ref. 1 leads to smaller values of $N_{eff}$. It was introduced to account for experimental results[3] which indicate a failure of the Ericson model. We have shown that the model of Ref. 1 does not provide a mathematically satisfying method to calculate $N_{eff}$ numerically.

In the Hauser-Feshbach expression for the energy-averaged statistical-model cross section, the sum over the capture cross sections from all open channels is required. This quantity, in general poorly known, can be obtained only at the expense of extensive computations. We have pointed out that according to Ref. 11 these sums can be avoided in the calculation of $N_{eff}$. Thus the procedure is considerably simpler than that of Refs. 3–5.

ACKNOWLEDGMENTS

We thank Dr. N. Rosenzweig for critical comments concerning the manuscript and Dr. M. Böhning and Professor H. A. Weidenmüller for helpful discussions.

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

†Present address: Institut de Physique Nucléaire, 91-Orsay, France.

‡On leave of absence from the Max-Planck Institut für Kernphysik, Heidelberg, Germany.

[1]W. R. Gibbs, Phys. Rev. 153, 1206 (1967).

[2]J. Bondorf and R. B. Leachman, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 34, No. 10 (1965).

[3]G. Dearnaley, W. R. Gibbs, R. B. Leachman, and P. C. Rogers, Phys. Rev. 139, B1170 (1965).

[4]G. G. Seaman, R. B. Leachman, and G. Dearnaley, Phys. Rev. 153, 1194 (1967).

[5]G. A Lock, J. R. Curry, P. J. Riley, and C. G. Shugart, Phys. Rev. 176, 1293 (1968).

[6]T. Ericson, Ann. Phys. (N.Y.) 23, 390 (1963).

[7]D. M. Brink, R. O. Stephen, and N. W. Tanner, Nucl. Phys. 54, 577 (1964).

[8]T. Ericson and T. Mayer-Kuckuk, Ann. Rev. Nucl. Sci. 16, 183 (1966).

[9]W. Hauser and H. Feshbach, Phys. Rev. 81, 366 (1952).

[10]Equation (12) of Ref. 11 has to be modified (Ref. 12) by replacing the spin-cutoff parameter $\sigma^2$ by ${\sigma_{res}}^2$, the spin cutoff for the residual nuclei predominantly formed by the decay of the compound nucleus.

[11]W. von Witsch, P. von Brentano, T. Mayer-Kuckuk, and A. Richter, Nucl. Phys. 80, 394 (1966).

[12]K. A. Eberhard, M. Böhning, P. von Brentano, and R. O. Stephen, Nucl. Phys. A125, 673 (1969).

[13]A. Bottega, W. R. Murray, and W. J. Naudé, Nucl. Phys. A136, 265 (1969).

[14]M. Bormann, F. Dreyer, V. Seebeck, and W. Voigt, Z. Naturforsch. 21a, 988 (1966).

---

PHYSICAL REVIEW C VOLUME 2, NUMBER 2 AUGUST 1970

# Assignment of $J^\pi = \frac{3}{2}^-$ for the 8.11-MeV Level of $^{11}C$†

H. T. Fortune,* J. R. Comfort, J. V. Maher, and B. Zeidman
*Argonne National Laboratory, Argonne, Illinois 60439*

(Received 16 February 1970)

A study of the $^{10}B(^3He,d)^{11}C$ and $^{12}C(^3He,\alpha)^{11}C$ reactions leading to the 8.11-MeV state in $^{11}C$ yields an unambiguous assignment of $J^\pi = \frac{3}{2}^-$ for that state.

## I. INTRODUCTION

The spin and parity of the 8.11-MeV level of $^{11}C$ (and its probable mirror at 8.57 MeV in $^{11}B$) have long been assigned[1] as $J^\pi \leq \frac{5}{2}^-$. The principal evidence has been derived from an internal-pair-correlation study in $^{11}B$ by Olness *et al.*[2] This study concluded that the 8.57-MeV → g.s. transition has multipolarity $M1+E2$, which led to an assignment of $J \leq \frac{7}{2}$ and negative parity for the 8.57-MeV state. (The spin $\frac{7}{2}$ is allowed because the extracted $M1$-$E2$ mixture allows pure $E2$.) The lifetime for decay to the $\frac{1}{2}^-$ state at 2.12 MeV allows the elimination of the $\frac{7}{2}^-$ possibility; hence $J^\pi \leq \frac{5}{2}^-$. The negative-parity assignments are also consistent with angular distributions from the $^9Be(^3He,p)^{11}B$(8.57 MeV) reaction[3] and the $^9Be(^3He,n)^{11}C$(8.11 MeV) reaction[4] at energies below 10 MeV, where an $l=0$ component was evident. In addition, the $q^2$ dependence of the $^{11}B(e,e')^{11}B$ transition probability[5] implies an $E2$-$M1$ mixture and thus an assignment $J^\pi \leq \frac{5}{2}^-$.

The results of previous studies of single-nucleon stripping and pick-up reactions to this state have been ambiguous. A plane-wave analysis[6] of $^{10}B(d,p)^{11}B$(8.57 MeV) data at $E_d = 10$ MeV gave a tentative $l=2$ assignment to the stripping pattern. Almost as good a fit, however, is obtained for $l=1$ in the plane-wave analysis; and $l=1$ is definitely favored in a later distorted-wave analysis.

Previous publications[7] on the $^{10}B(^3He,d)^{11}C$ reaction have reported an excitation energy for the 8.11-MeV state but no angular distributions. In the $^{12}C(^3He,\alpha)^{11}C$ reaction, angular distributions have