Nuclear Surface Properties with a Simple Effective Interaction*

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A simple three-part density-dependent effective interaction is used to calculate several general properties of the semiinfinite nuclear surface. This interaction, which is referred to here as the modified δ interaction (MDI), is quite similar to one introduced by Skyrme. The three parameters of the MDI are fixed by fitting the binding energy and density of infinite nuclear matter with $N=Z$ and also the ground-state energy of ^{16}O, all in first-order perturbation theory.

The nuclear surface properties are calculated in several different ways. First, they are extracted directly from the single-particle wave functions of a one-dimensional static Woods-Saxon potential. The diffuseness and depth of the potential are obtained by minimizing the surface energy. This procedure is called the independent particle model (IPM). The resulting surface thickness of 2.2 fm and surface energy of 19.3 MeV and the approximately Fermi shape of the density distribution are in good agreement with empirical results.

The calculations were then repeated using the Thomas-Fermi approximation to obtain the density. The resultant surface thickness and surface energy, 2.0 fm and 16.0 MeV, are considerably smaller than the IPM results. Furthermore, the calculated density distribution has a longer "shoulder" inside the nucleus and a shorter "tail" outside than the IPM distribution.

1. INTRODUCTION

A. Nuclear Surface

An important aspect of finite nuclei is the nuclear surface. There have been many studies of the nuclear surface, mainly using the Thomas-Fermi approximation (TFA). Bethe[1] has investigated the properties of the nuclear surface, namely, the mass density distribution and the surface energy. He starts with a realistic nucleon-nucleon interaction[2] but then makes various approximations. (Use of the semiinfinite-nuclear-matter model, replacement of the tensor interaction by an effective central interaction,[3] and the replacement of the short-range repulsion by a density-dependent δ interaction.) He then obtains a formulation quite similar to that used by Wilets[4] previously, but with the important difference that no free parameters are used.

Recently, Brueckner[5] and his collaborators also developed a Thomas-Fermi treatment of nuclei, and they obtain quite reasonable density distributions. Both Bethe and Brueckner *et al.* took the calculated binding energy of nuclear matter versus density as a starting point of their treatment of the nuclear surface. Bethe used the results from the Reid hard-core potential.[2] In Ref. 5 and a part of Ref. 1, the energy density is assumed to be of the form

$$e=\rho W(\rho)+\zeta(\nabla\rho)^2, \tag{1.1}$$

where $W(\rho)$ is the calculated energy per particle of nuclear matter at nucleon density ρ. The only explicit dependence on the nuclear force enters into the coefficient ζ of the gradient term. The density distribution can then be obtained by either a variational treatment or the solution of a differential equation.

A more detailed TFA of the nuclear surface, again starting from a realistic interaction, the Reid soft-core interaction,[2] but making fewer approximations than in Ref. 1, was given by Németh and Bethe.[6] In their paper, the density distribution at the surface is obtained as a solution of an integral equation (which was mentioned but not worked out in detail in Ref. 1). Their calculated surface thickness[7] 2.2 fm, and surface energy[8] 19.5 MeV, are in good agreement with empirical values.

There have also been some studies of the nuclear surface in which the TFA is not made. One such calculation using a realistic nucleon-nucleon *interaction was made by Brueckner et al.*[9] using the Gammel-Thaler interaction. However, most of the other non-TFA calculations do not use realistic interactions as a starting point, but rather, simplified effective interactions.

B. Effective Interactions

The use of effective interactions in nuclear-structure calculations is somewhat parallel to that of realistic interactions. The idea behind the former is the hope that by simplifying the interaction it becomes easier to calculate some nuclear properties in terms of others and also to gain better insight into the physics involved. Various kinds of effective interactions have been used in

recent calculations. In most of them, there is less emphasis on fitting nucleon-nucleon phase shifts than on fitting average nuclear properties, especially those of nuclear matter (binding energy and density). With such interactions it may be easier to do Hartree-Fock calculations. We just mention a few of the effective interactions which have been used. The group of Davies, Krieger, and Baranger[10] calculated properties of double-closed-shell nuclei (energy, density distribution, and single-particle energies.) They used interactions which fit the properties of nuclear matter (in first-order perturbation theory), and also the low-energy parameters of the two-nucleon interaction.[11]

Volkov[12] made use of a simple effective interaction (Gaussian with arbitrary spin and isospin dependence) whose parameters fit the binding energy of ^{4}He and ^{16}O, for calculations of properties of $1p$-shell nuclei. This interaction was generalized by Manning,[13] who made the interaction density dependent so as to also fit the properties of nuclear matter.

Pearson *et al.*[14] used effective interactions which fit, first of all, the properties of nuclear matter. In addition, instead of fitting the two-body data, they required that the long-range part of this effective potential in the $^1S_0$ state should coincide with the one-boson-exchange potential at distances ≥ 1 fm. For smaller distances, the one-boson potential breaks down, and Pearson *et al.* use a purely phenomenological potential.

A slightly different approach was taken by Brink and Boeker.[15] These authors examined several classes of density-independent interactions, for example, an attractive Gaussian plus a repulsive δ interaction. For almost every class, there are several interactions which differ only by the values of the parameters. The latter were chosen so as to fit the binding energy and density of nuclear matter, and the binding energy and rms radius of the α particle, all in first-order perturbation theory. Brink and Boeker then calculated the ground-state energy and harmonic-oscillator spacing (they use oscillator single-particle wave functions) for other light double-closed-shell $N=Z$ nuclei, namely, ^{16}O and ^{40}Ca. These interactions do not fit nucleon-nucleon scattering phase shifts, but this is hoped not to be serious. If the interaction fits ground-state properties of the α particle and of nuclear matter, it should be reasonable for intermediate $N=Z$ nuclei. Indeed, for most of the interactions used by Brink and Boeker, the agreement between calculated and empirical values is good.

Pandharipande[16] has used similar criteria for the interactions he considered. These are attractive Yukawa plus a density-dependent (or momentum-dependent) δ-repulsion interaction. Again, the binding energy and rms radii of the α particle, ^{16}O, and ^{40}Ca were fitted quite well. Seyler and Blanchard[17] used a momentum-dependent Yukawa interaction, whose three parameters (for $N=Z$ nuclei) were adjusted to fit the ground-state binding energy and density of nuclear matter and also the empirical surface energy. Treating the problem in the TFA they obtained quite realistic particle density distributions. This interaction was used by Myers[18] and Myers and Swiatecki[18a] to study correction terms to the semiempirical binding-energy formula. The Seyler-Blanchard interaction was also used by Kohler[19] in his calculation of the nuclear surface properties; he did calculations both with and without TFA and obtained values of 18 and $21A^{2/3}$ MeV, respectively. Other calculations using effective interactions but not with the TFA, were made by Tabakin and Amos[20] and by Donnelly and Naqvi.[21]

Recently, Vautherin and Veneroni[22] have made extensive Hartree-Fock calculations for the properties of closed-shell nuclei, especially ^{208}Pb, using mainly the Brink-Boeker[15,23] interaction $B1$ and an interaction proposed by Skyrme.[24]

The interaction which we use in this article is close to the latter. It is essentially a δ-function interaction whose strength is a function of the relative momentum of the interacting particles and also of the local density. The interaction is of the form:

$$V = a\delta(\vec{r}) + \tfrac{1}{2}\beta[\vec{p}^2\delta(\vec{r}) + \delta(\vec{r})\vec{p}^2] + \eta\vec{p}\delta(\vec{r})\cdot\vec{p}, \quad (1.2)$$

where a, β, and η are constants (which may be density dependent). The quantity $\vec{p}$ is the relative momentum of the interacting particles taken as an operator, $-i\nabla$.

For the density dependence, Skyrme assumed that a is a linear function of the local[25] density $\rho(R)$

$$a = -\alpha + \gamma\rho(R) \quad (1.3)$$

and β, η are independent of ρ. (r is the distance between the interacting particles and $\vec{R}$ is the position of their c.m.) The parameters were adjusted to fit the nuclear-matter binding energy and density of nuclear matter and also the nuclear surface energy. Skyrme used this interaction to calculate shell-model matrix elements in light $1p$-shell nuclei, and he obtained rather good agreement with empirical matrix elements deduced from experimental level spectra.

C. Summary of Present Paper

In this article, we use an effective interaction

which we call the modified δ interaction (MDI), of the same form as Skyrme's except that we set $\eta = 0$ (which implies no interaction in odd states) and the density-dependent part is assumed to be proportional to $\rho^{2/3}$, as suggested by Bethe.[1] It is readily seen that the MDI acts only in S states of relative motion. In this paper we consider only $N = Z$ nuclei, and we neglect any spin or isospin dependence of the interaction.

In the next section we discuss the form of the MDI and obtain explicit expressions for the potential energy for both infinite nuclear matter and finite nuclei.

In Sec. 3, we determine the values of the MDI parameters so as to fit the binding energy and density of nuclear matter and the binding energy of the the ground state of ^{16}O. These criteria are very similar to those of Brink and Boeker,[15] except that the latter used ^{4}He as an "anchor point," rather than ^{16}O. Using this interaction, it is easy to derive related properties of nuclear matter, such as the compression modulus and the depth of the one-body potential as a function of the momentum for a nucleon in nuclear matter.

Section 4 deals with the binding energy and rms radii of the ground states of ^{4}He, ^{16}O, and ^{40}Ca. The single-particle wave functions are assumed to be harmonic-oscillator wave functions, and the oscillator spacing is varied to minimize the energy. The calculated binding energies and rms radii come out in quite good agreement with empirical values.[26,27] This variational calculation, as well as others described in Secs. 5 and 6, are, of course, not Hartree-Fock calculations, in the sense that we do not attempt to reach self-consistency between the density and the one-particle potential by iteration.

In Sec. 5 we discuss the problem of the nuclear surface using a plane semi-infinite nuclear model. We solve the Schrödinger equation for a one-dimensional Woods-Saxon potential to obtain the one-particle wave functions, and then the particle density and kinetic energy density. Neglecting all dynamical correlations, we obtain a surface energy of 19.3 MeV and a 90 to 10% fall-off distance of the density equal to 2.2 fm. These results are quite close to the empirical values of 19.5 MeV[8] and $t = 2.4$ fm[7] for finite nuclei.

Section 6 deals with results obtained with the TFA. Just as in the independent-particle model (IPM) case, we minimize the calculated surface energy with respect to the diffuseness and depth of the generating nucleon-nucleus potential. However, in the TFA the particle and kinetic energy density may be obtained analytically in terms of the Fermi energy and potential. The main result is that the surface thickness and surface energy are significantly smaller than the IPM results, namely, 2.0 fm and 16.0 MeV. Three other TFA calculations were made, in which both the energy-density relation and the density distribution are taken to have a simple form, and the surface energy is calculated without using a generating potential.

The last section lists the main conclusions, chiefly the comparison of the results obtained with TFA or IPM. It is suggested that great care be taken in nuclear-structure calculations involving the TFA approximation.

2. MODIFIED δ INTERACTION

A. Form of Interaction

In this paper we will use an effective central nucleon-nucleon interaction which acts only in S states of relative motion. Then the general S-state matrix element of the interaction (with wave functions normalized to unity per unit volume), is

$$\langle k' | V | k \rangle = \int V(r) j_0(k'r) j_0(kr) d^3r . \tag{2.1}$$

For small momenta or for a short-range interaction we can expand in powers of k. This gives

$$\langle k' | V | k \rangle = \int V(r) d^3r - \tfrac{1}{6}(k^2 + k'^2) \int V(r) r^2 d^3r + \tfrac{1}{120}(k^4 + \tfrac{10}{3} k^2 k'^2 + k'^4) \int V(r) r^4 d^3r \cdots . \tag{2.2}$$

The main assumption in our effective interaction is to drop all terms of higher order than k^2. Thus for the diagonal matrix element we obtain

$$\langle k | V | k \rangle = \int V(r) d^3r - \tfrac{1}{3} k^2 \int V(r) r^2 d^3r . \tag{2.3}$$

In coordinate space, this interaction can be written as

$$V(\vec{p}, \vec{r}) = \left[\int V(r) d^3 r \right] \delta(\vec{r}) - \tfrac{1}{6} \left[\int V(r) \vec{r}^{\,2} d^3r \right] \left[\vec{p}^{\,2} \delta(\vec{r}) + \delta(\vec{r}) \vec{p}^{\,2} \right] \cdots . \tag{2.4}$$

Thus this interaction depends explicitly on the relative momentum of the interacting particles. The first term in (2.4) is a conventional δ interaction. The next term also looks like a δ function, but it appears only if the interaction has a finite range. It is explicitly dependent on the relative momentum of the interacting particles. Thus we denote this interaction as a modified δ interaction (MDI).

In addition, we allow for a density dependence of the interaction such as that suggested by Bethe.[1]

This term appears because of both the short-range repulsion (through the disperison[28] effect) and the tensor interaction. Both mechanisms give more repulsion (or less attraction) with increasing density. As a crude approximation, this density-dependent term can be written as $\gamma k_F^2(R) \times \delta(\vec{r})$, where $k_F(R)$ is the local Fermi momentum. R denotes distance from the center of the nucleus, and the dependence of the strength on density is $\rho^{2/3}$. It has, however, been pointed out by Nemeth and Bethe[6] that the density dependence should have a finite range, owing mainly to the action of the tensor force.

Altogether, the MDI has three independent parameters which we will denote by α, β, and γ. Thus

$$V(\vec{p}, \vec{R}, \vec{r}) = -\alpha\delta(\vec{r}) + \tfrac{1}{2}\beta[p^2\delta(\vec{r}) + \delta(\vec{r})p^2] + \gamma k_F^2(\vec{R})\delta(\vec{r}), \tag{2.5}$$

where

$$\alpha = -\int V(r)d^3r, \tag{2.6a}$$

$$\beta = -\tfrac{1}{3}\int V(r)r^2 d^3r. \tag{2.6b}$$

As can be seen by comparison with Eq. (1.2), this interaction is quite similar to Skyrme's.

B. Energy of Nucleus with MDI

Let us consider the interaction energy of a two-nucleon pair in a many-body system, such as the nucleus. The direct-interaction energy between a pair in states i and j is (in first-order perturbation theory)

$$\langle ij|V|ij\rangle = \int \rho_i(\vec{r}_1)\rho_j(\vec{r}_2)V(\vec{r}_1 - \vec{r}_2)d^3r_1 d^3r_2. \tag{2.7}$$

For the exchange term, one obtains:

$$\langle ij|V|ji\rangle = \int \rho_i(\vec{r}_1,\vec{r}_2)\rho_j(\vec{r}_1,\vec{r}_2)V(\vec{r}_1 - \vec{r}_2)d^3r_1 d^3r_2, \tag{2.8}$$

where $\rho_i(\vec{r}_1, \vec{r}_2)$ is the density matrix equal to $\psi_i^*(r_1)\psi_i(r_2)$.

For an interaction which is present in spatially even states only, such as the MDI, it is readily verified that the interaction energy of a pair of nucleons is (apart from a constant multiplying factor) just the sum of the direct and exchange terms, even when the wave functions used are antisymmetrized. Then

$$\Delta E_{ij} = \tfrac{3}{8}\langle ij|V|ij\rangle + \tfrac{3}{8}\langle ij|V|ji\rangle. \tag{2.9}$$

Next we expand the density in powers of $\vec{r} = \vec{r}_1 - \vec{r}_2$ up to second order in $\vec{r}$. This gives

$$\langle ij|V|ij\rangle = \int \rho_i(\vec{R})\rho_j(\vec{R})d^3R\int V(r)d^3r - \tfrac{1}{6}\int \nabla\rho_i(\vec{R})\cdot\nabla\rho_j(\vec{R})d^3R\int V(r)r^2d^3r, \tag{2.10}$$

where

$$\vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2), \tag{2.11}$$

and we have made use of the spherical symmetry of the interaction by averaging over the direction of $\vec{r}$. The density matrix $\rho_i(\vec{r}_1, \vec{r}_2)$ can also be expanded in powers of $\vec{r}$. One obtains, again after averaging over the direction of $\vec{r}$,

$$\rho_i(\vec{r}_1, \vec{r}_2) = \psi_i^*(\vec{r}_1)\psi_i(\vec{r}_2) = \psi_i^2(R) + \tfrac{1}{12}r^2[\tfrac{1}{2}\psi_i^*(\vec{R})\nabla^2\psi_i(\vec{R}) + \tfrac{1}{2}\nabla^2\psi_i^*(\vec{R})\psi_i(\vec{R}) - \nabla\psi_i^*(\vec{R})\cdot\nabla\psi_i(\vec{R})]. \tag{2.12}$$

The quantity in brackets is, apart from a constant factor $\hbar^2/2M$, the kinetic energy density. Let us define

$$\tau_i = -\tfrac{1}{4}\psi_i^*\nabla^2\psi_i - \tfrac{1}{4}(\nabla^2\psi_i^*)\psi_i + \tfrac{1}{2}\nabla\psi_i^*\cdot\nabla\psi_i. \tag{2.13}$$

Then the first two terms of the density matrix are

$$\rho_i(\vec{r}_1, \vec{r}_2) = \rho_i(\vec{R}) - \tfrac{1}{6}r^2\tau_i(\vec{R}), \tag{2.13a}$$

and the exchange contribution to the interaction energy of a pair is

$$\langle ij|V|ji\rangle = \int \rho_i(\vec{R})\rho_j(\vec{R})d^3R\int V(r)d^3r - \tfrac{1}{6}\int[\rho_i(\vec{R})\tau_j(\vec{R}) + \tau_i(\vec{R})\rho_j(\vec{R})] \times d^3\vec{R}\int V(r)r^2d^3r. \tag{2.14}$$

The kinetic density is often defined not by Eq. (2.13) but by

$$\tau_i = -\psi_i^*\nabla^2\psi_i. \tag{2.15}$$

It is readily verified that both forms give the

same total kinetic energy T_i.

$$T_i = \frac{\hbar^2}{2M}\int \tau_i \, d^3R. \tag{2.16}$$

However, for the calculation of the potential energy integrals, for example, $\int \rho_i \tau_i d^3R$, these two forms give different results. (Of course, in the TFA the two expressions give identical results for all the required integrals.) The form (2.13) used here appears to be the more natural one to use in an expansion of the density matrix as discussed here. Summing over all pairs of states i and j, we obtain the potential energy including both direct and exchange contributions:

$$\text{P.E.} = \tfrac{3}{8}\int \rho^2 d^3R \int V\, d^3r - \tfrac{1}{32}\int (\nabla\rho)^2 d^3R \int V r^2 d^3r - \tfrac{1}{16}\int \rho\tau d^3R \int V r^2 d^3r. \tag{2.17}$$

The first and third terms on the right-hand side appear also in the expression for the energy of nuclear matter. To obtain the total energy, we must add to Eq. (2.17) the kinetic energy and also a contribution from the density-dependent part of the δ interaction. This gives

$$E = \frac{\hbar^2}{2M}\int \tau\, d^3R - \tfrac{3}{8}\alpha\int \rho^2 d^3R + \tfrac{3}{16}\beta\int \rho\tau\, d^3R + \tfrac{3}{32}\beta\int (\nabla\rho)^2 d^3R + \tfrac{3}{8}\gamma\frac{k_F^{\,2}}{\rho_0^{2/3}}\int \rho^{8/3} d^3R, \tag{2.18}$$

where α, β, and γ have been defined in Eqs. (2.5) and (2.6), ρ_0 and k_F refer to nucleon density and Fermi momentum of nuclear matter at normal density, as discussed in the next section.

3. NUCLEAR MATTER

Using the MDI it is easy to obtain an analytic expression for the energy of nuclear matter (per particle) as a function of density (or more explicitly, as function of the Fermi momentum). In this paper we consider only nuclear matter with equal numbers of neutrons and protons. Also, only the first-order perturbation energy is calculated.

A. Saturation Conditions

To obtain the total energy of nuclear matter per particle, W, we use the well-known relations

$$\rho_0 = (2/3\pi^2)k_F^{\,3}, \tag{3.1a}$$

$$\tau_0 = \tfrac{3}{5}k_F^{\,2}\rho_0, \tag{3.1b}$$

which are both valid in the plane-wave approximation. The result is

$$W(k_F) = \tfrac{3}{10}\frac{\hbar^2}{M}k_F^{\,2} + \frac{k_F^{\,3}}{4\pi^2}\left[-\alpha + (\tfrac{3}{10}\beta + \gamma)k_F^{\,2}\right]. \tag{3.2}$$

One obtains two relations among the three parameters α, β, and γ by requiring that the interaction reproduces the correct empirical saturation conditions: binding energy and density of nuclear matter. We use

$$W_0 = -15.7 \text{ MeV},^{8} \tag{3.3a}$$

$$k_F = 1.36 \text{ fm}^{-1}.^{1} \tag{3.3b}$$

The latter corresponds to a radius constant $r_0 = 1.12$ fm for hypothetical finite nuclei with the same density as nuclear matter but with zero surface thickness,[7] namely, $\rho_0 = 0.170$ nucl/fm^3. The two relations are

$$\alpha = 1160 \text{ MeV fm}^3, \tag{3.4a}$$

$$0.3\beta + \gamma = 298.8 \text{ MeV fm}^5. \tag{3.4b}$$

As we have already seen, the MDI has both momentum-dependent terms and a density-dependent term. The coefficients of these two terms, β and γ, are not uniquely defined by the nuclear-matter conditions alone. This means that, as far as the fit to the nuclear matter energy is concerned, the density dependence of the interaction can be simulated by the momentum dependence, and vice versa. This is, however, not quite true for finite nuclei, especially light nuclei.[29]

B. Compression Modulus

The compression modulus[1] (often referred to as compressibility) of nuclear matter can be defined by

$$K = k_F^{\,2}\left(\frac{\partial^2 W}{\partial k_F^{\,2}}\right). \tag{3.5}$$

For a MDI, which fits the nuclear-matter saturation conditions, i.e., for which Eqs. (3.4) hold, one obtains $K = 306$ MeV, regardless of the values of β and γ separately. For more realistic interactions, K ranges from 150 to 200 MeV.[1] It should also be mentioned that a good approximation to the nuclear-matter energy as function of density is given by[24]

$$W(\rho) = |W_0|\left[-2\,\rho/\rho_0 + \rho^2/\rho_0^{\,2}\right]. \tag{3.6}$$

For this energy-density relation (and with $\rho_0 = 0.170$ nucl/fm^3 and $|W_0| = 15.7$ MeV) one finds

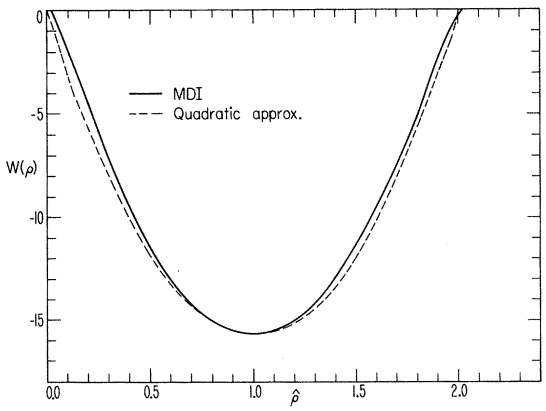

FIG. 1. Energy versus density relation of nuclear matter obtained for the MDI, and for the simple quadratic term Eq. (3.6).

$K = 283$ MeV quite close to the MDI value. The energy versus density relation for the MDI [Eq. (3.2)] is plotted in Fig. 1 and compared with the $W(\rho)$ relation (3.6).

C. Average Potential Felt by a Nucleon in Nuclear Matter

The formal expression for the average potential felt by a nucleon of momentum k_i, in nuclear matter at normal density, is

$$U(k_i, k_F) = A^{-1} \sum_{k_n \le k_F} \langle k | V | k \rangle + \Delta U_R , \tag{3.7}$$

where $k = \frac{1}{2}|\vec{k}_i - \vec{k}_n|$. The last term, the so-called rearrangement energy,[30] is nonzero if the interaction is density dependent, as is the case here, or if second or higher terms in the perturbation series are considered.

For the MDI, the calculation simplifies greatly. The rearrangement energy is given by

$$\Delta U_R = \tfrac{3}{4}\rho_0 \times \tfrac{1}{3}\gamma k_F^2 , \tag{3.8}$$

and the total nucleon-nucleus potential has the following simple form:

$$U(k_i, k_F) = \tfrac{3}{4}\rho_0 [-\alpha + (\tfrac{3}{20}\beta + \tfrac{4}{3}\gamma) k_F^2] + \tfrac{3}{16}\rho_0 \beta k_i^2 . \tag{3.9}$$

As is evident, the potential is momentum dependent unless $\beta = 0$. However, for the MDI, there are no terms of fourth or higher order in k. Thus this momentum dependence can be simulated by a reduction of the effective mass.

We have already shown that the fit to the binding energy and density of nuclear matter specifies the value of the parameter α and the linear combination $0.3\beta + \gamma$. In the following section, we obtain the values of β and γ separately, by requiring a fit to the ground-state energy of ^{16}O. Substituting these values into Eq. (3.9) we obtain

$$U(k_i, k_F = 1.36 \text{ fm}^{-1}) = -80.7 + 26.8\, k_i^2/k_F^2 . \tag{3.10}$$

The rearrangement contribution ΔU_R is 12.9 MeV. This value is in excellent agreement with the 13-MeV rearrangement energy obtained by Brueckner *et al.*[31] using a realistic but complicated interaction, but it is only about half as large as the values quoted by Masterson and Lockett.[32] Our result corresponds to an effective mass in nuclear matter $m^* = 0.59m$.

It is readily verified that expression (3.9) for $U(k_F, k_F)$ satisfies the Hugenholtz-Van Hove theorem: Single-particle energy at the Fermi surface equals the energy per particle of nuclear matter.

$$\epsilon(k_F) = T(k_F) + U(k_F, k_F) = -15.7 \text{ MeV} = W(k_F) . \tag{3.11}$$

For nucleons in the lowest state (the $1s$ state) the single-particle energy, i.e., minus the removal energy, should approach 81 MeV, at least in the limit $A \to \infty$. For finite nuclei, the removal energy is expected to be less, because of the finite kinetic energy of the nucleons, even those in the $1s$ state. The expectation value of this kinetic energy can be roughly estimated by replacing the potential well by a harmonic-oscillator potential. For example, for the heavy nucleus ^{208}Pb, the kinetic energy is

$$T(1s) = \tfrac{3}{4}\hbar\omega \sim 5 \text{ MeV} . \tag{3.12}$$

Thus in the present model, the $1s$ single-particle energy is $-81 + 5 = -76$ MeV. This value is the average of neutron and proton binding energies, neglecting the symmetry part of the potential and the Coulomb potential. The former raises the neutron energy by about 5 MeV in ^{208}Pb and lowers the proton energy by the same amount. The Coulomb potential raises the proton energy by about 20 MeV. Thus we obtain a $1s$ single-particle energy of -70 MeV for the neutrons and -60 MeV for the protons in ^{208}Pb. These values are intermediate between those obtained by two sets of authors, all of whom use density-dependent effective interactions: (a) the recently calculated results by Davies *et al.* and Meldner[33] $-(95$ to $92)$ MeV for neutrons and $-(85$ to $75)$ MeV for protons, but without explicitly taking into account the rearrangement effects, and (b) the results quoted in Ref. 32 as well as some recent work by Brink and Vauther-

in[23] and Negele,[34] all of whom obtain a $1s$ single-particle energy of $-(62$ to $64)$ MeV for neutrons and $-(52$ to $55)$ MeV for protons.

It may also be of interest to estimate the single-particle energy of a $1s$ nucleon in the light nucleus ^{16}O. For this case, the calculated average of neutron and proton binding energies is much smaller than in nuclear matter, so that we cannot simply use Eq. (3.10) with $k_F = 1.36$ fm^{-1}. However, Wong[35] has suggested using $k_F = 1.2$ fm^{-1} for this case. With the same parameters as for ^{208}Pb, which was treated like nuclear matter, we find for ^{16}O that $U(0, k_F = 1.2 \text{ fm}^{-1}) = -63$ MeV and the kinetic energy, using the oscillator model, is 12 MeV. Also it is probably more reasonable to use $k_i \neq 0$, in particular, the value 0.76 fm^{-1}, which corresponds to a kinetic energy of 12 MeV. When this term and the kinetic energy are added to the potential term in Eq. (3.10), we find the energy of a $1s$ nucleon in ^{16}O to be -45 MeV, again in rough agreement with empirical values and previous calculations.

We have already noted in Sec. 3 that the fit to nuclear matter alone fixes the value of the combination $0.3\beta + \gamma$. Thus if one sets $\beta = 996$ MeV fm^5 and $\gamma = 0$, the resultant momentum-dependent but density-independent potential gives too much binding of the $1s$ nucleon in nuclear matter, namely, 112.7 MeV, and an effective mass $m^* = 0.4m$. Conversely, if $\beta = 0$ and $\gamma = 298.8$ MeV fm^5, i.e., a strongly density-dependent but static potential, the $1s$ binding energy is 55 MeV, and the effective mass is exactly m.

4. DOUBLE-CLOSED-SHELL N=Z NUCLEI

An important test of an effective interaction, such as the MDI, is that it can account for the ground-state binding energies and density distributions of finite nuclei. In this paper we restrict ourselves to the double-closed-shell $N = Z$ nuclei ^{4}He, ^{16}O, and ^{40}Ca. (In a subsequent paper by J. Ehlers and the present author, we investigate $N \neq Z$ nuclei as well, and use a Woods-Saxon and also a momentum-dependent nucleon-nucleus interaction.) The single-particle wave functions are taken to be eigenfunctions of a harmonic-oscillator potential, and the total energy (with c.m. terms subtracted out), is minimized as a function of the oscillator length b.

$$b = [\hbar/(M\omega)]^{1/2} . \tag{4.1}$$

As we saw in the previous section, the fit of the MDI parameters α, β, and γ to the nuclear-matter saturation properties fixes the value of α and gives one relation between the coefficients β and γ. Another relation between β and γ can be obtained by requiring a fit to the empirical ground-state energy of any one of the three nuclei under consideration. In previous work,[36] the author fitted the ^{4}He ground state, following the treatment in Ref. 15, but a better over-all agreement with properties of light nuclei is probably obtained by requiring instead a fit to the ^{16}O ground-state energy[37] (with the Coulomb energy subtracted). This, together with Eq. (3.4b), gives $\beta = 454.3$ MeV fm^5 and $\gamma = 162.6$ MeV fm^5. The energies versus oscillator lengths for ^{4}He, ^{16}O, ^{40}Ca are:

$$E(^4\text{He}) = 2.25\,\frac{\hbar^2}{Mb^2} - \frac{6}{(2\pi)^{3/2}}\,\frac{\alpha}{b^3} + \frac{9\beta}{(2\pi)^{3/2}b^5} + \frac{18.84}{(2\pi)^{3/2}}\,\frac{\gamma}{b^5} , \tag{4.2a}$$

$$E(^{16}\text{O}) = 17.25\,\frac{\hbar^2}{Mb^2} - \frac{46.5}{(2\pi)^{3/2}}\,\frac{\alpha}{b^3} + \frac{78.75\,\beta}{(2\pi)^{3/2}\,b^5} + \frac{207.3}{(2\pi)^{3/2}}\,\frac{\gamma}{b^5} , \tag{4.2b}$$

$$E(^{40}\text{Ca}) = 59.25\,\frac{\hbar^2}{Mb^2} - \frac{182.4}{(2\pi)^{3/2}}\,\frac{\alpha}{b^3} + \frac{369.2\,\beta}{(2\pi)^{3/2}\,b^5} + \frac{1071}{(2\pi)^{3/2}}\,\frac{\gamma}{b^5} . \tag{4.2c}$$

We also calculated the rms charge radius of the closed-shell nuclei. This is related to the empirical rms charge radius by[38]

$$\langle r^2\rangle_{\text{emp}} = \langle r^2\rangle_{\text{MDI}} - \frac{3}{2}\,\frac{b^2}{A} + \langle r^2\rangle_{\text{prot}} . \tag{4.3}$$

The two corrections are due to (1) a c.m. term and (2) the finite extent of the proton charge distribution. The results are listed in Tables I and II. As can be seen, the calculated and empirical ground-state energies[39,40] of ^{4}He and ^{40}Ca are quite well reproduced. The fit to empirical rms radii is not as good, but still satisfactory in view of the uncertainty in the experimental results.

5. NUCLEAR SURFACE–SEMI-INFINITE-SLAB MODEL–IPM

In this and the next section we discuss the application of the MDI to the calculation of the nuclear-matter distribution at the surface, especially the surface thickness and the nuclear surface energy.

TABLE I. Ground-state energies in MeV. Comparison of MDI and empirical results.

Nucleus	E_{emp}[a]	E_{Coul}[b]	$E_{\text{emp}}-E_{\text{Coul}}$	E_{MDI}
^{4}He	−28.2	0.3	−28.5	−29.8
^{16}O	−127.6	11.9	−139.5	−139.4
^{40}Ca	−342.1	68.7	−410.8	−402.6

[a]See Ref. 39. [b]See Ref. 40.

We restrict ourselves to considering a semi-infinite slab of nuclear matter. Thus the curvature of the nucleus is neglected. The calculations are basically variational, i.e., we minimize the surface energy as a function of the diffuseness of the generating potential.

Turning briefly to the empirical results, the surface energy in the semiempirical mass formula is 19.5 MeV.[8] There is some ambiguity in the definition of the skin thickness. Following customary notation, we shall identify it as the 90 to 10% fall-off distance. The magnitude of the surface thickness, as deduced from empirical charge densities in many nuclei, is about 2.4 fm.[41]

Bethe[1] has proposed a different definition of the surface thickness, namely, the reciprocal of the maximum slope of the density distribution (normalized to unity in the interior). However, the usual definition may be more appropriate for a Fermi-type density distribution, which turns out to be close to the IPM density distribution.

A. Calculation of Nuclear Surface Energy

Let us now apply the MDI, with its three parameters fixed, to the calculation of the nuclear surface energy. We start with the general form of the energy as given in (2.18). This can be greatly condensed by introducing the energy density. Thus the total energy equals

$$E=\int e(\rho,\tau,\nabla\rho)\,d^3R, \tag{5.1}$$

i.e., the energy density is a function of the density and its gradient and of the kinetic energy density

TABLE II. rms charge radii in fm. Comparison of MDI and empirical results.

Nucleus	$r_{\text{emp}}^{\text{rms}}$	$b_{\text{MDI}}=(\hbar/Mw)^{1/2}$	$r_{\text{MDI}}^{\text{rms}}$
^{4}He	1.63 to 1.71[a]	1.47	1.75
^{16}O	2.70[b]	1.67	2.58
^{40}Ca	3.49[b]	1.87	3.32

[a]See Ref. 26.
[b]See Ref. 27.

alone. It should be pointed out again that this result holds only for a restricted class of interactions such as Skyrme's and the MDI. For a more general nucleon-nucleon interaction, e is a function of higher derivatives of ρ and also of other terms beyond ρ and τ in the expansion of the density matrix.

Furthermore, for the MDI, the energy density can be written as a sum of two terms:

$$e(\rho,\tau,\nabla\rho)=e_1(\rho,\tau)+\tfrac{3}{32}\beta(\nabla\rho)^2. \tag{5.2}$$

The energy, as obtained by integrating (5.2) over d^3R, contains contributions from both the interior and the surface, i.e., both volume and surface energy. To obtain the surface energy, we must subtract from (2.18) the volume energy.

$$E_{\text{vol}}=W_0A=W_0\int\rho(R)\,d^3R=e_0\int\hat{\rho}(R)\,d^3R, \tag{5.3}$$

where

$$\hat{\rho}=\rho/\rho_0 \tag{5.4}$$

and ρ_0 denotes the energy density of nuclear matter at normal density. Thus the surface energy is

$$E_{\text{surf}}=\int[e_1(\rho,\tau)-e_0\hat{\rho}]\,d^3R+\tfrac{3}{32}\beta\int(\nabla\rho)^2\,d^3R. \tag{5.5}$$

The first integral denotes the loss of binding energy because of the lower density at the surface, while the second integral is due to the apparent finite range of the interaction.[42] The main contribution to both integrals comes from the region of the nuclear surface.

For the semi-infinite-slab model, the density ρ and the kinetic energy density τ are functions of only one coordinate, which we denote by z. This model corresponds to the limiting case $A\to\infty$. The integration d^3R over volume can be reduced to a one-dimensional integral over z. Restricting ourselves to a spherical density distribution, we obtain

$$d^3R=4\pi R^2dR\to 4\pi r_0^2A^{2/3}\,dz\to\frac{3A^{2/3}}{\rho_0r_0}\,dz. \tag{5.6}$$

Thus the expression for the surface energy becomes

$$E_{\text{surf}}=\frac{3A^{2/3}}{r_0}\left[\int\left(\frac{[e_1(\hat{\rho},\hat{\tau})]}{\rho_0}-e_0\hat{\rho}\right)dz+\tfrac{3}{32}\beta\rho_0\int\left(\frac{d\hat{\rho}}{dz}\right)^2dz\right], \tag{5.7}$$

where $\hat{\tau}=\tau/\tau_0$, τ_0 being the kinetic energy density

of nuclear matter of density ρ_0. For the actual calculation of the surface energy, we express the energy density e_1 as a sum of terms each of which is a function of $\hat{\rho}$ and $\hat{\tau}$. The contribution of the gradient term $(d\hat{\rho}/dz)^2$ can be evaluated directly. Also, since (at least in the TFA discussed in Sec. 6) $\hat{\rho}$ and $\hat{\tau}$ are functions of z/a_U, we introduce a new variable $s=z/a_U$, where a_U is the diffuseness parameter in the generating potential. Thus for the kinetic energy terms, we have

$$e_1^{\tau}=(\hbar^2/2M)\tau, \tag{5.8a}$$

$$e_0^{\tau}=\rho_0 T_{\mathrm{Av}}, \tag{5.8b}$$

where T_{Av} is the average kinetic energy per particle in nuclear matter. The contribution of this term to the surface energy is

$$E_{\mathrm{surf}}^{\tau}=\frac{3}{r_0}A^{2/3}\,\frac{\hbar^2}{2M}\,\frac{\tau_0}{\rho_0}\int(\hat{\tau}-\hat{\rho})\,dz=\frac{3}{r_0}A^{2/3}\,T_{\mathrm{Av}}I_0 a_U, \tag{5.9}$$

where

$$I_0=\int(\hat{\tau}-\hat{\rho})\,ds. \tag{5.10}$$

The surface contribution from the potential energy can also be obtained easily. The complete expression for the surface energy is

$$E_{\mathrm{surf}}=\frac{3A^{2/3}}{r_0}\,a_U\left\{\tfrac{3}{10}\frac{\hbar^2}{M}k_F{}^2 I_0+\tfrac{3}{8}\rho_0\left[-\alpha I_1+(\tfrac{3}{10}\beta I_2+\gamma I_3)k_F{}^2\right]\right\}+\frac{3A^{2/3}}{r_0 a_U}\,\tfrac{3}{32}\,\rho_0\beta I_g, \tag{5.11}$$

where

$$\begin{aligned} I_r&=\int_{-\infty}^{\infty}[f_r(s)-\hat{\rho}(s)]\,ds,\\ f_0&=\hat{\tau},\\ f_1&=\hat{\rho}^2,\\ f_2&=\hat{\rho}\hat{\tau},\\ f_3&=\hat{\rho}^{8/3}, \end{aligned} \tag{5.12}$$

and

$$I_g=\int_{-\infty}^{\infty}\left(\frac{d\hat{\rho}}{ds}\right)^2 ds. \tag{5.13}$$

B. Independent Particle Model (IPM)

In the calculations discussed here, we used a one-dimensional Woods-Saxon potential to generate the single-particle wave functions. Thus we do not attempt Hartree-Fock self-consistency. On the other hand, this IPM calculation should give more accurate results than the Thomas-Fermi approximation discussed in Sec. 6. The potential is written as

$$U(s)=-U_0/(1+e^s). \tag{5.14}$$

The single-particle wave functions are, of course, separable in Cartesian coordinates, and thus they can be written as

$$\psi(\vec{\mathrm{k}},\vec{\mathrm{r}})=e^{i(k_x x+k_y y)}\,\phi(k_z,z), \tag{5.15}$$

where the ϕ are the eigenfunctions of the one-dimensional potential-well problem. We set $U_0=50$ MeV and assume that all states up to an energy -11.5 MeV are occupied. This corresponds to a Fermi kinetic energy $T_F=38.5$ MeV, which is the value in nuclear matter at normal density ρ_0. Each single-particle wave function $\phi(k_z,z)$ is normalized such that in the interior it approaches $\sin[k_z z+\delta(k_z)]$ without an additional normalization factor.

The particle density and kinetic energy density may be obtained directly from the wave functions by integrating over momenta. Thus

$$\begin{aligned} \hat{\rho}(z)&=2\int_0^{k_F}\phi^2(k_z,z)\,d^3k\Big/\int_0^{k_F}d^3k\\ &=3k_F{}^{-3}\int_0^{k_F}(k_F{}^2-k_z{}^2)\,\phi^2(k_z,z)\,dk_z, \end{aligned} \tag{5.16}$$

$$\hat{\tau}(z)=\hat{\tau}_x(z)+\hat{\tau}_y(z)+\hat{\tau}_z(z), \tag{5.17}$$

$$\hat{\tau}_x(z)=\hat{\tau}_y(z)=\frac{2\int_0^{k_F}k_x{}^2\,\phi^2(k_z,z)\,d^3k}{\int_0^{k_F}k^2 d^3k} \tag{5.18a}$$

$$=\tfrac{5}{4}k_F{}^{-5}\int_0^{k_F}(k_F{}^2-k_z{}^2)^2\phi^2(k_z,z)\,dk_z,$$

$$\hat{\tau}_z(z)=\frac{\int_0^{k_F}[\phi'^2(k_z,z)-\phi(k_z,z)\phi''(k_z,z)]\,d^3k}{\int_0^{k_F}k^2 d^3k}$$

$$=\tfrac{5}{2}k_F{}^{-5}\int_0^{k_F}(k_F{}^2-k_z{}^2)(\phi'^2-\phi\,\phi'')\,dk_z, \tag{5.18b}$$

$$\phi'=\partial\phi/\partial z. \tag{5.19}$$

The next step is to calculate the various integrals

I_0, I_1, etc., and sum the various contributions to the surface energy. The one-particle potential used here has an arbitrary parameter a_U which specifies the diffuseness. In order to calculate the correct surface energy, we must of course, minimize the calculated value of E_{surf} with respect to a_U.

This procedure was followed and the resulting minimum surface energy of 19.4 MeV was obtained at a diffuseness parameter $a_U = 0.60$ fm. The calculated E_{surf} is in excellent agreement with the empirical value[8] and also with the results obtained by Németh and Bethe.[6] This latter agreement is probably coincidental, since these authors used a realistic interaction radically different from the MDI, and also since they used the TFA, which was not done in the present work. The present particle density function obtained differs little from the Fermi form (same as Woods-Saxon). This is illustrated in Fig. 2. Thus if we fit $\hat{\rho}$ to the Fermi form,

$$\hat{\rho} = (1 + e^{z/a_\rho})^{-1} . \tag{5.20}$$

Then,

$$a_\rho \approx 0.8 a_U \approx 0.5 \text{ fm} . \tag{5.21}$$

This corresponds to a 90-to-10% surface thickness of 2.2 fm, in good agreement with the empirical values for finite nuclei, especially when the finite radius of the proton is taken into account.

It is also evident that the potential extends further out than the density, even apart from its larger surface thickness. This well-known feature is a consequence of the saturation feature (momentum and density dependence) of the MDI, or of any other saturating effective interaction.[4]

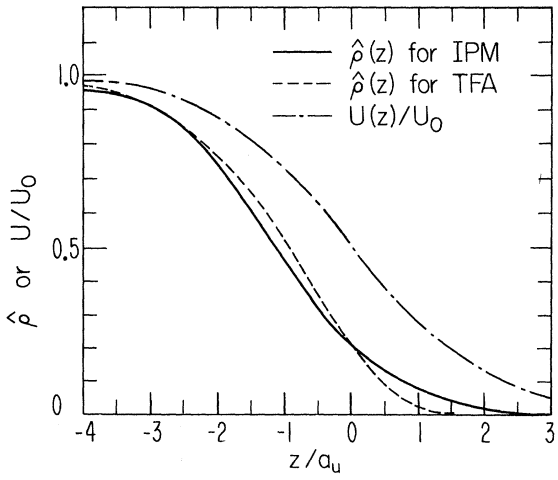

FIG. 2. Particle density and potential at the nuclear surface (semi-infinite-slab model), both in units of the nuclear-matter value, assuming static generating Woods-Saxon potential of depth 50 MeV and diffuseness constant 0.6 fm.

Finally, it has been verified that these results are quite insensitive to the depth of the Woods-Saxon potential. The minimum surface energy of 19.3 MeV (instead of 19.4) is obtained for a well depth of 57.5 MeV and diffuseness constant 0.6 fm. Of course the results may be different if the generating potential is momentum dependent, such as the one discussed in Sec. 3C. This point was not investigated for the semi-infinite-slab model. However, in our current calculations both static and momentum-dependent Woods-Saxon potentials are being treated.

6. THOMAS-FERMI APPROXIMATION (TFA)– SEMI-INFINITE-SLAB MODEL

A. Calculations Using a Single-Particle Potential as Starting Point

The Thomas-Fermi approximation was discussed in some detail by Bethe.[1] This approximation is valid to the extent that the density and one-particle potential are slowly varying functions of position. In the TFA, the particle density ρ can be immediately calculated in terms of the Fermi energy E_F and the single-particle potential $U(R)$. If U is static, then ρ is just the density

$$\rho(R) = \frac{2}{3\pi^2} k_F^3(R) = \frac{2}{3\pi^2} \left\{ \frac{2M[E_F - U(R)]}{\hbar^2} \right\}^{3/2} \tag{6.1}$$

which would be obtained if U were a constant. Even if U is momentum dependent, i.e., of the form (3.9), the first part of relation (6.1) still holds.[1]

The second simplification in the TFA is that the kinetic energy density is taken to be a simple function of ρ. Specifically

$$\tau(R) = \tfrac{3}{5} \rho(R) k_F^2(R) = \tfrac{3}{5} (\tfrac{3}{2} \pi^2)^{2/3} \rho^{5/3}(R) . \tag{6.2}$$

Once ρ and τ are calculated, the surface energy may be obtained by varying the diffuseness parameter so as to minimize the energy. Actually the value of a_U which gives minimum energy is close to 0.6 fm in both the IPM (Sec. 5) and the TFA. We chose this value in order to make a direct comparison between results obtained by these two methods.

The values of the integrals I_0, I_1, etc., defined in Sec. 5, are listed in Table III.

Thus the kinetic-energy and potential-energy terms are about 10% smaller than the IPM values, while the gradient term is about 10% larger. It

TABLE III. Values of surface integrals defined in Sec. 5 for IPM and TFA methods, assuming Woods-Saxon potential with $U_0 = 50$ MeV and $a_U = 0.6$ fm.

Integral	Term	IPM	TFA
I_0	τ	−0.582	−0.544
I_1	ρ^2	−0.842	−0.738
I_2	$\rho\tau$	−1.176	−1.042
I_3	$\rho^{8/3}$	−1.147	−1.042
I_g	$\nabla\rho$	+0.204	+0.231

should also be mentioned that all I_τ integrals calculated in the TFA are independent of the diffuseness parameter a_U. Thus from Eq. (5.11) it is clear that the two parts of the surface energy, the terms giving e_1 and those proportional to $(\nabla\rho)^2$, contribute equally. This is not the case in the IPM, where the e_1 term is about twice as large as the gradient term.

The TFA and IPM particle densities are plotted in Fig. 2. As can be seen, the TFA density (unlike the Fermi distribution), is not symmetric about its half-density point. Its maximum slope occurs at about $\hat{\rho} \sim \frac{1}{3}$ rather than $\frac{1}{2}$ as for the Fermi distribution. This asymmetry of the TFA density distribution was already noted by Bethe.[1] Also, $\hat{\rho}$ obtained in the TFA vanishes at the point where the classical kinetic energy vanishes, i.e., at the point R where $U(R) = E_F$. By comparison, the $\hat{\rho}$ obtained in the IPM has a longer tail. Bethe estimated that the TFA breaks down significantly where $\hat{\rho} < 0.15$, and he modified his TFA densities so as to roughly match the more realistic IPM values in the outer part of the nuclear surface. In comparing the results obtained using the IPM or TFA, we assumed a diffuseness parameter of the potential equal to 0.6 fm. However, for the TFA, the minimum surface energy = 16.0 MeV occurs at a slightly smaller diffuseness, namely $a_U = 0.56$ fm. This is equivalent to a surface thickness of the density equal to about $\underline{t} \sim 1.8$ to 2.1 fm, depending how this "thickness" is defined. Thus, for the MDI interaction considered here, the IPM treatment gives good agreement with the empirical surface energy and surface thickness, while the TFA underestimates both quantities significantly. In the Thomas-Fermi approximation the energy density is a function only of $\hat{\rho}$ and $(\nabla\hat{\rho})^2$. In fact, it is readily seen that the energy densities given in Eq. (5.5) are

$$e_1(\hat{\rho}) = \rho_0 \hat{\rho}\, W(\hat{\rho}) \tag{6.3}$$

and

$$e_0 = \rho_0 W(\rho_0). \tag{6.4}$$

Thus, in the TFA the surface energy can be expressed in the form:

$$E_{\text{surf}} = \frac{3A^{2/3}}{r_0}\left\{\int \hat{\rho}\,[W(\hat{\rho}) - W(\rho_0)]\,dz\right\} + \frac{3A^{2/3}}{r_0}\,\tfrac{3}{32}\beta\rho_0 \int \left(\frac{d\hat{\rho}}{dz}\right)^2 dz\,. \tag{6.5}$$

It was attempted by Weizsäcker[43] to simulate the deviation of the kinetic energy density τ from the TFA value by adding a term equal to

$$(\hbar^2/8M)(\nabla\rho)^2/\rho \tag{6.6}$$

to the TFA value of τ. Later Berg and Wilets[44] showed that such correction should be considerably smaller in order to fit the results of a more exact calculation. Thus for particles moving in a periodic potential such a gradient correction does appear, but with a coefficient of only $\frac{1}{9}$ the Weizsäcker term.[45] This result has also been obtained by several authors in work on atomic physics.[46] In the case of the nuclear surface, however, the deviation of τ from the TFA value as a function of position is more complicated. In fact the deviation is not even positive everywhere.[47]

B. Phenomenological TFA Calculations

We have seen that, for a modified δ interaction, the IPM reproduces the empirical surface thickness and surface energy better than the TFA based on a one-particle potential. It may be of interest to compare the IPM results also with those from other TFA calculations using analytical density distributions directly. This comparison will be made for the simple energy-density relation

$$W(\hat{\rho}) = |W_0|(-2\hat{\rho} + \hat{\rho}^2), \tag{6.7}$$

which reproduces Eq. (3.2) quite well, and for three different TFA models. The gradient term in the energy is, according to Eq. (5.5),

$$E_{\text{grad}} = \mu\rho_0^{-1}\int (\nabla\rho)^2 d^3R, \tag{6.8}$$

where

$$\mu = \tfrac{3}{32}\beta\rho_0 \tag{6.9}$$

has the value of 7.2 MeV fm^2 for the chosen value of β in the MDI. If the density distribution is a Fermi distribution [Eq. (5.20)], the surface energy is minimized when the diffuseness parameter a_ρ, defined by Eq. (5.20), is

$$a_\rho = (\mu/3|W_0|)^{1/2}. \tag{6.10}$$

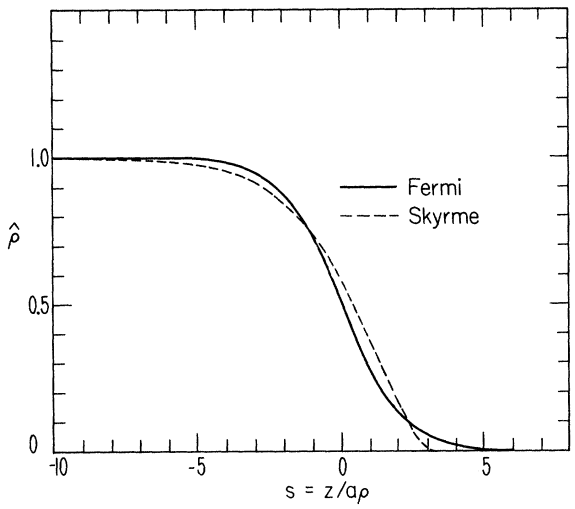

FIG. 3. Comparison of Fermi and Skyrme particle density distributions at the nuclear surface.

This model will be called the Fermi model. According to Skyrme,[24] the optimum form of the density distribution with Eqs. (6.7) and (6.8) is

$$\hat{\rho} = \tanh^2\left(1 - \frac{z - z_0}{\sqrt{12}\, a_\rho}\right) \quad \text{for } z < z_0 + \sqrt{12}\, a_\rho ,$$

$$= 0, \qquad \text{for } z > z_0 + \sqrt{12}\, a_\rho ,$$

where a_ρ is given by (6.10), and z_0 is the point where $\hat{\rho} = \tanh^2(1) = 0.580$. This distribution resembles that of Bethe[1] in the sense that it is not symmetrical about the half-density point. Figure 3 shows a comparison of the Fermi and Skyrme distributions. Finally, as the third model, we use Wilets's result[4] that the optimum distribution is just the Fermi form (5.20) if in the gradient term (6.8) $(\rho_0)^{-1}$ is replaced by $(3\rho)^{-1}$.

Using the MDI values of the parameters, one finds a surface parameter a_ρ of 0.39 fm, i.e., a surface thickness of only 1.7 fm, for all three models. The calculated surface energy,

$$E_{\mathrm{surf}} = (3A^{2/3}/r_0)\,|W_0|\,a_\rho , \tag{6.12}$$

is 16.5 MeV for the Fermi and Wilets models. For the Skyrme model, E_{surf} is less than (6.12) by 7.5%, i.e., 15.3 MeV. Thus we see that all three phenomenological TFA models give surface thickness and surface energy considerably lower than the value obtained with the IPM for the same compression modulus K and inhomogeneity coefficient μ.

The TFA results can be brought closer to the IPM values ($a_\rho = 0.5$ fm, $E_{\mathrm{surf}} = 19.3$ MeV) if one adjusts the values of μ and K. Thus if μ is multiplied by a factor 1.8, we obtain the much more reasonable values of $a_\rho = 0.5$ fm, $E_{\mathrm{surf}} = 21.0$ MeV (Fermi and Wilets) and 19.4 MeV (Skyrme). If the compression modulus K is also varied as in Wilets, by using a more general form of the energy-density relation than Eq. (6.7), then a fit to the IPM result can be obtained in the Wilets model with $\mu = 12$ MeV fm^2 and $K = 210$ MeV. However, these parameters are significantly different from those deduced from the MDI.

7. CONCLUSION

We have seen that a simple phenomenological three-parameter effective nucleon-nucleon interaction can correlate a number of nuclear properties surprisingly well. It is perhaps not unexpected that if one fits the ^{16}O ground-state energy, in addition to the nuclear matter properties, one also gets good agreement with the ground-state energies and rms radii of ^{4}He and ^{40}Ca. What is surprising, however, is that this interaction also fits the surface energy and nuclear density distribution of semi-infinite nuclear matter quite well, even though the effective interaction leads to a compression modulus of nuclear matter about twice as large as the conventional value of 150 to 200 MeV, given by most Brueckner-type calculations of nuclear matter. It is also significant that the calculated value of the surface energy (and surface thickness) using the Thomas-Fermi approximation is 10 to 15% less than the value obtained in the independent-particle calculation. Regarding the last point, it has, of course, not been proved here that the 3-MeV difference between the IPM and TFA results for the surface energy holds for an arbitrary effective interaction, even one which fits the saturation properties of nuclear matter. However, there is some indication from work by Köhler[19] that this difference may indeed be more general. This probably means that care has to be exercised in the use of the Thomas-Fermi approximation as applied to nuclear-structure problems.

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