Extension of Two-Nucleon Transfer Theory to Include Inelastic Processes*

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The theory of two-nucleon transfer reactions is extended to include higher-order transitions going through intermediate nuclear states produced by inelastic scattering of the ingoing or outgoing particle. The reaction, as is usual, is treated to first order, but the inelastic processes are treated to all orders among the retained channels. The theory is formulated so that it is very easily applied to any microscopic structure calculation, since the relevant content of such calculations appear in the theory only through the values of the matrix elements of the pair-creation $[a^\dagger a^\dagger]_J$ and scattering $[a^\dagger \tilde{a}]_J$ operators.

1. INTRODUCTION

The existent treatments of the two-nucleon transfer reaction all neglect the effect of inelastic processes. There are two circumstances where this neglect is not justified. The first concerns a question of parentage. If in the reaction

$$t+(A)\rightarrow p+(A+2),$$

the configuration of the group of (A) nucleons is the same in the residual state of interest as in the target ground state, then the usual treatment *may* be valid. However, if the state of motion of any of the core nucleons is different, then a description of the inelastic processes that produced this difference becomes essential, if in fact the state is excited in the reaction.

The second circumstance when inelastic processes are crucial for a correct description arises when some inelastic transitions in either the target or residual system, or both, are so strong as to produce significant deexcitation back into the elastic channel. In this circumstance the usual one-channel optical potential will not provide a good description of the relative motion in the vicinity of the nucleus, just where it is important for the description of the transfer reaction.

In this paper we present the formalism by which inelastic processes can be incorporated into the theory of two-nucleon transfer reactions. The method that we use was described and justified in an earlier publication,[1] and may be referred to briefly as the source-term method. It is in a source term, which appears as an inhomogeneity in the coupled differential equations describing inelastic scattering, that the nature of the transfer reaction is specified. We treat the reaction under the usual assumptions made in two-nucleon transfer theory.[2] The theory is formulated in such a way that the particular structure of the nuclear states enters the description of the reaction *only* through certain generalized two-particle coefficients of fractional parentage. *As an example* these are calculated for a particular nuclear model, namely one in which the ground state is the BCS vacuum, and which has excited two-quasiparticle states as well as a "two-phonon" triplet which is built from the operator which describes the collective 2_1^+ state.

For definiteness our notation refers to the (p,t) reaction, but the results are general, and we indicate at appropriate places how to interpret the formalism for other two-nucleon transfer reactions.

2. COUPLED EQUATIONS

Our method has been discussed in detail for (d,p) reactions in an earlier publication, and the notation which we use here for the (p,t) reaction will be analogous.[1] The method consists of writing down the coupled equations which describe the inelastic scattering of the projectile by the target, and a second system of coupled equations which would describe the scattering in the residual partition, in this case the triton and final nucleus, except that the second system is coupled to the first by a source term which describes the transfer process. Thus the inelastic processes are carried to all orders among the retained channels, but the asymmetric way that the source appears means that the reaction itself is treated as a weak process only in first order. This corresponds to the fact that (p,t) cross sections are typically an order of magnitude smaller than the strongest inelastic ones.

We will describe the scattering in the initial partition of the system consisting of proton plus nucleus $(A+2)$ by a system of N equations which includes the strongly coupled channels and any others of interest: For each parity π and total angular momentum I of the system, they are

$$(T_{p'}+U_{p'}(r)-E_{p'})u_{p'}^{p\pi I}(r)+\sum_{p''\neq p'}V_{p'p''}^{\pi I}(r)u_{p''}^{p\pi I}(r)=0. \tag{2.1}$$

The rationale for use of such a system for a microscopic description of inelastic scattering has been discussed in detail elsewhere.[3] Here U stands for a complex optical potential, which is a parametrization of the diagonal matrix elements of the effective interaction which enters the problem because we intend to solve it in a highly truncated space of nuclear states, and V is the direct interaction between the nuclear nucleons and the exterior particle. Matrix elements are taken with respect to the channel wave functions

$$\phi_{p\pi I}^{M}(\hat{r}, A+2) = [\mathcal{Y}_{l_p s_p j_p}(\hat{r}, \sigma)\Phi_{\alpha_p J_p}(A+2)]_I^M , \quad (2.2)$$

where Φ is a nuclear wave function, $\mathcal{Y}$ is a spin-orbit function for the scattered proton,[1] and the square bracket denotes vector coupling. We use p to denote the whole collection of quantum numbers defining a proton channel

$$p \equiv l_p s_p j_p \alpha_p J_p . \quad (2.3)$$

When *both* p and p' appear in an equation, p is understood to refer to an entrance channel, and p' to any channel. The superscript p on u in Eq. (2.1) signifies that the system is subject to the boundary condition that only channels p, in which the nucleus is in its ground state, have incoming waves.

$$u_{p'}^{p\pi I}(r) \to \delta_{pp'} I_{l_p}(k_p r) - (v_p/v_{p'})^{1/2} S_{p',p}^{\pi I} O_{l_{p'}}(k_{p'} r) . \quad (2.4)$$

Here O and I are outgoing and incoming spherical waves[1] and k and v are wave number and velocity, respectively.

The equations describing the final partition of the system are

$$[T_{t'} + U_t(R) - E_{t'}] w_{t'}^{p\pi I}(R) + \sum_{t'' \neq t'} V_{t',t''}^{\pi I}(R) w_{t''}^{p\pi I}(R) = \sum_{p'} \rho_{p',t'}^{p\pi I}(R) , \quad (2.5)$$

which are to be solved subject to the condition that there are only outgoing waves in triton channels:

$$w_t^{p\pi I}(R) \to -(v_p/v_t)^{1/2} S_{t,p}^{\pi I} O_{l_t}(k_t R) . \quad (2.6)$$

These equations differ in structure from the preceding ones only in the addition of the source term ρ which represents the appearance of tritons in the channel t' due to the transfer process in the various channels p' of the initial partition. The rest of the paper is essentially devoted to the description of how this source is constructed, given the detailed microscopic structure of the states in the target and residual nuclei.

As we remarked, the asymmetric way in which the source appears in our equations means that the reaction is treated in first order only. One could consider introducing a source for the inverse reaction into (2.1), but if in fact the two-nucleon transfer reaction is not strong, this would be imprudent, as the additional numerical work involved in solving the system of equations with the additional coupling is enormous.

3. SOURCE TERM

The source for tritons in the channel t' is equal to the sum of matrix elements describing the transfer processes leading to it from the various channels p',

$$\rho_{t'}^{p\pi I}(R) = \sum_{p'} \rho_{p',t'}^{p\pi I}(R) , \equiv R \sum_{p'} \left\langle \varphi_{t',\pi I}^{M}(\hat{R}, A) \varphi_0 \middle| v \middle| \phi_{p',\pi I}^{M}(\hat{r}, A+2) \frac{u_{p'}^{p\pi I}(r)}{r} \right\rangle . \quad (3.1)$$

All coordinates in the matrix element are integrated except R. Here v represents the stripping interaction. In Ref. 1 we showed that this form for the source leads precisely to the usual distorted-wave Born approximation (DWBA) in the event that in our equations, the off-diagonal matrix elements of V describing the inelastic processes are dropped, and it leads to the result of Penny and Satchler[4] when they are kept.

We will evaluate the matrix elements appearing in the source term under the assumptions usually made in two-nucleon transfer theory.[2] Thus the interaction v is taken to be a δ function in the coordinate $\vec{\rho}$ describing the displacement of the proton from the center of mass of the two neutrons

$$v = v_{p,n_1} + v_{p,n_2} \simeq g\delta(\rho) . \quad (3.2)$$

We assume that the relative motion between pairs in the triton is S state and that the radial part is Gaussian in form. Then the triton radial function ϕ_0, separates in the coordinates ρ and the relative coordinate $\vec{r}$ between the neutrons, so that[2]

$$\varphi_0 = \phi_{10}(3\eta^2 r^2)\phi_{10}(4\eta^2\rho^2) . \quad (3.3)$$

Here ϕ_{10} is the oscillator function and η is the triton size parameter defined in Ref. 2.

Employing the assumptions leading to (3.2) and (3.3), the evaluation of (3.1) proceeds along the lines used in Ref. 1. The result can be written in the form[5]

$$\rho_{p,t}^{\pi I}(R) = \frac{A+2}{A} D_0 \sum_{LSJ} A_{LSJ}(p,t) \tilde{u}_{LSJ}^{\alpha_p \alpha_t}(R) u_p^{\pi I}\left(\frac{A}{A+2}R\right) . \quad (3.4)$$

This is written more generally than required for the (p,t) reaction (where only $S=0$ transfer is al-

lowed) so that we can cover other two-nucleon transfer reactions. Here L, S, J, denote the orbital, intrinsic, and total angular momentum carried by the transferred pair. The quantity A_{LSJ} is a geometrical factor defined by

$$A_{LSJ}(p,t)=(-)^{l_t+j_p+J_p+I}\left(\frac{\hat{j}_t\hat{l}_p\hat{J}_p\hat{L}}{4\pi}\right)^{1/2}\begin{pmatrix} l_p & L & l_t\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} j_p & J & j_t\\ J_t & I & J_p\end{Bmatrix}\begin{bmatrix} l_p & s_p & j_p\\ L & S & J\\ l_t & s_t & j_t\end{bmatrix}\Lambda_S(p,t), \tag{3.5}$$

where $\hat{l}=2l+1$. This factor is general for any particle transfer reaction when p is interpreted to refer to the lighter and t to the heavier of the light nuclides.[6] D_0 is a constant which collects together two normalization factors,

$$D_0=g[\phi_{10}(4\eta^2\rho^2)]_{\rho=0}=g\left(\frac{4\eta^2}{\pi}\right)^{3/4}, \tag{3.6}$$

Λ_S is the overlap matrix element for spins

$$\Lambda_S=\langle\chi^{m_t}_{s_t}(\vec{\sigma}_p\vec{\sigma}_1\vec{\sigma}_2)|[\chi_{s_p}(\vec{\sigma}_p)\chi_S(\vec{\sigma}_1\vec{\sigma}_2)]^{m_t}_{s_t}\rangle. \tag{3.7}$$

Here χ_{s_t} and χ_{s_p} are the spin functions of the triton and the proton and

$$\chi_S(\vec{\sigma}_1\vec{\sigma}_2)=[\chi_{1/2}(\vec{\sigma}_1)\chi_{1/2}(\vec{\sigma}_2)]_S \tag{3.8}$$

is the singlet or triplet spin function of the two transferred particles. For other reactions, see the Appendix. [For the reaction (d,α) an additional overlap occurs, Ω_d, defined in the Appendix of Ref. 2. An analogous overlap in the internal motion of the light nuclides should be inserted whenever neither the incoming, outgoing, or transferred object is a single nucleon.]

The function $\tilde{u}^{\alpha_p\alpha_t}_{LSJ}(R)$ in (3.4) is a projected wave function which defines the radial motion of the center of mass of the transferred pair in the state α_p of the nucleus $(A+2)$ *when* their correlation corresponds to that in the triton, given that the remaining nucleons are in the state α_t of nucleus (A).

To see how it can be evaluated in terms of known quantities, consider first an example where (A) is in its ground state having zero spin, and $(A+2)$ is in a state having a pure parentage based on (A), with two neutrons in a pure shell-model state $\psi_{(a,b)J}$ (where $a\equiv n_a l_a j_a$).

According to its meaning, the projected wave function is defined by the equation

$$\sum_L\tilde{u}^{ab}_{LSJ}(R)Y^0_L(\hat{R})=\langle\varphi_{10}(3\eta^2r^2)\chi^0_S(\vec{\sigma}_1,\vec{\sigma}_2)|\psi^0_{abJ}(\vec{r}_1,\vec{r}_2)\rangle. \tag{3.9}$$

If we use oscillator radial functions for the shell-model states in ψ_{abJ}, the right side can be evaluated as in Ref. 2 to yield

$$\tilde{u}^{ab}_{LSJ}(R)=\sum_N G^{ab}_{NLSJ}u_{NL}(2\nu R^2), \tag{3.10}$$

where u_{NL} is an oscillator radial function as defined in the Appendix of Ref. 2. (The incorrect asymptotic behavior of these functions may be corrected in one of several ways suggested in that Appendix.) The structure amplitudes G are defined in terms of known quantities[2] and have been tabulated.[7]

In the general case where the spin of (A) is not zero and the parentage of $(A+2)$ is not pure, the particles may be transferred in various ways consistent with the structure of the states. Nevertheless, we can write the general result in terms of (3.10) as

$$\tilde{u}^{\alpha_p\alpha_t}_{LSJ}(R)=\sum_{a\le b}\beta_{ab}(\alpha_p,\alpha_t)\tilde{u}^{ab}_{LSJ}(R), \tag{3.11}$$

where $\beta(\alpha_p,\alpha_t)$ are generalized two-particle parentage factors.[2]

In this way of formulating the problem, the structure of the nuclei (A) and $(A+2)$ enters the description of the reaction *only* through the values of the parentage coefficients $\beta(\alpha_p,\alpha_t)$. For a complicated structure calculation, these would depend upon intimate details of the calculation. However, for a simple model they can be exhibited explicitly, as we do below.

We note as a matter of computational convenience, that if the bound-state single-particle wave functions are expressed on a basis of harmonic-oscillator functions ($e^{-\nu r^2/2}$), then the source term connecting any two channels can be written in the form

$$\rho^{\pi I}_{p,t}(R)=e^{-\nu R^2}P^{\pi I}_{p,t}(R^2)u^{\pi I}_p\left(\frac{A}{A+2}R\right), \tag{3.12}$$

where P is a finite polynomial in R^2. Therefore it can be stored in a computer according to the coefficients of the polynomial rather than as a table in R.

The amplitude for outgoing tritons can be written in terms of the S matrix elements obtained by imposing the boundary conditions (2.6) on the solutions to (2.5). We must, as in Ref. 1, take account of the normalization (2.4) used for the proton radial functions in the source term. The result can be written

$$f(J_p M_p \mu_p \to J_t M_t \mu_t) = \frac{1}{2ik_p} \sum_{l_p j_p l_t j_t I} [4\pi(2l_p+1)]^{1/2} i^{l_p - l_t} e^{i(\sigma_p+\sigma_t)} C(l_p s_p j_p; 0\,\mu_p \mu_p) C(j_p J_p I; \mu_p M_p, M_p+\mu_p)$$

$$\times \sum_{m_t, m} C(l_t s_t j_t; m_t \mu_t m) C(j_t J_t I; m M_t, M_p+\mu_p) S^{\pi I}_{t,p} Y^{m_t}_{l_t}(\hat{k}_t) . \tag{3.13}$$

The cross section for the p, t reaction is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2(2J_p+1)} \sum_{M_p \mu_p M_t \mu_t} |f|^2 . \tag{3.14}$$

4. INVERSE REACTION

By using the inverse property of the amplitudes, it is possible to compute the inverse reaction. However, of the elements of S obtained in Sec. 2, the only interesting ones for the inverse are the ground-to-ground elements. The others can be obtained by solving a series of boundary-condition equations in which the incident wave is inserted in the various p channels. However, if one is primarily interested in the (t, p) reaction, it is more convenient to set up the procedure of Sec. 2 explicitly for this reaction. The (t, p) reaction can be treated in an analogous way. In this case the homogenous system corresponding to (2.5) is solved for the scattering. These solutions are used to construct the proton sources which now are inserted as an inhomogeneity into (2.1). The source term is otherwise constructed from the *same* ingredients as the triton source,

$$\rho^{\pi I}_{t,p}(r) = \frac{A}{A+2} D_0 \sum_{LSJ} A_{LSJ}(p, t) \times \tilde{u}^{\alpha_p \alpha_t}_{LSJ}\left(\frac{A+2}{A} r\right) u^{\pi I}_t\left(\frac{A+2}{A} r\right), \tag{4.1}$$

where A_{LSJ} and $\tilde{u}$ are as defined previously for the (p, t) reaction. The amplitude for outgoing protons can be obtained from (3.13) by interchanging p and t.

5. TWO-NUCLEON PARENTAGE COEFFICIENTS

The parentage coefficient given in Ref. 2 (there in L-S coupling) may be expressed in second-quantization notation, for equivalent particles, as

$$\beta_{abJ}(\alpha_p, \alpha_t) \equiv [\hat{J}_p(1+\delta_{ab})]^{-1/2} \langle \Phi_{\alpha_p J_p}(A+2) \| [d^\dagger_a d^\dagger_b]_J \| \Phi_{\alpha_t J_t}(A) \rangle . \tag{5.1}$$

Here $d^\dagger_{am_a}$ creates a particle in the state $n_a l_a j_a m_a$. The definition adopted for reduced matrix elements is that of Racah.[8] However for transfer of a neutron and proton pair, when the isospin formalism is *not* used, the δ_{ab} factor should be omitted and $d^\dagger_a$ and $d^\dagger_b$ commute.

If the structure of the nuclear states is defined in terms of quasiparticles $\alpha^\dagger$ related to particles through,[9]

$$d^\dagger_{am} = U_a \alpha^\dagger_{am} + V_a \tilde{\alpha}_{am} , \qquad \tilde{\alpha}_{am} = (-)^{j_a - m} \alpha_{a,-m} , \tag{5.2}$$

where U and V are coefficients of the Bogolyubov-Valatin transformation, then

$$[d^\dagger_a d^\dagger_b]^M_J = -U_a U_b A^\dagger_{JM}(a, b) + (-)^{J+M} V_a V_b A_{J-M}(a, b) - U_a V_b N^\dagger_{JM}(a, b) - (-)^{J+M} V_a U_b N_{J-M}(a, b) + V_a U_b (\hat{j}_a)^{1/2} \delta_{ab} \delta_{J0} , \tag{5.3}$$

where

$$A^\dagger_{JM}(a, b) = -[\alpha^\dagger_a \alpha^\dagger_b]^M_J , \qquad N^\dagger_{JM}(a, b) = (-)^{j_a - j_b + M} N_{J-M}(b, a) = -[\alpha^\dagger_a \tilde{\alpha}_b]^M_J . \tag{5.4}$$

A. Two-Quasiparticle States

As an example of the calculation of the parentage coefficients, we consider a simple model. If the ground state is described as the BCS vacuum, and excited states as two-quasiparticle states, then four types of parentage coefficients enter the problem, illustrated by Fig. 1. We write the two-quasiparticle wave functions for nucleus (A) as

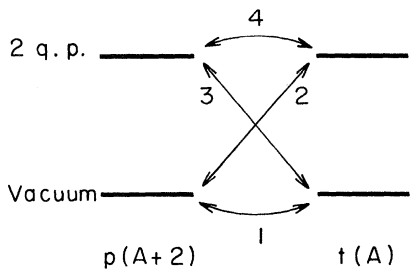

FIG. 1. Four types of parentage coefficients occur in a nucleus with the vacuum ground state and two-quasiparticle excited states.

$$|\alpha(A)JM\rangle = \tfrac{1}{2}\sum_{a,b}\eta_{ab}^{\alpha J}(A)A_{JM}^{\dagger}(a,b)|\omega_A\rangle, \qquad (5.5)$$

where the sum on a, b is *not* ordered (i.e., both a, b and b, a occur).[10] Here $|\omega_A\rangle$ is the BCS vacuum, and similarly for nucleus $(A+2)$. Then the parentage coefficients for the four types of transitions numbered in Fig. 1 are[11]:

$$\beta_{abJ}(1) = (\tfrac{1}{2}\hat{j}_a)^{1/2}V_a(A+2)U_b(A)\delta_{ab}\delta_{J0}, \qquad (5.6)$$

$$\beta_{abJ}(2) = \left(\frac{\hat{J}}{1+\delta_{ab}}\right)^{1/2}\eta_{ab}^{\alpha_t J_t}(A)V_a(A+2)V_b(A+2)\delta_{JJ_t}, \qquad (5.7)$$

$$\beta_{abJ}(3) = -\left(\frac{1}{(1+\delta_{ab})}\right)^{1/2}\eta_{ab}^{\alpha_p J_p}(A+2)U_a(A)U_b(A)\delta_{JJ_p}, \qquad (5.8)$$

$$\beta_{abJ}(4) = (\hat{j}_a/2)^{1/2}\Theta(\alpha_p,\alpha_t)V_a(A+2)U_b(A)\delta_{ab}\delta_{J0}\delta_{J_pJ_t} + \left(\frac{\hat{J}\hat{J}_t}{1+\delta_{ab}}\right)^{1/2}(-)^{J_t+J-J_p}[U_a(A)V_b(A+2)X_{ab}(\alpha_p,\alpha_t)$$
$$+(-)^{j_a-j_b+J}U_b(A)V_a(A+2)X_{ba}(\alpha_p,\alpha_t)], \qquad (5.9)$$

where

$$\Theta(\alpha_p,\alpha_t) = \tfrac{1}{2}\sum_{cd}\eta_{cd}^{\alpha_p J_p}(A+2)\eta_{cd}^{\alpha_t J_t}(A), \qquad (5.10)$$

$$X_{ab}(\alpha_p,\alpha_t) = \sum_{d}\eta_{da}^{\alpha_p J_p}(A+2)\eta_{bd}^{\alpha_t J_t}(A)\begin{Bmatrix} J_t & J & J_p \\ j_a & j_d & j_b \end{Bmatrix}. \qquad (5.11)$$

Note that Θ represents the overlap of a two-quasiparticle state in (A) with one in $(A+2)$. For the lowest collective 2_1^+ states in adjacent nuclei, this overlap will usually be close to unity.

B. Two-Phonon States

In analogy with the vibrational model, we may use the collective operator

$$Q_{\alpha 2M}^{\dagger}(A) \equiv \tfrac{1}{2}\sum_{a,b}\eta_{ab}^{\alpha}(A)A_{2M}^{\dagger}(a,b) \qquad (5.12)$$

corresponding to the lowest 2_1^+ state to generate a triplet of "two-phonon" states

$$|\alpha_p(A)J_pM_p\rangle = \frac{1}{\sqrt{2}}[Q_2^{\dagger}(A)Q_2^{\dagger}(A)]_{J_p}^{M_p}|\omega_A\rangle, \qquad (5.13)$$

and similarly for the nucleus $(A+2)$. The quasiboson commutation relations are approximately [assuming identical BCS vacuum for (A) and $(A+2)$],

$$[Q_{\alpha_p J_p M_p}(A+2), Q_{\alpha_t J_t M_t}^{\dagger}(A)] \cong \Theta(\alpha_p,\alpha_t)\delta_{J_pJ_t}\delta_{M_pM_t}. \qquad (5.14)$$

The two-particle operator $[d^{\dagger}d^{\dagger}]$ does not connect the vacuum ground state to the two-phonon states. The additional parentage factors needed are illustrated in Fig. 2. They are computed in the quasiboson approximation to be

$$\beta_{abJ}(5) = [(2\hat{J}_t)^{1/2}/\hat{J}]\Theta\beta_{\mathrm{col}}(2)\delta_{J2}\delta_{J_p2}, \qquad (5.15)$$

$$\beta_{abJ}(6) = \sqrt{2}\,\Theta\beta_{\mathrm{col}}(3)\delta_{J2}\delta_{J_t2}, \qquad (5.16)$$

$$\beta_{abJ}(7) = \Theta_{\mathrm{col}}{}^2\beta(1)\delta_{J_pJ_t}, \qquad (5.17)$$

where the subscripts "col" denote that these quantities involve the amplitudes η of the collective 2_1^+ states in (A) and $(A+2)$. In (5.15) and (5.16), Θ is the overlap between the two-quasiparticle state and the *collective* 2_1^+ state from which the two-photon states are built, whereas in (5.17) it is the overlap between the collective 2_1^+ states in the two nuclei. Therefore the coupling is weak except with another two-phonon state or with the collective 2_1^+ state, in which cases $\Theta \simeq 1$.

7. COMMENT ON INELASTIC SCATTERING

Beside the source terms, the structure of the nuclear states enters the evaluation of the matrix elements of V in (2.1) and (2.5). The point at which a knowledge of nuclear structure enters can be reduced finally, as was shown in Ref. 3, to the evaluation of a certain set of amplitudes which express the nuclear form factors in terms of those for single-particle transitions. These amplitudes (see Sec. 3 of Ref. 3) can be written as

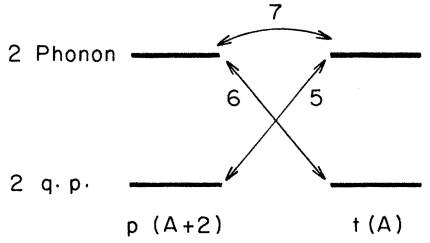

FIG. 2. Microscopic two-phonon states have parentage coefficients connecting them to other two-quasiparticle states or to each other.

$$S_J^{ab}(\alpha_1\alpha_2) = -(\hat{J})^{-1/2}\langle \alpha_1 J_1 \| [d_a^\dagger \tilde{d}_b]_J \| \alpha_2 J_2\rangle ,$$

that is to say, they are reduced matrix elements of the scattering operators $[d_a^\dagger \tilde{d}_b]_J$ for the various shell-model states a, b that enter the nuclear description. The evaluation of these amplitudes for the same nuclear model described in Sec. 5 A appears in Ref. 3.

8. SUMMARY

A theory of two-nuclear transfer reactions which includes the effects of inelastic processes has been formulated. In addition to the direct production of the final state from the target ground state, these processes allow for its production through intermediate states produced by the inelastic scattering of the incoming or outgoing particle. The reaction is treated as the weak process it is, only in first order, but the inelastic processes are treated to all orders among the retained channels. The theory can be applied easily to microscopic calculations of nuclear structure, because the entire content of such descriptions can be inserted into our formulation very concisely through reduced matrix elements of two types of operators; namely, the transfer operators $[d_a^\dagger d_b^\dagger]_J$ and the scattering $[d_a^\dagger \tilde{d}_b]_J$, evaluated between the nuclear states for the various configurations a, b entering the structure calculation. This obviously makes all structure calculations readily accessible to use in calculations of transfer and scattering. However, for many-particle shell-model calculations the job requires an intimate knowledge of the conventions employed, and generally can be done best (or only) by the structure theorist himself.

APPENDIX

We consider the spin overlap Λ_S in a more general two-nucleon transfer reaction

$$a + A \rightarrow (a - \nu - \pi) + (A + \nu + \pi) ,$$

where a contains n and p neutrons and protons, and ν and π of them are transfered ($\nu + \pi = 2$).

Although our theory does not contain exchange processes implied by antisymmetrization, we do carry the statistical factors that weight the direct integrals considered.[12]

Thus we define Λ_S to contain those factors corresponding to the light nuclides [as β does for the nuclei (Sec. 5 and Ref. 2)].

$$\Lambda_S = \left[\binom{n}{\nu}\binom{p}{\pi}\right]^{1/2} \times \langle \chi^m_{S_1}(a) | [\chi_{S_2}(a - \nu - \pi)\chi_S(\nu + \pi)]^m_{S_1}\rangle .$$

[Resolve any ambiguity in interpretation by comparisons with (3.7) and (3.8) for the (p, t) reaction.]

For various light particles

$$\begin{aligned} \chi_S(a) &= \chi_0(\vec{\sigma}_{n_1}\vec{\sigma}_{n_2})\chi_0(\vec{\sigma}_{p_1}\vec{\sigma}_{p_2}), && \alpha , \\ &= \chi_{1/2}(\vec{\sigma}_p)\chi_0(\vec{\sigma}_{n_1}\vec{\sigma}_{n_2}), && \text{triton} , \\ &= \chi_1(\vec{\sigma}_n\vec{\sigma}_p), && \text{deuteron} . \end{aligned}$$

Thus we find

$$\begin{aligned} \Lambda_S &= \delta_{S0}, && (p, t) \text{ or } (n, \mathrm{He}^3) , \\ &= \sqrt{3}\,\delta_{S1}, && (\alpha, d) , \\ &= -(1/\sqrt{2})(\delta_{S0} - \sqrt{3}\,\delta_{S1}), && (p, \mathrm{He}^3) , \\ &= (1/\sqrt{2})(\delta_{S0} + \sqrt{3}\,\delta_{S1}), && (n, t) . \end{aligned}$$

However there is some evidence[13] that the strength of the stripping interaction is different for $S = 0$ and 1. This may be taken into account for example for (p, He^3) by rewriting

$$\Lambda_S = -(1/\sqrt{2})(a_0\delta_{S0} - \sqrt{3}\,a_1\delta_{S1}) ,$$

where a_1/a_0 gives the relative strengths.

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[1]R. J. Ascuitto and N. K. Glendenning, Phys. Rev. 181, 1396 (1969).

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[3]N. K. Glendenning, in *Proceedings of the International School of Physics "Enrico Fermi," Course XL, 1967*, edited by M. Jean (Academic Press Inc., New York, 1969), p. 332; and Nucl. Phys. A117, 49 (1968).

[4]S. K. Penny and G. R. Satchler, Nucl. Phys. 53, 145 (1964).

[5]Since the coordinate $\vec{r}$ in (2.1) and (3.1) defines the displacement of proton from nucleus $(A+2)$, and $\vec{R}$ the displacement of the triton from (A), the effect of the δ function in $\vec{\rho}$ sets $\vec{r} = A/(A+2)\vec{R}$.

[6]The square bracket is a recoupling coefficient related to the 9-j symbol, cf. Eq. (3.4) of Ref. 2.

[7]N. K. Glendenning, tables of structure amplitudes for (p,t), (p, He^3), and (α, d) are available as Lawrence Radiation Laboratory Report Nos. UCRL-18268, UCRL-18269, and UCRL-18270, respectively (unpublished).

[8]We use Racah's original definition, cf. A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 75.

[9]Condon-Shortley phases are used with a consequence that $UV(-)^l \geq 0$.

[10]Usually wave functions are tabulated according to amplitudes C appearing in $|\alpha JM\rangle = \sum_{a\leq b} C_{ab}^{\alpha J}(1+\delta_{ab})^{-1/2} \times A_{JM}^{+}(ab)|\omega\rangle$ with $\sum_{a\leq b}|C|^2 = 1$. However, for formal manipulations the unrestricted form (5.5) is often more convenient. The connection between the amplitudes is $\eta_{ab}^{\alpha J} = (-)^{j_a - j_b + J}\eta_{ba}^{\alpha J} = C_{ab}^{\alpha J}(1+\delta_{ab})^{1/2}$.

[11]The first three results were given, aside from the configuration amplitude, by S. Yoshida, Nucl. Phys. 33, 685 (1962).

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Comment on the Number of Degrees of Freedom in Fluctuation Analysis*

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Different methods of calculating the number of degrees of freedom N_{eff} in statistical cross-section fluctuations are compared. The underlying physical assumptions of the methods are discussed critically, and numerical examples for N_{eff} are given.

I. INTRODUCTION

The number of degrees of freedom N_{eff} is a fundamental quantity in analyses of cross-section fluctuations, since its knowledge allows one to estimate the relative values of the direct and compound-nucleus cross sections. In Sec. II we investigate the "basic cross-section method" for the calculation of N_{eff} and the modifications of it due to Gibbs. We show that the lack of statistical independence of the basic cross sections raises difficulties when one solves for N_{eff} within the Gibbs model. In Sec. III, we present calculations of N_{eff} for recently published experiments on $^{27}\text{Al}(\alpha, p)^{30}\text{Si}$ and $^{31}\text{P}(d, \alpha)^{29}\text{Si}$ in order to demonstrate the differences between the approaches discussed in Sec. II.

In addition, it is pointed out that a realistic calculation of the fluctuation damping coefficient does not require the knowledge of the capture cross sections of all open compound-nucleus decay channels.

II. COMPARISON OF DIFFERENT FORMALISMS

In an article on the "Limitation of the Number of Degrees of Freedom in Fluctuation Analysis," Gibbs[1] discusses the applicability of an expression for this number given by Bondorf and Leachman.[2] These authors decompose the differential cross section $\sigma(\theta)$ into a number N of "basic cross sections" σ_μ and write

$$\sigma = \sum_\mu \sigma_\mu , \tag{1}$$

where the μ are the four spin projections of the colliding and outgoing particles. The variance of the fluctuations of each σ_μ is

$$(\langle\sigma_\mu^2\rangle - \langle\sigma_\mu\rangle^2)/\langle\sigma_\mu\rangle^2 = 1 , \tag{2}$$

where the angular brackets represent energy averages. In Ref. 2, the σ_μ are assumed to be statistically independent; and hence the number N_{eff} of degrees of freedom, which is defined as

$$N_{\text{eff}}(\theta) = \langle\sigma(\theta)\rangle^2/[\langle\sigma^2(\theta)\rangle - \langle\sigma(\theta)\rangle^2] , \tag{3}$$

reduces to

$$N_{\text{eff}}^{\text{all}}(\theta) = [\sum_{\mu=1}^{N}\langle\sigma_\mu(\theta)\rangle]^2/[\sum_{\mu=1}^{N}\langle\sigma_\mu(\theta)\rangle^2] . \tag{4}$$

For computational purposes the averaged basic cross sections $\langle\sigma_\mu\rangle$ may be identified with Hauser-Feshbach expressions. Thus the analysis of several fluctuation experiments[3-5] has been based on Eq. (4). However, the σ_μ need not be statistically independent. Indeed, if their number N is larger than the number Λ of statistically independent scattering matrix elements $U_{ij,i'j'}^{J\pi}$ responsible for the reaction cross section, the assumption of independence of the σ_μ must fail. The purpose of the paper by Gibbs[1] is to give an appropriate correction to $N_{\text{eff}}^{\text{all}}$ in this case. From a schematic model, in which all basic cross sections are assumed to be equal and each of them is constructed out of Λ indepen-