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Meson Dynamics and the Nuclear Many-Body Problem. I. The Free-Nucleon Problem*

W. D. Brown,† R. D. Puff, and L. Wilets
University of Washington, Seattle, Washington 98105
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Relativistic nucleon-meson field theory is cast in a form appropriate to the nuclear many-body problem. Martin-Schwinger thermodynamic Green's functions are employed. The n-nucleon Green's functions satisfy a set of coupled equations which is formally identical to those of the usual potential many-body problem. Here, however, the two-body potential is replaced by a time-dependent interaction in which the mesons are formally eliminated in favor of higher-order nucleon correlations. A general program of increasing complexity for obtaining approximate solutions to the equations is discussed. It is necessary to solve the "vacuum" (one-, two-, etc., nucleon propagator) first. This has been done in the "Hartree-Fock" approximation which sums all diagrams containing a single continuous nucleon line with all possible uncrossed π and ω meson lines. Peaks in the spectral function are identified with masses of known N^* resonances. The appearance of ghost states, which arise in the process of mass and wave-function renormalization, is discussed. The resultant Green's function is used to calculate the magnetic moment of the nucleon, yielding a significantly better isovector component of the magnetic moment than the usual perturbation theory, although the isoscalar component is poor in both cases.

I. INTRODUCTION

The impressive accomplishments of two-body potential theories in the nuclear many-body problem, initiated by Brueckner, have yielded considerable insight into the structure of nuclear matter.[1] Numerical progress during the last decade, however, has been uneven. At present, the most sophisticated calculations employing the best phenomenological potentials still lack several MeV in the mean binding energy per nucleon.[2] The search for these "final" few MeV has concentrated on improvement of the nucleon-nucleon interaction, and the hunt for important neglected perturbation terms.[3]

There are three basic assumptions which define the starting point for nearly all current nuclear-matter calculations:

(1) Only nucleon dynamics are considered, usually nonrelativistically.

(2) Only two-body interactions are considered. There has been some investigation of three-body forces, but the results are inconclusive.[4]

(3) The two-body interactions are phenomenological fits to experimental two-body scattering and bound-state data, with guidance from theory. There have been moderately successful attempts to obtain similar potentials from meson theories directly. The potential matrix elements required in the many-body theories differ from those involved in scattering in that they are off the energy shell, and in that the Pauli principle alters the weights of the various intermediate states.[5]

The program presented in the present paper seeks to study the validity of each of these assumptions by circumventing them. The approach treats the relevant mesons and nucleons in a relativistic field theory without the introduction of two-nucleon forces explicitly. The problem can be expressed in a form which has the same formal *structure* as the many-nucleon (potential) problem.

We employ the Martin-Schwinger[6] Green's-function formalism in the present development. The method leads to an infinite set of coupled equations for successively higher-order Green's functions, completely equivalent to the field equations. The problem can be reduced to tractable form by approximating the higher-order Green's function in terms of the lower-order Green's functions. Various factorizations which preserve the important two-body correlations have been investigated,[7,8] the most attractive of which, for the nonrelativistic two-body potential problem, is obtained schematically from the factorization $G_3 \sim G_2G_1$, and is denoted by Λ_{10}. A recent study by Fiset[9] indicates that preserving three-nucleon correlations results in a correction to the Λ_{10} theory of less than 0.1 MeV per nucleon in the binding energy. This lends encouragement to a similar factorization in the present program which would again terminate the set at a tractable stage. The limitation to two-nucleon correlations does not exclude all the many-body force effects, and we anticipate that the effective two-nucleon potential depends upon properties of the many-body medium.

A system described by a relativistic field theory is intrinsically a many-body system; this is, of course, why field-theoretic methods have been extensively adopted by many-body theorists. Although the concept of a Dirac sea is of limited utility in a relativistic theory, there is a nice parallel with the Fermi sea of nuclear matter. One can view the Fermi sea as filled Fermi *and* Dirac seas of positive and negative energy levels. It should be emphasized, however, that the relativistic vacuum is the ground state of the system and that there is no need to view it as a filled sea of negative-energy nucleons. Charge-conjugation invariance implies that it would be equally valid to view it as a filled sea of negative-energy antinucleons.

The program, in stages of increasing complexity, may be outlined as follows:

A. Hartree-Fock Factorization, Coupling to Pions

Only pions are included with the Hartree-Fock factorization because they represent the longest-range forces and are relatively weak. The more massive mesons may be included later along with internucleon correlations. This phase is further divided:

(i) *One-nucleon propagator in vacuum.* At this stage, the one-particle Green's-function equation is derived and renormalization procedures are utilized to remove divergences in favor of the following finite quantities: nucleon mass, coupling constant, and wave-function normalization. The solution of the vacuum equation is required for the finite-density problem, but one also obtains single-nucleon properties, such as electromagnetic form factors and magnetic moments. We include the ω meson here because it presents no special problem.

(ii) *The finite-density problem.* The off-mass-shell (energy-shell) one-particle Green's function derived in the vacuum is required as input to the finite-density problem. It is here that many-body effects first appear, even though Hartree-Fock factorization is used: roughly speaking, one is including one-pion exchange, which involves intermediate-state pion and nucleon propagators which are, in turn, obtained self-consistently in the medium.

B. Two-Body Correlations, Various Mesons

The factorization $G_3 \sim G_2G_1$ preserves two-body correlations and leads to a T-matrix equation. The heavier (vector) mesons can be included here, since the resulting strong internucleonic interactions can be handled through the correlations. Again we can consider two cases:

(i) *Two-body propagator in vacuum.* G_2 describes two-body scattering and the bound state of the deuteron. The coupling constants and masses of the various mesons (the only parameters of the theory) can be adjusted to fit two-body scattering data and the deuteron binding energy. The deuteron magnetic moment, quadrupole moment, and form factors can then be obtained for comparison with experiment. Also available at this point would be improved neutron and proton moments and form factors.

(ii) *The finite-density problem.* This is the ultimate goal of the program, which includes the calculation of binding energy, density, symmetry energy, compressibility, etc. A comparison will be made with a similar Green's-function factorization theory which utilizes a nucleon-nucleon potential, in order to exhibit the effects of meson-nucleon dynamics.

The present paper, which is the first in a series, presents the general formulation of the program and the implementation of that program through (A.i), that is, the Hartree-Fock one-body problem.

A nonrelativistic approach to the same problem has been presented by Dover and Lemmer.[10] The philosophy is similar in that they also obtain time-dependent interactions to replace meson dynamics.

II. GENERAL FORMULATION

We consider a system of nucleons and, for example, π mesons described by the Hamiltonian

$$H = H_N + H_\pi + H', \tag{1}$$

where

$$H_N = \tfrac{1}{2}\int d^3r\{\bar\psi_\zeta(x)\,(\vec\gamma\cdot\vec{\mathrm{p}}+M_0)_{\zeta\zeta'}\psi_{\zeta'}(x) - [(\vec\gamma\cdot\vec{\mathrm{p}}+M_0)_{\zeta\zeta'}\psi_{\zeta'}(x)]\bar\psi_\zeta(x)\}\,, \tag{2}$$

$$H_\pi = \frac{1}{2}\int d^3r\{\pi^j\pi^j + \vec{\nabla}\phi^j\cdot\vec{\nabla}\phi^j + m_{0\pi}{}^2\phi^j\phi^j\}, \tag{3}$$

$$H' = \frac{1}{2}g_{0\pi}\int d^3r[\bar{\psi}_\zeta(x)(\tau_j\gamma_5)_{\zeta\zeta'}\psi_{\zeta'}(x)\phi^j(x) - \psi_{\zeta'}(x)(\tau_j\gamma_5)_{\zeta\zeta'}\bar{\psi}_\zeta(x)\phi^j(x)]. \tag{4}$$

The following notation is used: ζ, ζ', etc., refer to both Dirac spin and isospin indices of the nucleons. Where they are required separately, we will use α, α', etc., for spin and β, β', etc., for isospin.

The j index refers to the three (Hermitian) components of the meson field [the neutral pion field $\phi_0 = \phi^3$ and the charged pion fields $\phi_\pm = 2^{-1/2}(\phi^1 \pm i\phi^2)$].

We use the summation convention for repeated indices. π^j is the operator canonically conjugate to the field operator ϕ^j.

We further employ the following conventions:

$$AB = A^\mu B_\mu = A_\mu B_\nu g^{\mu\nu} = \vec{A}\cdot\vec{B} - A_0B_0,$$

$$(g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix},$$

$$x = (t, \vec{r}),\quad \partial^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2,\quad \hbar = c = 1.$$

$$\bar{\psi} = \psi^\dagger\gamma_0,$$

$$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3,$$

$$\{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu},$$

$$\gamma_0{}^2 = 1,\quad \gamma_i{}^2 = -1,\quad \gamma_5{}^2 = -1,$$

$$\{\gamma_5, \gamma_\mu\} = 0,$$

$$\int_{q^3} = \int\frac{d^3q}{(2\pi)^3},\quad \int_{q^4} = \int\frac{d^4q}{(2\pi)^4}. \tag{5}$$

Notice that we have assumed nucleon-meson coupling, but no direct meson-meson coupling. Meson-meson coupling can appear through nucleon antinucleon channels.

The fields satisfy the equal-time commutation relations

$$\{\psi_\zeta(\vec{r}, t), \bar{\psi}_{\zeta'}(\vec{r}', t)\} = (\gamma_0)_{\alpha\alpha'}\delta_{\beta\beta'}\delta^3(\vec{r} - \vec{r}'), \tag{6}$$

$$[\pi^j(\vec{r}, t), \phi^{j'}(\vec{r}', t)] = -i\delta_{jj'}\delta^3(\vec{r} - \vec{r}'), \tag{7}$$

with other combinations commuting or anticommuting as usual.

The field equations follow from the above relations and $i\dot{\Theta} = [\Theta, H]$.

$$(\gamma p + M_0)_{\zeta\zeta'}\psi_{\zeta'}(x) = -g_{0\pi}(\tau_j\gamma_5)_{\zeta\zeta'}\psi_{\zeta'}(x)\phi^j(x),$$

$$(-\partial^2 + m_{0\pi}{}^2)\phi^j(x) = -\frac{1}{2}g_{0\pi}[\bar{\psi}_\zeta(x), (\tau_j\gamma_5)_{\zeta\zeta'}\psi_{\zeta'}(x)],$$

$$\pi^j = \dot{\phi}^j. \tag{8}$$

At this point we note that the above equations may be generalized to include a wide class of mesons interacting through nonderivative nucleon-meson coupling. In a compact notation, the field equations become

$$(\gamma p + M_0)_{\zeta\zeta'}\psi_{\zeta'}(x) = -\Omega^j_{\zeta\zeta'}\psi_{\zeta'}(x)\phi^j(x),$$

$$(-\partial^2 + m_{0j}{}^2)\phi^j(x) = -\frac{1}{2}[\bar{\psi}_\zeta(x), \Omega^j_{\zeta\zeta'}\psi_{\zeta'}(x)]. \tag{9}$$

(Note that for vector mesons there is also a modification of the commutation relations and a subsidiary condition.) Here, the index j refers not only to the field component of a meson family but also to the family itself. We list in Table I the forms of Ω for the various mesons which are supposed to contribute to nuclear forces. In what follows, we will use the Ω notation wherever the equations are of sufficient generality to warrant it.

Nucleon Green's functions are defined by

$$G_n(1\cdots n; 1'\cdots n') = i^n\langle T\{\psi(1)\cdots\psi(n)\bar{\psi}(n')\cdots\bar{\psi}(1')\}\rangle, \tag{10}$$

where T is the Wick time-ordering symbol, which includes a factor $(-)^p$, where p is the number of permutations in going from the time-ordered sequence to that shown above. The index (n) refers to $(\vec{r}_n, t_n, \zeta_n)$, and the expectation value of some operator X is

$$\langle X\rangle = \frac{\mathrm{tr}\, e^{-\beta(H-\mu N-\nu I)}X}{\mathrm{tr}\, e^{-\beta(H-\mu N-\nu I)}}; \tag{11}$$

in the usual thermodynamic limit $\langle N\rangle = A \to \infty$, $\langle I\rangle = \frac{1}{2}A - Z \to \infty$, with the volume $V \to \infty$ in such a way that the total particle density A/V and proton density Z/V are finite. We will eventually take the (zero-temperature) limit $\beta \to \infty$, so as to obtain ground-state averages for a system with fixed A/V and Z/V. The quantities μ and ν play the roles of the chemical potentials or Lagrange multipliers related to the average densities. The number and isospin operators are defined by

TABLE I. Forms of Ω for various mesons.

$\Omega^j_{\zeta_1\zeta_2}$	j values	Meson	J^P	I
$g_{0\pi}(\gamma_5)_{\alpha_1\alpha_2}(\tau_j)_{\beta_1\beta_2}$	1, 2, 3	π	0^-	1
$g_{0\eta}(\gamma_5)_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}$	4	η	0^-	0
$g_{0\omega}(\gamma_\mu)_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}$	$5+\mu$	ω	1^-	0
$g_{0\phi}(\gamma_\mu)_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}$	$9+\mu$	ϕ	1^-	0
$g_{0\rho}(\gamma_\mu)_{\alpha_1\alpha_2}(\tau_k)_{\beta_1\beta_2}$	$13+4(k-1)+\mu$	ρ	1^-	1
$g_{0\sigma}\delta_{\zeta_1\zeta_2}$	25	σ?	0^+	0

$$N=\tfrac{1}{2}(\gamma_0)_{\alpha\alpha'}\delta_{\beta\beta'}\int d^3r[\bar{\psi}_\zeta(x),\psi_{\zeta'}(x)],$$

$$I_3=\tfrac{1}{4}(\gamma_0)_{\alpha\alpha'}(\tau_3)_{\beta\beta'}\int d^3r[\bar{\psi}_\zeta(x),\psi_{\zeta'}(x)]+\int d^3r\, i\pi^j(x)t_3^{jj'}\phi^{j'}(x). \quad (12)$$

The vacuum problem is defined by $A/V=Z/V=0$, or $\mu=\nu=0$.

Since N and I commute with the Hamiltonian, we may choose to employ the operator $\mathcal{H}=H-\mu N-\nu I$ in the dynamical equation $i\dot{\mathcal{O}}=[\mathcal{O},\mathcal{H}]$. This introduces a phase factor in the field operators and produces a trivial change in the field equations (8) or (9):

$$\gamma p+M_0\to\gamma p+M_0-\gamma_0\mu-\tfrac{1}{2}\gamma_0\tau_3\nu$$

and

$$-\partial^2+m_{0j}{}^2\to-\partial^2+m_{0j}{}^2-\nu^2(\delta_{j1}+\delta_{j2})\equiv\partial^2+\bar{m}_{0j}{}^2.$$

The function G_1 then satisfies

$$\begin{aligned}(\gamma p_1-\gamma_0\mu-\tfrac{1}{2}\gamma_0\tau_3\nu+M_0)_{\zeta_1\zeta_2}G_1(x_1,\zeta_2;x_{1'}\zeta_{1'})&=\delta(11')-i\Omega^j_{\zeta_1\zeta_3}\langle T\{\psi_{\zeta_3}(x_1)\phi^j(x_1)\bar{\psi}_{\zeta_{1'}}(x_{1'})\}\rangle\\&=\delta(11')-i\int d^4x_3\Omega^j_{\zeta_1\zeta_3}\delta(x_1-x_3)\\&\quad\times\langle T\{\psi_{\zeta_3}(x_3)\phi^j(x_1)\bar{\psi}_{\zeta_{1'}}(x_{1'})\}\rangle.\end{aligned}\quad (13)$$

The meson field equation then produces a similar expression for the quantity on the right-hand side of (13),

$$\begin{aligned}(-\partial_1^2+\bar{m}_{0j}{}^2)\,\langle T\{\psi_{\zeta_3}(x_3)\phi^j(x_1)\bar{\psi}_{\zeta_{1'}}(x_{1'})\}\rangle&=-\tfrac{1}{2}\Omega^j_{\zeta_2\zeta_4}\langle T\{\psi_{\zeta_3}(x_3)[\bar{\psi}_{\zeta_2}(x_1),\psi_{\zeta_4}(x_1)]\bar{\psi}_{\zeta_{1'}}(x_{1'})\}\rangle\\&=\int d^4x_2d^4x_4\Omega^j_{\zeta_2\zeta_4}\delta(x_1-x_4)\delta(x_2-x_4)G_2(34;2^\pm1'),\end{aligned}\quad (14)$$

where

$$G_2(34;2^\pm1')=\tfrac{1}{2}[G_2(34;2^+1')+G_2(34;2^-1')],$$

and

$$n^\pm\equiv(\vec{r}_n,t_n\pm0^+,\zeta_n).$$

We also define the one-meson Green's function

$$\mathcal{G}^{jj'}(x;x')=i\langle T\{\phi^j(x)\phi^{j'}(x')\}\rangle, \quad (15)$$

and note that for Bose fields the Wick time-ordering symbol does not include the factor $(-)^P$. The equation of motion for $\mathcal{G}$ is

$$(-\partial_1^2+\bar{m}_{0j}{}^2)\mathcal{G}^{jj'}(x_1;x_1{}')=\delta_{jj'}\delta(x_1-x_{1'})-i\Omega^j_{\zeta_1\zeta_2}\times\langle T\{\tfrac{1}{2}[\bar{\psi}_{\zeta_1}(x_1),\psi_{\zeta_2}(x_1)]\phi^{j'}(x_{1'})\}\rangle. \quad (16)$$

[No sum over j is implied on the left-hand side of (14).] In order to invert (16), we introduce the noninteracting meson function $\delta_{jj'}\mathcal{G}_0^j(x_1x_{1'})$, which satisfies the equation

$$(-\partial_1^2+\bar{m}_{0j}{}^2)\,\mathcal{G}_0^j(x_1x_{1'})=\delta(x_1,x_{1'}), \quad (17)$$

subject to the same boundary conditions as the general $\mathcal{G}^{jj}$, namely periodicity in the time variable,

$$\mathcal{G}^{jj'}(\vec{r},t-i\beta;\vec{r}',t')=\mathcal{G}^{jj'}(\vec{r},t;\vec{r}',t'), \quad (18)$$

as can be demonstrated directly from its definition. If we introduce the space Fourier transform[11]

$$\mathcal{G}^{jj'}(x_1x_{1'})=\int_{p^3}e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_{1'})}\mathcal{G}^{jj'}(\vec{p},t_1-t_{1'}), \quad (19)$$

then the explicit solution which satisfies the boundary condition (18) is

$$\mathcal{G}_0^j(p,t-t')=\frac{i}{2w_{0j}}\left(\frac{e^{iw_{0j}|t-t'|}}{e^{\beta w_{0j}}-1}-\frac{e^{-iw_{0j}|t-t'|}}{e^{-\beta w_{0j}}-1}\right),$$

$$\xrightarrow[\beta\to\infty]{}\frac{i}{2w_{0j}}\,e^{-iw_{0j}|t-t'|}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{e^{-iw(t-t')}}{w_{0j}{}^2-\omega^2-i\epsilon}, \quad (20)$$

where $w_{0j}=(\vec{p}^{\,2}+\bar{m}_{0j}{}^2)^{1/2}$. We define the Fourier transform

$$\mathcal{G}_0^j(\vec{p},p_0)=[\vec{p}^{\,2}-p_0^2+\bar{m}_{0j}{}^2-i\epsilon]^{-1},$$

which is valid for scalar or pseudoscalar mesons. If we return now to the space-time function, utilizing the final form in (20), we find

$$\mathcal{G}_0^j(\vec{r}t;\vec{r}'t')=-\frac{\bar{m}_{0j}}{8\pi^2}\,\frac{1}{|r^2-\bar{t}^{\,2}|}\times\begin{cases}-2iK_1(\bar{m}_{0j}(r^2-\bar{t}^{\,2})^{1/2}), & r>|\bar{t}|,\\ \pi H_1^{(2)}(\bar{m}_{0j}(\bar{t}^{\,2}-r^2)^{1/2}), & r<|\bar{t}|,\end{cases} \quad (21)$$

with

$$r=|\vec{r}-\vec{r}'| \quad\text{and}\quad \bar{t}=t-t'.$$

We see that $\mathcal{G}_0^j$ is an even function of $t-t'$, and is thus symmetric advanced-retarded.

It is a simple matter to show that the function

$$\langle T\{\psi_{\zeta_3}(x_3)\phi^j(x_1)\bar{\psi}_{\zeta_{1'}}(x_{1'})\}\rangle,$$

satisfies the periodic boundary condition in the index t, exhibited by (18). Therefore, the inversion of (14) yields

$$\langle T\{\psi_{\zeta_3}(x_3)\phi^j(x_1)\bar{\psi}_{\zeta_{1'}}(x_{1'})\}\rangle=\Omega^j_{\zeta_2\zeta_4}\delta(x_2-x_4)\mathcal{G}_0^j(x_1-x_4)G_2(34;2^\pm1'), \quad (22)$$

where integrations over repeated space-time indices, as well as summations over repeated subscripts, are implied. The integrations over the time index t_4 may run between any surfaces on which the boundary conditions may be imposed; in particular, we choose the range 0 to $-i\beta$. The same limits can be assigned to *all* repeated time index integrations, since those not related to the boundary conditions involve δ functions. Substitution of (22) into (13) now gives

$$G_1^{0\,-1}(12)G_1(21') = \delta\,(11') + i\langle 12|v|34\rangle G_2(34;\,2^{\pm}1'), \tag{23}$$

with

$$G_1^{0\,-1}(12) = (\gamma p_1 - \gamma_0\mu - \tfrac{1}{2}\gamma_0\tau_3\nu + M_0)_{\zeta_1\zeta_2}\delta(x_1 - x_2), \tag{24}$$

and

$$\langle 12|v|34\rangle \equiv -\sum_j \Omega^j_{\zeta_1\zeta_3}\Omega^j_{\zeta_2\zeta_4}\delta(x_1 - x_3)\mathcal{G}^j_0(x_1 - x_2)\delta(x_2 - x_4). \tag{25}$$

We note at this point that the meson field has been eliminated formally from the problem. The one-nucleon Green's function is coupled directly to the two-nucleon Green's function by an explicit time-dependent "potential" $\langle 12|v|34\rangle$. If $\langle 12|v|34\rangle$ contained a $\delta(t_1 - t_4)$ instead of the $(t_1 - t_4)$ dependence given by (20) or (21), (23) would correspond to the first equation in the Green's-function hierarchy for particles interacting through an ordinary two-body potential. The higher-order equations of this hierarchy are familiar in the static potential problem, and our notation in (23)–(25) is such that the same equations appear here. Thus, we find, by an analysis similar to that above, the result

$$G_1^{0\,-1}(11'')G_n(1''2\cdots n;\,1'\cdots n') = \sum_{l=1}^{n}(-1)^{l+1}\delta(1l')G_{n-1}(2\cdots n;\,1'\cdots \text{omit } l'\cdots n')$$
$$+\,i\langle 1, n+1|v|n+2, n+3\rangle G_{n+1}[23\cdots n, n+2, n+3;\,(n+1)^{\pm}1'\cdots n']. \tag{26}$$

It is this set of nuclear Green's-functions equations which we must solve in some approximation scheme.

Although we have formally eliminated the meson field to find the nucleon propagators, it will be necessary to calculate the one-meson propagator $\mathcal{G}^{jj'}$ in order to evaluate the meson contribution to the energy. We may do this in terms of nucleon propagators alone. The quantity appearing on the right-hand side of Eq. (16) satisfies

$$(-\partial_{1'}^2 + \overline{m}_{0j'}{}^2)\tfrac{1}{2}\langle T\{[\overline{\psi}_{\zeta_1}(x_1),\,\psi_{\zeta_2}(x_1)]\phi^{j'}(x_{1'})\}\rangle = -\Omega^{j'}_{\zeta_3\zeta_4}\tfrac{1}{4}\langle T\{[\overline{\psi}_{\zeta_1}(x_1),\,\psi_{\zeta_2}(x_1)][\overline{\psi}_{\zeta_3}(x_{1'}),\,\psi_{\zeta_4}(x_{1'})]\}\rangle. \tag{27}$$

It can be shown that the boundary conditions on the time $t_{1'}$ are exactly those of (18) (i.e., periodic), so that the inversion of (27) can be carried out immediately. There is no inhomogeneous term in this inversion as a result of the fact that the ϕ and ψ fields commute at equal times. Inserting the result of this inversion into (16), and further inverting the resulting expression, we obtain

$$\mathcal{G}^{jj'}(x_1 - x_{1'}) = \delta_{jj'}\,\mathcal{G}^j_0(x_1 - x_{1'}) - i\,\mathcal{G}^j_0(x_1 - x_2)\Omega^j_{\zeta_1\zeta_2}\,\Omega^{j'}_{\zeta_3\zeta_4}\,G_2(x_2\zeta_2, x_3\zeta_4; x_2{}^{\pm}\zeta_1, x_3{}^{\pm}\zeta_3)\,\mathcal{G}^{j'}_0(x_3 - x_{1'}). \tag{28}$$

III. EXPRESSIONS FOR THE ENERGY AND DENSITY

We can now express the energy and particle densities in terms of nucleon Green's functions. From (12) the nucleon number is given by

$$\frac{\langle N\rangle}{V} = \tfrac{1}{2}i\sum_{\zeta_1\zeta_2}(\gamma_0)_{\alpha_2\alpha_1}\delta_{\beta_2\beta_1}[G_1(x_1\zeta_1;\,x_1{}^{+}\zeta_2) + G_1(x_1{}^{+}\zeta_1;\,x_1\zeta_2)]. \tag{29}$$

In this section we consider the energy for the special case of π mesons only and $Z = N(\nu = 0)$. The energy is conveniently broken into two parts. The first of these involves the nucleon and interaction terms. Utilizing the equations of motion, these can be shown to be given by

$$\frac{\langle H_N\rangle + \langle H'\rangle}{V} = \tfrac{1}{2}\sum_{\zeta_1\zeta_2}(\gamma_0)_{\alpha_2\alpha_1}\delta_{\beta_2\beta_1}\left(i\frac{\partial}{\partial t_1} + \mu\right)\langle\overline{\psi}_{\zeta_2}(r_1t_2)\psi_{\zeta_1}(r_1t_1) - \psi_{\zeta_1}(r_1t_1)\overline{\psi}_{\zeta_2}(r_1t_2)\rangle\Big|_{t_2 = t_1}$$
$$= \tfrac{1}{2}i\sum_{\zeta_1\zeta_2}(\gamma_0)_{\alpha_2\alpha_1}\delta_{\beta_2\beta_1}\left\{\left[\left(i\frac{\partial}{\partial t_1} + \mu\right)G_1(x_1\zeta_1;\,x_2\zeta_2)\right]_{x_2 = x_1{}^{+}} + \left[\left(i\frac{\partial}{\partial t_1} + \mu\right)G_1(x_1\zeta_1;\,x_2\zeta_2)\right]_{x_2 = x_1{}^{-}}\right\}. \tag{30}$$

The meson energy, according to (3), is given by

$$\langle H_\pi\rangle = \tfrac{1}{2}\sum_j\left\langle\int d^3r(\pi^j\pi^j + \vec{\nabla}\phi^j\cdot\vec{\nabla}\phi^j + m_{0\pi}{}^2\phi^j\phi^j)\right\rangle.$$

Using the expression $\pi^j = \dot{\phi}^j$, together with the time- and space-translation invariance of the expectation value, we may rewrite this energy in the form

$$\langle H_\pi \rangle = \tfrac{1}{2}\sum_j \int d^3r \left[-\frac{\partial^2}{\partial t^2} - \nabla^2 + m_{0\pi}{}^2 \right] \langle \phi^j(r't')\phi^j(rt)\rangle \Big|_{x'=x}$$

$$= -\tfrac{1}{2} i \sum_j \left(-\partial^2 + m_{0\pi}{}^2 - 2\frac{\partial^2}{\partial t^2} \right) \mathcal{G}^{jj}(\vec{r} - \vec{r}', t - t')\Big|_{x'=x^+} . \quad (31)$$

Utilizing (28) to eliminate the unknown meson Green's function, we arrive at

$$\langle H_\pi \rangle = i \int d^3r_1 \frac{\partial^2}{\partial t_1{}^2} \mathcal{G}_0^j(x_1 - x_{1'})\Big|_{x_1' = x_1^+} + \sum_j \int d^3 r_1 \left[\frac{\partial^2}{\partial t_1{}^2} \mathcal{G}_0^j(x_1 - x_2) - \tfrac{1}{2}\delta(x_1 - x_2) \right]$$

$$\times \Omega^j_{\zeta_1\zeta_2} \Omega^j_{\zeta_3\zeta_4} G_2(x_2\zeta_2, x_3\zeta_4; x_2{}^+\zeta_1, x_3\zeta_3)\, \mathcal{G}_0^j(x_3 - x_{1'})\Big|_{x_{1'} = x_1{}^+} . \quad (32)$$

The first term on the right-hand side of (32) is just the zero-point (vacuum) energy for the noninteracting meson field. Using (20), this can be shown to be

$$3V\int_{p^3} \tfrac{1}{2}(\vec{p}^{\,2} + m_{0\pi}{}^2)^{1/2} .$$

Since we are concerned only with energies relative to the vacuum, this (infinite) term may be ignored immediately.

Combining (30) and (32), we obtain

$$\frac{\langle H \rangle}{V} = \tfrac{1}{2} i \sum_{\zeta_1\zeta_2} (\gamma_0)_{\alpha_2\alpha_1} \delta_{\beta_2\beta_1} \left\{ \left[\left(i\frac{\partial}{\partial t_1} + \mu \right) G_1(x_1\zeta_1; x_2\zeta_2) \right]_{x_2 = x_1+} + \left[\left(i\frac{\partial}{\partial t_1} + \mu \right) G_1(x_1\zeta_1; x_2\zeta_2) \right]_{x_2 = x_1-} \right\}$$

$$+ \sum_{\substack{j \\ \zeta_1\zeta_2\zeta_3\zeta_4}} \left[\frac{\partial}{\partial t_1{}^2} \mathcal{G}_0^j(x_1 - x_2) - \tfrac{1}{2}\delta(x_1 - x_2) \right] \Omega^j_{\zeta_1\zeta_2} \Omega^j_{\zeta_3\zeta_4} G_2(x_2\zeta_2, x_3\zeta_4; x_2{}^+\zeta_1, x_3{}^+\zeta_3)\, \mathcal{G}_0^j(x_3 - x_{1'})\Big|_{x_{1'} = x_1{}^+} . \quad (33)$$

All of the terms on the right-hand side of (33) are, in fact, infinite in the vacuum limit. These infinities are associated with the occupation of the negative-energy nucleon states, and with the renormalization of the meson masses. In the calculations on finite-density systems, we will subtract out the vacuum ($\mu = \nu = 0$) energy, and will deal only with $(\langle H \rangle_{\mu,\nu} - \langle H \rangle_{0,0})/V$, which, after sufficient renormalization, should be finite in a consistent theory.

IV. SPECTRAL REPRESENTATION OF G_1

A. General Properties

G_1 may be decomposed into components $G^>$ and $G^<$ according to

$$G_1(x - x') = \theta(t - t')G^>(x - x') + \theta(t' - t)G^<(x - x') , \quad (34)$$

where

$$G^>_{\zeta\zeta'}(x - x') = i\langle \psi_\zeta(x)\bar{\psi}_{\zeta'}(x')\rangle ,$$

$$G^<_{\zeta\zeta'}(x - x') = -i\langle \bar{\psi}_{\zeta'}(x')\psi_\zeta(x)\rangle . \quad (35)$$

From the definition of the thermodynamic average $\langle\ \rangle$, it can be shown that $G^>$ and $G^<$ satisfy the boundary condition

$$G^>(\vec{r}, -i\beta - t') = -G^<(\vec{r}, 0 - t') . \quad (36)$$

This immediately permits a Fourier analysis of $G^>$ and $G^<$ in terms of a single function, G_a:

$$G^>(\vec{p}, t) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_+(-\omega) G_a(\vec{p}, \omega) e^{-i\omega t} ,$$

$$G^<(\vec{p}, t) = -i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_+(\omega) G_a(\vec{p}, \omega) e^{-i\omega t} , \quad (37)$$

where the spatial coordinate has also been Fourier transformed, although the equations are of identical form in $(\vec{r}, t)$ space. The Fermi function is given by

$$f_\pm(\omega) = [e^{\beta\omega} \pm 1]^{-1} \xrightarrow[\beta\to\infty]{} \pm\theta(-\omega) . \quad (38)$$

Since $f_+(\omega) + f_+(-\omega) = 1$, it follows that $G_a(\vec{p}, \omega)$ is the Fourier transform of $\langle\{\psi(x), \bar{\psi}(x')\}\rangle$. We use the identity

$$\theta(t) = i\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{e^{-i\omega' t}}{\omega' + i\epsilon}$$

and (34) and (37) to obtain G_1 in terms of $G_a(\beta\to\infty)$

$$G_1(\vec{p},p_0)=\int_{-\infty}^{\infty} G_1(\vec{p},t)e^{ip_0t}dt$$
$$=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\frac{\theta(-\omega)}{\omega-p_0+i\epsilon}+\frac{\theta(\omega)}{\omega-p_0-i\epsilon}\right]G_a(\vec{p},\omega)$$
$$=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{G_a(\vec{p},\omega)}{\omega-p_0(1+i\epsilon)}. \tag{39}$$

This suggests introducing a function which analytically continues G_1 in p_0

$$\hat{G}(\vec{p},z)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{G_a(\vec{p},\omega)}{\omega-z}, \tag{40}$$

$$G_a(\vec{p},\omega)=-i[\hat{G}(\vec{p},\omega+i\epsilon)-\hat{G}(\vec{p},\omega-i\epsilon)], \tag{41}$$

$$G_1(\vec{p},p_0)=\hat{G}(\vec{p},p_0(1+i\epsilon)). \tag{42}$$

From the anticommutation relations, it follows that

$$G_a(\vec{r}-\vec{r}',t=t')=\gamma_0\delta(\vec{r}-\vec{r}'),$$

or

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_a(\vec{p},\omega)=\gamma_0. \tag{43}$$

B. Special Vacuum Representation

Because of the Lorentz and time-reversal invariance of the vacuum, the Dirac matrices can enter into G_1 only in the combination γp and 1. As an alternative to (40), we can write[12]

$$\hat{G}(\vec{p},z)=\int_{-\infty}^{\infty}d\kappa\frac{A(\kappa)}{\vec{\gamma}\cdot\vec{p}-\gamma_0z+\kappa}$$
$$=\int_{-\infty}^{\infty}d\kappa A(\kappa)\frac{\vec{\gamma}\cdot\vec{p}-\gamma_0z-\kappa}{z^2-\vec{p}^2-\kappa^2}, \tag{44}$$

or

$$G_1(p)=\hat{G}(\vec{p},p_0(1+i\epsilon))$$
$$=\int_{-\infty}^{\infty}d\kappa A(\kappa)\frac{\gamma p-\kappa}{-p^2-\kappa^2+i\epsilon}. \tag{45}$$

These equations essentially represent the Green's functions as a sum over all intermediate states; the states with $\kappa>0$ have normal parity, those with $\kappa<0$ have abnormal parity. At this point it is convenient to introduce the projection operators

$$P_\pm(p)=\tfrac{1}{2}\left(1\mp\frac{\gamma p}{w_p}\right)\text{ or }\begin{cases}\gamma p=(P_--P_+)w_p\\ 1=P_++P_-\end{cases}. \tag{46}$$

The sign of $(-p^2)^{1/2}$ is defined by convention to be

$$w_p\equiv(-p^2)^{1/2}\equiv\begin{cases}|(-p^2)^{1/2}|, & -p^2>0,\\ i\,|(p^2)^{1/2}|, & -p^2<0.\end{cases} \tag{47}$$

Thus we can write

$$G_1(p)=\int_{-\infty}^{\infty}d\kappa A(\kappa)\left(\frac{(w_p-\kappa)P_--(w_p+\kappa)P_+}{(w_p+i\epsilon)^2-\kappa^2}\right)$$
$$=\int_{-\infty}^{\infty}d\kappa A(\kappa)\left(\frac{P_-}{\kappa+(w_p+i\epsilon)}+\frac{P_+}{\kappa-(w_p+i\epsilon)}\right)$$
$$=P_-\tilde{G}(-w_p-i\epsilon)+P_+\tilde{G}(w_p+i\epsilon), \tag{48}$$

where

$$\tilde{G}(z)=\int_{-\infty}^{\infty}d\kappa\frac{A(\kappa)}{\kappa-z}. \tag{49}$$

It follows from the commutation relations that

$$\int_{-\infty}^{\infty}d\kappa A(\kappa)=1. \tag{50}$$

The projection operators permit a ready inversion of $G_1(p)$ in terms of $\tilde{G}$:

$$G_1^{-1}(p)=P_+\tilde{G}^{-1}(w_p+i\epsilon)+P_-\tilde{G}^{-1}(-w_p-i\epsilon). \tag{51}$$

In the noninteracting case,

$$\tilde{G}_0^{-1}(z)=M_0-z, \tag{52}$$

and it follows from the Lehmann-Källen form for $\tilde{G}$, (49), that

$$\tilde{G}^{-1}(z)=M_0-z+\Sigma(z), \tag{53}$$

where

$$\Sigma(z)=-\int_{-\infty}^{\infty}d\kappa\frac{T(\kappa)}{\kappa-z}. \tag{54}$$

We know that $\tilde{G}(z)$ must have a pole at $z=M$

$$0=M_0-M-\int_{-\infty}^{\infty}d\kappa\frac{T(\kappa)}{\kappa-M}. \tag{55}$$

Subtracting (55) from (54), we obtain

$$\tilde{G}^{-1}(z)=(M-z)\left[1+\int_{-\infty}^{\infty}d\kappa\frac{T(\kappa)}{(\kappa-M)(\kappa-z)}\right]. \tag{56}$$

Until this point we have not discussed the necessity of introducing a cutoff in order to insure convergence of the quantities with which we are dealing. We anticipate here the possibility that, for certain approximations at least, A and G may vanish as the cutoff recedes to infinity. In that case, the unrenormalized quantities will not exist, but the renormalized quantities may. Then, the above discussion goes through as before, except that the Lehmann-Källen form may need subtractions; however, the dispersion relation for Σ *cannot possibly need a subtraction unless*

$$\int_{-\infty}^{\infty}d\kappa A(\kappa)=0.$$

But $A(\kappa)$ is supposed to be positive definite; thus, a second subtraction for Σ implies that the sum over states is not positive definite, i.e., we have a ghost.

It is convenient, in any case, to use renormalized quantities such that G_R has a pole with unit residue at $z=M$. We have demanded that $G(z)$ have a pole at $z=M$. The residue of the pole is

$$Z_2=\frac{M-z}{\tilde{G}^{-1}(z)}\bigg|_{z=M}=\left[1+\int_{-\infty}^{\infty}d\kappa\,\frac{T(\kappa)}{(\kappa-M)^2}\right]^{-1}. \tag{57}$$

We define a renormalized $\tilde{G}$

$$\begin{aligned}\tilde{G}_R^{-1}(z)&\equiv Z_2\tilde{G}^{-1}(z)\\&=(M-z)\left[1-(M-z)\int_{-\infty}^{\infty}d\kappa\,\frac{T_R(\kappa)}{(\kappa-M)^2(\kappa-z)}\right],\end{aligned} \tag{58}$$

where $T_R(\kappa)=Z_2T(\kappa)$. The spectral form for $\tilde{G}_R$ is

$$\tilde{G}_R(z)=\frac{\tilde{G}(z)}{Z_2}=\int_{-\infty}^{\infty}d\kappa\,\frac{A_R(\kappa)}{\kappa-z}, \tag{59}$$

where $A_R(\kappa)=A(\kappa)/Z_2$. It follows from the canonical commutation relations and (48) that

$$\int_{-\infty}^{\infty}d\kappa\,A_R(\kappa)=Z_2^{-1}. \tag{60}$$

V. HARTREE-FOCK IN VACUO

A. Approximation

Equation (23) for G_1 in terms of G_2 can be written

$$G_1^{-1}(11')=G_1^{0\,-1}(11')+i\langle 12|T|1'3\rangle G_1(32^{\pm}), \tag{61}$$

where T is defined by

$$\langle 12|v|34\rangle G_2(34;56)\equiv\langle 12|T|34\rangle G_1(35)G_1(46). \tag{62}$$

In the Hartree-Fock approximation, we replace T by $v-v_{ex}$

$$\langle 12|T_{\rm HF}|34\rangle=\langle 12|v|34\rangle-\langle 12|v|43\rangle, \tag{63}$$

with v given by (25).

We restrict ourselves in the remainder of the paper to π mesons except in Secs. V.D and VI.E, where the ω meson is included. The direct term in the potential gives no contribution (because of the pseudoscalar nature of the pions), and we find, by Fourier transformation

$$\begin{aligned}[G_1^{-1}(p)]_{\zeta_1\zeta_{1'}}&=[G_1^{0\,-1}(p)]_{\zeta_1\zeta_{1'}}+i\sum_j\Omega^j_{\zeta_1\zeta_3}\Omega^j_{\zeta_2\zeta_{1'}}\\&\quad\times\int_{q^4}\mathcal{G}_0^j(p-q)[G_1(q)]_{\zeta_3\zeta_2}\cos(q_0 0^+).\end{aligned} \tag{64}$$

We note that $[G_1]_{\zeta_3\zeta_2}$ is diagonal in the isospin indices and independent of the β index $(G_{\zeta_3\zeta_2}=G_{\alpha_3\alpha_2}\delta_{\beta_3\beta_2})$. The combination of Ω's appearing in (64) then becomes

$$\sum_j\Omega^j_{\zeta_1\zeta_3}\Omega^j_{\zeta_2\zeta_{1'}}\delta_{\beta_3\beta_2}=3g_{0\pi}{}^2(\gamma_5)_{\alpha_1\alpha_3}(\gamma_5)_{\alpha_2\alpha_{1'}}\delta_{\beta_1\beta_{1'}}. \tag{65}$$

The matrix equation in spin indices now becomes

$$\begin{aligned}G_1^{-1}(p)&=G_1^{0\,-1}(p)+3ig_{0\pi}{}^2\int_{q^4}\mathcal{G}_0^j(q)\gamma_5\\&\quad\times G_1(p-q)\gamma_5\cos(q_0-p_0)0^+.\end{aligned} \tag{66}$$

Equation (66) can be expressed in terms of $\tilde{G}$, which is free of Dirac algebra, by noting the following relationships:

$$\left.\begin{aligned}&\gamma_5P_{\pm}\gamma_5=-P_{\mp},\\&{\rm tr}P_+(p)P_{\pm}(p-q)=1\pm\frac{p(p-q)}{w_pw_{p-q}},\\&{\rm tr}P_-(p)P_{\pm}(p-q)=1\mp\frac{p(p-q)}{w_pw_{p-q}}.\end{aligned}\right\} \tag{67}$$

We express G_1 in terms of $\tilde{G}$, operate on (66) with $P_{\pm}(p)$, and then take the trace to obtain

$$\tilde{G}^{-1}(w_p+i\epsilon)=M_0-w_p-\tfrac{3}{2}ig_{0\pi}{}^2\int_{q^4}\frac{\cos(q_0-p_0)0^+}{q^2+m_{0\pi}{}^2-i\epsilon}\left[\left(1+\frac{p(p-q)}{w_pw_{p-q}}\right)\tilde{G}(w_{p-q}+i\epsilon)+\left(1-\frac{p(p-q)}{w_pw_{p-q}}\right)\tilde{G}(-w_{p-q}-i\epsilon)\right], \tag{68}$$

and a similar expression for $\tilde{G}^{-1}(-w_p-i\epsilon)$. This expression can be simplified somewhat if we select the Lorentz frame in which three vector $\vec{p}=0$. This yields the equation

$$\tilde{G}^{-1}(p_0(1+i\epsilon))=M_0-p_0-\tfrac{3}{2}ig_{0\pi}{}^2\int_{q^4}\frac{\cos(q_0-p_0)0^+}{q^2+m_{0\pi}{}^2-i\epsilon}\left[\left(1-\frac{p_0-q_0}{w_{p-q}}\right)\tilde{G}(w_{p-q}+i\epsilon)+\left(1+\frac{p_0-q_0}{w_{p-q}}\right)\tilde{G}(-w_{p-q}-i\epsilon)\right], \tag{69}$$

which is valid for $p_0\gtrless 0$.

We can now utilize (49) to express $\tilde{G}$ in terms of the spectral function A and (58) to express $\tilde{G}^{-1}$ in terms of its renormalized spectral function T_R

$$\frac{M-p_0}{Z_2}\left[1-(M-p_0)\int_{-\infty}^{\infty}d\kappa\,\frac{T_R(\kappa)}{(\kappa-M)^2[\kappa-p_0(1+i\epsilon)]}\right]=M_0-p_0+3\left(\frac{g_{0\pi}}{4\pi}\right)^2\int_{-\infty}^{\infty}d\kappa\,\bar{K}(p_0,\kappa)A(\kappa)\,, \tag{70}$$

where the kernel on the right-hand side is given by

$$\bar{K}(p_0,\kappa)=-\frac{2i}{\pi}\int_0^{\infty}q^2dq\int_{-\infty}^{\infty}dq_0\,\frac{\cos(q_0-p_0)0^+}{q^2-q_0^2+m_{0\pi}^2-i\epsilon}\left[\frac{1-(p_0-q_0)/w}{\kappa-w-i\epsilon}+\frac{1+(p_0-q_0)/w}{\kappa+w+i\epsilon}\right], \tag{71}$$

where, here $w=[(p_0-q_0)^2-\vec{q}^{\,2}]^{1/2}$. The integral involving the real part of K on the right-hand side of (70) is infinite (in the absence of a cutoff), but the imaginary part is finite. We take the imaginary part of (70) and multiply through by $-\pi^{-1}Z_2\mathrm{sgn}(p_0)$ to obtain

$$T_R(p_0)=3\left(\frac{g_{0\pi}}{4\pi}\right)^2\int_{-\infty}^{\infty}d\kappa\,K(p_0,\kappa)A(\kappa)Z_2$$
$$=3\left(\frac{g_{R\pi}}{4\pi}\right)^2\int_{-\infty}^{\infty}d\kappa\,K(p_0,\kappa)A_R(\kappa)\,, \tag{72}$$

where $g_{R\pi}=g_{0\pi}Z_2$ is the renormalized coupling constant, and

$$K(p_0,\kappa)=-\frac{\mathrm{sgn}(p_0)}{\pi}\mathrm{Im}\,\bar{K}(p_0,\kappa)$$
$$=[p_0^4-2p_0^2(\kappa^2+m_{0\pi}^2)+(\kappa^2-m_{0\pi}^2)^2]^{1/2}$$
$$\times\frac{1}{2|p_0|^3}[(p_0-\kappa)^2-m_{0\pi}^2]\theta(p_0^2-(|\kappa|+m_{0\pi})^2)\,. \tag{73}$$

The complete set of equations now reads (with a change of variable)[13]:

$$T_R(\kappa)=3\left(\frac{g_{R\pi}}{4\pi}\right)^2\int_{-\infty}^{\infty}d\kappa'\,K(\kappa,\kappa')A_R(\kappa')\,, \tag{74a}$$

$$\tilde{G}_R^{-1}(\kappa(1+i\epsilon))=(M-\kappa)\left[1-(M-\kappa)\times\int_{-\infty}^{\infty}d\kappa'\,\frac{T_R(\kappa')}{(\kappa'-M)^2[\kappa'-\kappa(1+i\epsilon)]}\right], \tag{74b}$$

$$A_R(\kappa)=|\tilde{G}_R^{-1}(\kappa(1+i\epsilon))|^{-2}T_R(\kappa)+\delta(\kappa-M)\,. \tag{74c}$$

The integral in (74a) is finite, for fixed κ, because the kernel has a cutoff at $|\kappa'|=(\kappa^2-m_{0\pi}^2)^{1/2}$. Specifically, for large κ or κ', the kernel has the asymptotic form

$$K(\kappa,\kappa')\to\frac{1}{2|\kappa|^3}(\kappa^2-\kappa'^2)(\kappa-\kappa')^2\theta(\kappa^2-\kappa'^2)\,. \tag{75}$$

We observe that $\int d\kappa\,A_R(\kappa)$ is finite by the following consistency argument. Assuming that this is the case, it follows from (74a) and (75) that

$$T_R(\kappa)\xrightarrow[|\kappa|\to\infty]{}C|\kappa|\,, \tag{76}$$

where

$$C=\frac{3}{2}\left(\frac{g_{R\pi}}{4\pi}\right)^2Z_2^{-1}\,.$$

Then (74b) yields

$$\tilde{G}_R^{-1}(\kappa(1+i\epsilon))\xrightarrow[|\kappa|\to\infty]{}2C\kappa\ln|\kappa|\,. \tag{77}$$

From (74c), (76), and (77) we find immediately that

$$A_R(\kappa)\xrightarrow[|\kappa|\to\infty]{}\frac{(4C)^{-1}}{|\kappa|(\ln|\kappa|)^2}\,. \tag{78}$$

The integral of A_R is thus finite, consistent with the original assumption; however, the asymptotic behavior of $\tilde{G}_R^{-1}$ is inconsistent with the spectral form (59) for G_R together with the asymptotic form (78) for A_R. There must be a ghost, and the integral equations (74) for T must be modified to include the corresponding complex poles in G_R.

B. Numerical Results

Equations (74a)–(74c) were solved self-consistently for the spectral functions $A_R(\kappa)$ and $T_R(\kappa)$ on an IBM-7094 computer. $m_{0\pi}$ was replaced by the physical mass m_π (this represents an additional renormalization of the meson propagator which is not formally contained in Hartree-Fock theory). The pion-nucleon coupling constant was taken to be $g_\pi^2/4\pi=14.6$. Figure 1 displays the spectral function $A_R(\kappa)$; the δ function of unit weight at the physical nucleon mass is indicated. Z_2^{-1}, as calculated from (60), has the value $Z_2^{-1}=1.44$. The peak in $A_R(\kappa)$ for positive κ occurs at 1.43 nucleon mass units (nmu). This peak is suggestive of the 1470 MeV (1.555 nmu) N^* resonance, which has the same quantum numbers as the nucleon.

C. Comments on Ghost States

In this section we exhibit inconsistencies which are present in the solutions for $\tilde{G}_R^{-1}(z)$ [Eq. (58)] and $\tilde{G}_R(z)$ [Eq. (59)]. Such inconsistencies, which generally plague relativistic field theories (see, however, Ref. 13) are manifest here by the presence of ghost states.

In order to understand the nature of the inconsistency, let us first observe that the theory demands certain formal properties of the solution. Since

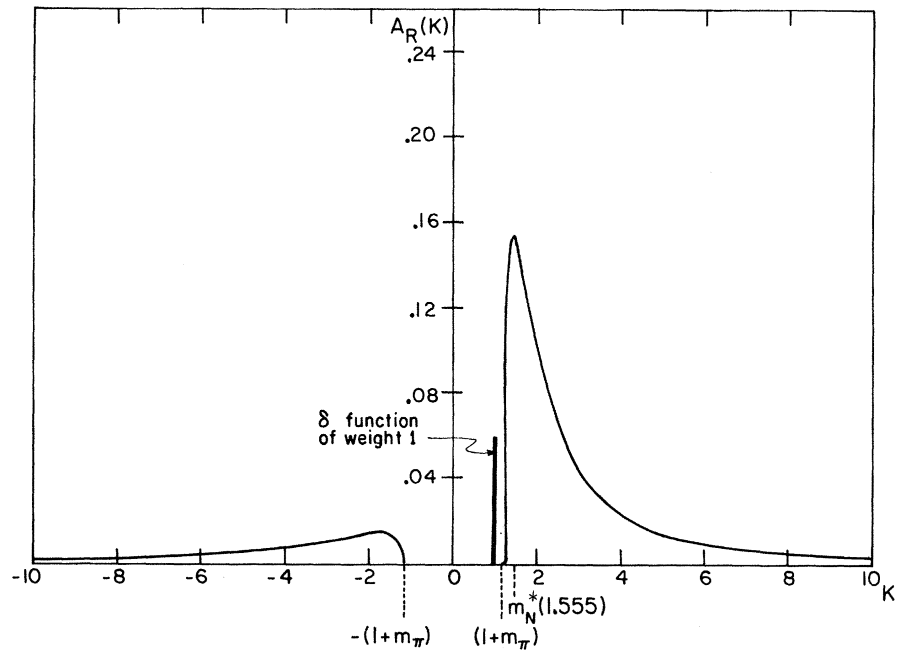

FIG. 1. Spectral function for the renormalized single-nucleon Green's function. Pion-nucleon coupling only.

$A(\kappa)$ is non-negative, it follows that $\tilde{G}(z)$ can have no poles *or zeros* off the real axis (Herglotz property). The absence of zeros follows by noting that

$$\tilde{G}(x+iy)=\int_{-\infty}^{\infty} d\kappa\, \frac{(\kappa-x+iy)A(\kappa)}{(\kappa-x)^2+y^2},$$

or

$$\mathrm{Im}\tilde{G}(x+iy)=y\int_{-\infty}^{\infty} d\kappa\, \frac{A(\kappa)}{(\kappa-x)^2+y^2}\neq 0 \quad \text{for } y\neq 0. \tag{79}$$

This is a necessary condition for writing a spectral representation for $\tilde{G}^{-1}(z)$, which also possesses the Herglotz property, as can be demonstrated from (54) or (56). Since the spectral functions are related by

$$T(\kappa)=|\tilde{G}^{-1}(\kappa)|^2 A(\kappa), \tag{80}$$

$T(\kappa)$ is also non-negative,

The renormalized quantities should possess all of the above properties, since they differ from the unrenormalized quantities by Z_2 or Z_2^{-1}, where Z_2 is the weight of the δ function in $A(\kappa)$, and since $\int A(\kappa)d\kappa=1$, it follows that

$$0<Z_2<1.$$

We can evaluate Z_2 either from (60),

$$Z_2^{-1}=\int_{-\infty}^{\infty} d\kappa\, A_R(\kappa), \tag{81}$$

which is finite in our calculations, or from a regrouping of (57),

$$Z_2=1-\int\frac{d\kappa\, T_R(\kappa)}{(\kappa-M)^2}. \tag{82}$$

But from the asymptotic form of $T_R(\kappa)$ as given by (76), this last expression is divergent and gives $Z_2=-\infty$. Clearly, we have an inconsistency.

Since Z_2 is negative, at least according to (82), we can expect further violations of the formal properties we assumed. For example, even though T_R is non-negative, T is not, and we cannot establish the Herglotz property for $\tilde{G}^{-1}$ or $\tilde{G}_R^{-1}$. In fact, $\tilde{G}_R^{-1}(z)$ evaluated according to (58) does have complex zeros. This means that G_R is not the inverse of what we have denoted by $\tilde{G}_R^{-1}$. [We have been careful not to write $\tilde{G}_R$ for $(\tilde{G}_R^{-1})^{-1}$.] The inverse of $\tilde{G}_R^{-1}$ can be written

$$[\tilde{G}_R^{-1}(z)]^{-1}=\int_{-\infty}^{\infty} d\kappa\, \frac{A_R(\kappa)}{\kappa-z}+\sum_j \frac{R_j}{z_j-z}, \tag{83}$$

where z_j is the location of the jth zero in $\tilde{G}_R^{-1}$ and R_j is the corresponding residue. These complex poles are referred to as ghost states, and they appear in conjugate pairs. Our prescription in this paper, which follows that of Redmond,[14] is to simply drop the poles when they occur in $\tilde{G}_R$.

The location and (minus) the residue of the upper member of a conjugate pair of ghosts is shown in Fig. 2. (There may be further pairs but we did not locate them.) The "trajectory" of the ghost is given as a function of the coupling constant, with unity corresponding to the physical value.

It is worth noting that the introduction of a cutoff does not automatically remove the ghost prob-

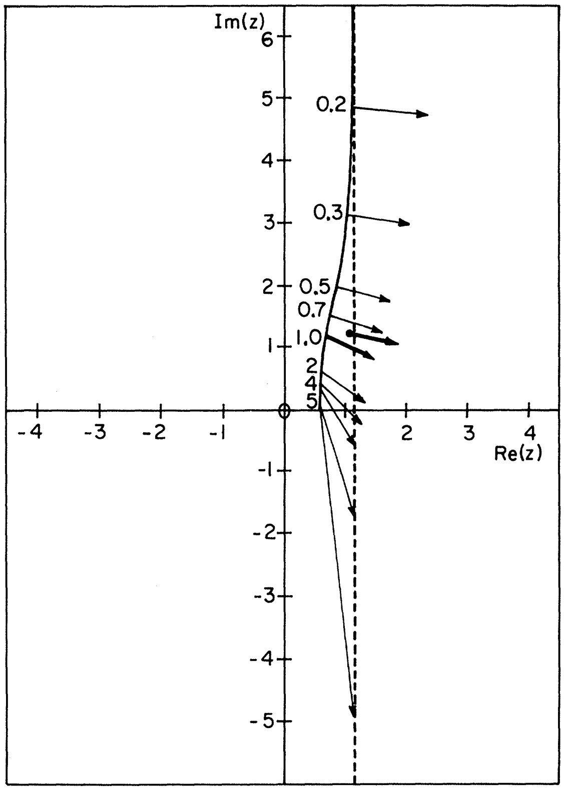

FIG. 2. Trajectory of the ghost state as a function of pion-nucleon coupling constant. The location (in nucleon mass units) of the ghost in the upper half complex z plane is shown. The vertical dashed line occurs at $(1+m_\pi)$ nucleon mass units. The arrow lengths and angles with the horizontal indicate the magnitudes and phase angles plus 180°, respectively, of the residues. The number at each arrow refers to the ratio of the chosen coupling constant to the physical value used in Sec. VB. The single point, together with its arrow, indicates the location and residue of the upper-half-plane ghost state, for the physical values of the coupling constants, when the ω meson is included.

lem. The origin of the problem lies in the negative value of Z_2 [according to (82)], not in the divergence of Z_2. We found that the cutoff necessary to keep Z_2 positive was undesirably small.

We paraphrase here three frequently expressed view points on ghost states (see, e.g., Ford[15]):

(a) The appearance of ghost states is caused by the inadequancy of the approximations. Improvement of the approximations would cause the ghosts to recede or weaken. An exact solution would be devoid of ghosts.

(b) Ghost states exist in nature (i.e., in a proper theory), but we do not know how to interpret them.

(c) Ghost states are a manifestation of the intrinsic inconsistency of local, renormalizable field theories.

There is a fairly long history of attempts to incorporate ghost states in quantum mechanics.[16-18] Until recently, such theories have suffered from the apparent defect that the ghost states had to be included in the unitarity sum over states, so that the physical particles alone would not satisfy unitarity. Lee and Wick[17] have recently shown that, at least in some diagrams, if the ghost mass is complex, the Feynman rules can be modified so that unitarity holds among the physical particles alone. The price that is paid for this is acausal properties of the theory; however, it is not known if such acausality can be manifested on a macroscopic level.

D. Inclusion of the ω Meson

The vector meson ω was also included in the vacuum Hartree-Fock problem, along with the pions, and the resulting equations were solved numerically. The coupling is given in Table I. The direct term in the Hartree-Fock factorization (63) also vanishes for the ω meson.

The propagator for vector mesons is given by

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu}+k^\mu k^\nu/m_{0\omega}^{\;2}}{k^2+m_{0\omega}^{\;2}-i\epsilon} \equiv \mathcal{G}^{5+\mu,5+\nu}(k)\,.$$

As is well known,[19] the longitudinal term $k^\mu k^\nu/m_{0\omega}^{\;2}(k^2+m_{0\omega}^{\;2}-i\epsilon)$ can be removed by a canonical transformation in the case of neutral vector mesons (the ω) coupled to a conserved current, and therefore, except for a change in the γ algebra, the derivations for ω coupling proceed in the same manner as those for pions in Part A of this section.

The inclusion of the ω meson generalizes (74a) as follows:

$$T_R(\kappa)=\int_{-\infty}^{\infty} d\kappa' A_R(\kappa') \times\left[3\left(\frac{g_\pi}{4\pi}\right)^2 K_\pi(\kappa,\kappa')+2\left(\frac{g_\omega}{4\pi}\right)^2 K_\omega(\kappa,\kappa')\right], \tag{84a}$$

where K_π is given by (73) and the ω kernel is

$$K_\omega(p_0,\kappa)=[p_0^{\;4}-2p_0^{\;2}(\kappa^2+m_{0\omega}^{\;2})+(\kappa^2-m_{0\omega}^{\;2})^2]^{1/2}[(p_0-\kappa)^2-2\kappa p_0-m_{0\omega}^{\;2}]\frac{1}{2|p_0|^3}\theta[p_0^{\;2}-(|\kappa|+m_{0\omega})^2]\,. \tag{84b}$$

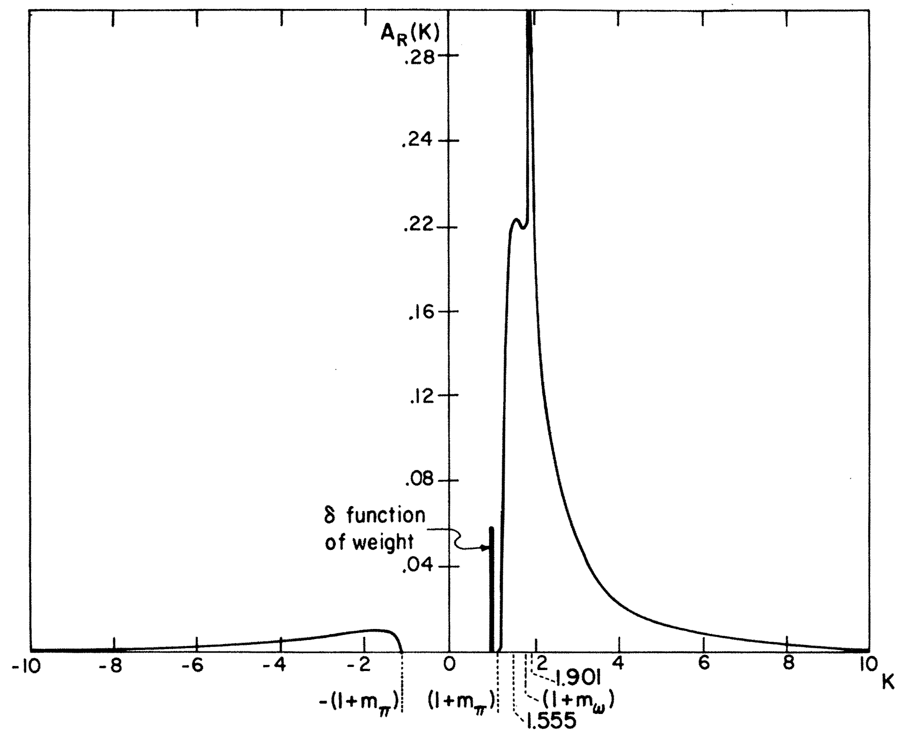

FIG. 3. Spectra function for the renormalized single-nucleon Green's function when both pion-nucleon and ω-nucleon couplings are present.

Equations (84), (74b), and (74c) were solved self-consistently on an IBM-7094 computer. $m_{0\omega}$ was replaced by the physical ω mass, and the coupling constant was assumed to have the value

$$g_\omega{}^2/4\pi = 6.36.$$

Figure 3 displays $A_R(\kappa)$ for the case where the nucleons interact via the exchange of π and ω mesons. There is a very narrow peak in the spectral function close to the mass of the N^* resonance at 1785 MeV = 1.901 nmu. The earlier, less pronounced peak near the N^* resonance at 1460 MeV = 1.555 nmu remains. The two N^* resonances referred to are the only ones known which have the same quantum numbers as the nucleon. It is dangerous to interpret the peaks with the physical resonances until we have examined N-π scattering in the model. The second peak in particular, however, is too narrow to be attributed to kinematical effects.

VI. MAGNETIC MOMENTS OF THE NUCLEON

In this section the formalism is used to describe electromagnetic properties of the nucleon. The renormalized equations are solved to obtain the magnetic moments.

A. Integral Equations for the Vertex Function

We make the gauge invariant replacement $p_\mu \to p_\mu - qA_\mu$, for all momenta in the field equations (8), resulting in

$$\overline{\mathcal{G}}_0{}^{-1}(x_1x_2)^{ij}\phi^j(x_2) = -g_{0\pi}\overline{\psi}_{\zeta_1}(x_1)(\tau_j\gamma_5)_{\zeta_1\zeta_2}\psi_{\zeta_2}(x_1)\,, \quad (85)$$

$$\overline{G}^{0\,-1}(12)\psi_{\zeta_2}(x_2) = -g_{0\pi}\,\phi^j(x_1)(\tau_j\gamma_5)_{\zeta_1\zeta_2}\psi_{\zeta_2}(x_1)\,, \quad (86)$$

where

$$\overline{\mathcal{G}}_0{}^{-1}(x_1x_2)^{ij} = [-\partial_1^2 + iet_3\{\partial_1^\mu, A_\mu(x_1)\} + m_{0\pi}{}^2]^{ij}\delta(x_1 - x_2)\,, \quad (87)$$

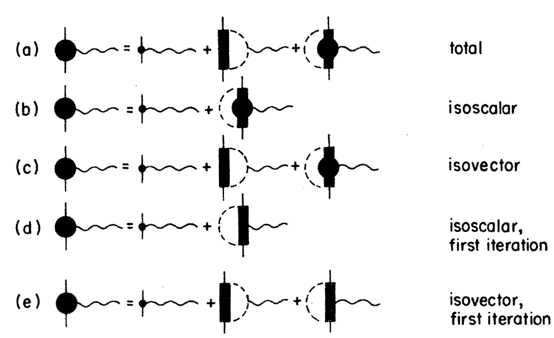

FIG. 4. Diagrams representing the integral equations for the photon-nucleon vertex function. (a) Diagram representing integral equation for the entire vertex function. (b) Integral equation for the isoscalar component. (c) Integral equation for the isovector component. (d) First iteration of the integral equation for the isoscalar component. (e) First interation of the integral equation for the isovector component.

$$\overline{G}^{0\,-1}(12) = \left[\frac{1}{i}\gamma^\mu \partial_{1\mu} - e\frac{1+\tau_3}{2}\gamma^\mu A_\mu(x_1) + M_0\right]_{\zeta_1\zeta_2}\delta(x_1 - x_2). \quad (88)$$

The term quadratic in eA_μ has been dropped. t_3 is the third component of the isovector operator which describes the three charge states of the pion. Proceeding as in the derivation of (23)–(25), one obtains from (85)–(88) and the Hartree-Fock factorization of G_2,

$$\overline{G}^{0\,-1}(12)\overline{G}(21') = \delta(11') + i\langle 12|\overline{v}|34\rangle \overline{G}(32^+)\overline{G}(41'), \quad (89)$$

where

$$\langle 12|\overline{v}|34\rangle = -g_{0\pi}{}^2(\gamma_5\tau_j)_{\zeta_1\zeta_3}\delta(x_1 - x_3)(\gamma^5\tau_j)_{\zeta_2\zeta_4} \times \delta(x_2 - x_4)\,\mathcal{G}_0^{ij}(x_1, x_4).$$

Equation (89) can be written as

$$\overline{G}^{-1}(11') = \overline{G}^{0\,-1}(11') + i\langle 14|\overline{v}|31'\rangle \overline{G}(34^+). \quad (90)$$

Taking the functional derivative of (90) with respect to $A_\mu(x_2)$ yields

$$\frac{\delta\overline{G}^{-1}(11')}{\delta A_\mu(2)} = \frac{\delta\overline{G}^{0\,-1}(11')}{\delta A_\mu(2)} - i\langle 14|\overline{v}|31'\rangle\frac{\delta\overline{G}(34^+)}{\delta A_\mu(2)} - i\,\frac{\delta\langle 14|\overline{v}|31'\rangle}{\delta A_\mu(2)}\overline{G}(34^+). \quad (91)$$

Substituting into (91) the definition of the vertex function for the nucleon-photon interaction

$$e\Gamma^\mu(11';2) = -\frac{\delta\overline{G}^{-1}(11')}{\delta A_\mu(2)}, \quad (92)$$

together with several algebraic manipulations, results in the following integral equation for the vertex function in the limit $A_\mu \to 0$:

$$\Gamma^\mu(11';2) = \tfrac{1}{2}(1+\tau_3)\gamma^\mu\delta(12)\delta(21') + i\langle 14|v|31'\rangle G(35)\Gamma^\mu(56;2)G(64^+) + 2g_{0\pi}{}^2\tau_3[\mathcal{G}_0(x_1 - x_2)\partial_2^\mu\,\mathcal{G}_0(x_2 - x_{1'}) - \mathcal{G}_0(x_2 - x_{1'})\partial_2^\mu\,\mathcal{G}_0(x_1 - x_2)]\gamma_5 G(11')\gamma_5. \quad (93)$$

Equation (93) is represented by the diagram in Fig. 4(a).

Decomposing the vertex function into its isoscalar and isovector components,

$$\Gamma^\mu = \tfrac{1}{2}[\Gamma_s^\mu + \tau_3\Gamma_v^\mu], \quad (94)$$

we find that (93) decouples into isoscalar and isovector integral equations represented by Figs. 4(b) and 4(c). The vertex function $\Gamma^\mu(11';2)$ depends only on the differences $x_1 - x_2$ and $x_2 - x_{1'}$. Therefore, the Fourier transform is a function of two four-momenta,

$$\Gamma^\mu(p,q) = \int d^4x\,d^4y\,e^{-ipx - iqy}\,\Gamma^\mu(11';2), \quad (95)$$

where

$$x = x_1 - x_2,$$

$$y = x_2 - x_{1'}.$$

The integral equations in momentum space for the isoscalar and isovector components are ($i = s, v$)

$$\Gamma_i^\mu(p,q) = \gamma^\mu + i\epsilon_i g_{0\pi}{}^2\int_{k^4}\mathcal{G}_0(k)\gamma_5 G(p-k)\Gamma_i^\mu(p-k)\Gamma_i^\mu(p-k,\ q-k)G(q-k)\gamma_5 - 4i\,\delta_{iv}g_{0\pi}{}^2\int_{k^4}\mathcal{G}_0(p-k)\,\mathcal{G}_0(q-k)(p+q-2k)^\mu\gamma_5 G(k)\gamma_5, \quad (96)$$

where $\epsilon_s = -3$ and $\epsilon_v = +1$.

B. Ward Identity, Renormalization, and Subtraction

The Ward identity[20] in momentum space is

$$(p-q)_\mu\,\Gamma_i^\mu(p,q) = G^{-1}(p) - G^{-1}(q). \quad (97)$$

It can be readily verified that both the isoscalar and isovector vertex functions which satisfy (96) also satisfy the Ward identity if Hartree-Fock Green's functions are used on the right-hand side of (97).

Taking the gradient of (97) with respect to p and going to the limit $p \to q$ results in

$$\Gamma_i^\mu(q,q) = \partial^\mu G^{-1}(q). \quad (98)$$

Lehmann, Symanzik, and Zimmermann[21] have shown that the unrenormalized inverse Green's function is of the form

$$G^{-1}(q)=Z_2^{-1}(\gamma q+M)+(\gamma q+M)^2\mathfrak{F}(q). \tag{99}$$

Comparing the gradient of (99) evaluated on the mass shell with (98) yields the following condition which must be satisfied by the unrenormalized vertex functions on the mass shell

$$\Gamma_i^\mu(q,q)\big|_{q^2=-M^2}=Z_2^{-1}\gamma^\mu. \tag{100}$$

However, the renormalized vertex functions on the mass shell at zero momentum transfer must equal γ^μ, which, together with (100), implies that the renormalized and unrenormalized vertex functions are related by

$$\Gamma_{iR}^\mu(p,q)=Z_2\Gamma_i^\mu(p,q). \tag{101}$$

In order that (100) be satisfied, we perform a subtraction on the mass shell at zero momentum transfer for both the isoscalar and isovector integral equations, and renormalize the resulting equations by multiplying by Z_2.

This yields

$$\Gamma_{iR}^\mu(p,q)=\gamma^\mu+i\epsilon_i g_\pi^2\int_{k^4}\mathcal{G}_0(k)\gamma_5\left[G_R(p-k)\Gamma_{iR}^\mu(p-k,q-k)G_R(q-k)-\left\{\begin{matrix}p=q\\ q^2=-M^2\end{matrix}\right\}\right]\gamma_5$$
$$-4i\delta_{iv}g_\pi^2\int_{k^4}\left[\mathcal{G}_0(p-k)\,\mathcal{G}_0(q-k)(p+q-2k)^\mu-\left\{\begin{matrix}p=q\\ q^2=-M^2\end{matrix}\right\}\right]\gamma_5 G_R(k)\gamma_5, \tag{102}$$

where

$$g_\pi=Z_2 g_{0\pi}\quad\text{and}\quad G_R=Z_2^{-1}G \tag{103}$$

are the renormalized coupling constant and nucleon Green's function.

The renormalized vertex functions are related to the charge and moment form factors by

$$\Gamma_{iR}^\mu(p,q)\big|_{p^2=q^2=-M^2}=\left\{F_1^i[(p-q)^2]\gamma^\mu+F_2^i[(p-q)^2]\frac{1}{i}\sigma^{\mu\nu}(p-q)\nu/2M\right\}. \tag{104}$$

In the limit of zero momentum transfer,

$$F_1^i(0)=1, \tag{105}$$

$$F_2^i(0)=\mu_i=\mu_p+\mu_n\quad\text{for } i=s,$$
$$=\mu_p-\mu_n\quad\text{for } i=v, \tag{106}$$

where μ_p and μ_n are the anomalous magnetic moments of the proton and neutron, respectively, measured in nuclear Bohr magnetons.

C. Reduction of the Vertex Equations and the Magnetic Moments

It is clear from inspection (96) that the momentum transfer $(p-q)$ is conserved in the integral equations. Therefore, the integral equations for different values of the momentum transfer are not coupled.

In order to take advantage of the conservation momentum transfer, the change of variables $P^\mu\equiv\frac{1}{2}(p+q)^\mu$, $Q^\mu\equiv\frac{1}{2}(p-q)^\mu$ is convenient. In terms of these new variables, the integral equations are

$$\tilde{\Gamma}_i^\mu(P,Q)=\gamma^\mu+i\epsilon_i g_\pi^2\int_{k^4}\mathcal{G}_0(k)\gamma_5\left[G(P+Q-k)\tilde{\Gamma}_i^\mu(P-k,Q)G(P-Q-k)-\left\{\begin{matrix}Q=0\\ P^2=-M^2\end{matrix}\right\}\right]\gamma_5$$
$$-8i\delta_{iv}g_\pi^2\int_{k^4}\left[\mathcal{G}_0(P+Q-k)\mathcal{G}_0(P-Q-k)(P-k)^\mu-\left\{\begin{matrix}Q=0\\ P^2=-M^2\end{matrix}\right\}\right]\gamma_5 G(k)\gamma_5. \tag{107}$$

All of the quantities in (107) and hereafter are understood to be renormalized; and hence the subscript R which appears in (102) has been dropped.

If the vertex functions on the mass shell are expanded in a momentum-transfer power series, the magnetic moments are given by the coefficients of the linear terms. In order to single out these linear terms, we take the gradient of (107) with respect to Q_β and let $Q_\beta\to 0$. Defining

$$\Gamma_i^{\mu\beta}(P)\equiv\partial_Q^\beta\tilde{\Gamma}_i^\mu(P,Q)\big|_{Q=0},$$

$$\Gamma_{0i}^\mu(P)\equiv\tilde{\Gamma}_i^\mu(P,Q=0),$$

we obtain

$$\Gamma_I^{\mu\beta}(P)=i\epsilon_I g_\pi{}^2\int_{k^4}\mathcal{G}_0(P-k)\gamma_5[\{\partial^\beta G(k)\}\Gamma_{0I}^{\mu}(k)G(k)+G(k)\Gamma_I^{\mu\beta}(k)G(k)-G(k)\Gamma_{0I}^{\mu}(k)\{\partial^\beta G(k)\}]\gamma_5 . \quad (108)$$

Γ_{0I}^{μ} can be expressed in terms of the gradient of the inverse Green's function by using the Ward identity. In terms of P and Q, the Ward identity is

$$2Q_\mu\tilde{\Gamma}_I^{\mu}(P,Q)=G^{-1}(P,Q)\;-G^{-1}(P-Q) . \quad (109)$$

Taking the gradient with respect to Q_β and letting $Q_\beta\to 0$ results in

$$\Gamma_{0I}^{\mu}(P)=\partial^\mu G^{-1}(P) . \quad (110)$$

Substituting this into (108) yields some simplification, and the inhomogeneous terms can be written in a more symmetrical form by using the relations $\{\partial^\mu G^{-1}\}G=-G^{-1}\partial^\mu G$ and $G\,\partial^\mu G^{-1}=-\{\partial^\mu G\}G^{-1}$. Thus, we have

$$\Gamma_I^{\mu\beta}(P)=i\epsilon_I g_\pi{}^2\int_{k^4}\mathcal{G}_0(P-k)\gamma_5[\{\partial^\mu G(k)\}G^{-1}(k)\{\partial^\beta G(k)\}-\{\partial^\beta G(k)\}G^{-1}(k)\{\partial^\mu G(k)\}+G(k)\Gamma_I^{\mu\beta}(k)G(k)]\gamma_5 . \quad (111)$$

A second gradient of the Ward identity implies that $\Gamma_I^{\mu\beta}$ is an antisymmetric tensor. That is, we have

$$2\partial_Q^\alpha\tilde{\Gamma}_I^{\beta}(P,Q)+2\partial_Q^\beta\tilde{\Gamma}_I^{\alpha}(P,Q)+2Q_\mu\partial_Q^\alpha\partial_Q^\beta\Gamma_I^{\mu}(P,Q)=\partial_P^\alpha\partial_P^\beta G^{-1}(P+Q)-\partial_P^\alpha\partial_P^\beta G^{-1}(P-Q) .$$

In the limit $Q\to 0$, this becomes

$$\Gamma_I^{\beta\alpha}(P)+\Gamma_I^{\alpha\beta}(P)=0 .$$

The most general second-rank antisymmetric tensor which can be constructed from the two four vectors P^μ and γ^μ is

$$\Gamma^{\mu\beta}(P)=g_1[\gamma^\mu P^\beta-\gamma^\beta P^\mu]+g_2\sigma^{\mu\beta} ,$$

where the g_i's are functions of the invariants P^2 and γP. Since $(\gamma P)^2=-P^2$, γP can occur only linearly,

$$\Gamma^{\mu\beta}(P)=[h_1(P^2)+h_2(P^2)\gamma P][\gamma^\mu P^\beta-\gamma^\beta P^\mu]+[h_3(P^2)+h_4(P^2)\gamma P]\frac{1}{i}\sigma^{\mu\beta} . \quad (112)$$

It is not difficult to show that invariance under TCP and Hermiticity of the electromagnetic current operator together imply that, in the Majorana representation, $\Gamma^{\mu\beta}(P)$ must satisfy the relation

$$[\Gamma^{\mu\beta}(P)\beta]^T=\gamma_5\Gamma^{\mu\beta}(P)\beta\gamma_5 . \quad (113)$$

Substituting (112) into (113) yields

$$h_1(P^2)=-h_4(P^2) .$$

Therefore, the most general form of the gradient of the vertex function consistent with TCP and Hermiticity is

$$\Gamma^{\mu\beta}(P)=\sum_{j=2}^{4}h_j(P^2)\Omega_j^{\mu\beta}(P) , \quad (114)$$

where

$$\Omega_2^{\mu\beta}(P)\equiv\gamma P[\gamma^\mu P^\beta-\gamma^\beta P^\mu] ,$$

$$\Omega_3^{\mu\beta}(P)\equiv\frac{1}{i}\sigma^{\mu\beta},$$

$$\Omega_4^{\mu\beta}(P)\equiv\frac{1}{i}\gamma P\sigma^{\mu\beta}-[\gamma^\mu P^\beta-\gamma^\beta P^\mu]=\epsilon^{\mu\beta\alpha\delta}P_\alpha\gamma_\delta\gamma^5 .$$

Of course, the form (114) holds for both the isoscalar and isovector components. Since the integral equations for the two components of the gradient of the vertex function are very similar, we shall hereafter write equations only for the isoscalar component; the generalization to the isovector component will be obvious.

Substituting (114) into (111) yields three coupled integral equations for three scalar functions of the single scalar variable P^2. We now make the assumption that these scalar functions $h_i(P^2)$ have no singularities except for a branch cut beginning at the lowest multiparticle strong-interaction threshold, i.e., at the threshold for pion production. Thus, we have,

$$h_i(P^2)=\int_{(M+m_\pi)^2}^{\infty}\frac{g_i(t)dt}{t+P^2-i\epsilon}, \quad (115)$$

where the spectral functions $g_i(t)$ are real.

Substituting (114) and (115) into (111), and taking the imaginary part, yields three coupled integral equations for the spectral functions. We have

$$g_i(t)=I_i^\pi(t)+\sum_{j=2}^{4}\int ds\,K_{\pi i}^{j}(t,s)g_j(s)\quad i=2,3,4, \quad (116a)$$

where

$$(M+m_\pi)^2\leq t\leq\infty ,$$

and

$$I_i^\pi(t)=\frac{1}{\pi}\mathrm{Im}\,\tilde{I}_i^\pi(-t)\,, \tag{116b}$$

$$\sum_i \tilde{I}_i^\pi(P^2)\Omega_i^{\mu\beta}(P)=\frac{3g_\pi{}^2}{i}\int_{k^4}\mathcal{G}_0(P-k)\gamma_5[\{\partial^\mu G(k)\}G^{-1}(k)\{\partial^\beta G(k)\}-\{\partial^\beta G(k)\}G^{-1}(k)\{\partial^\mu G(k)\}]\gamma_5\,, \tag{116c}$$

$$K_{\pi i}^j(t,s)=\frac{1}{\pi}\mathrm{Im}\int_{k^4}\frac{\tilde{H}_{\pi i}^j(-t,k)}{s+k^2-i\epsilon}, \tag{116d}$$

$$\sum_{i,j=2}\Omega_i^{\mu\beta}(P)\int_{k^4}\tilde{H}_{\pi i}^j(P^2,k)h_j(k^2)=\frac{3g_\pi{}^2}{i}\int_{k^4}\mathcal{G}_0(P-k)\gamma_5 G(k)\Gamma^{\mu\beta}(k)G(k)\gamma_5\,. \tag{116e}$$

We now investigate the question of how to obtain the magnetic moment after solving the integral equations (116). Equation (114) evaluated on the mass shell is

$$\Gamma^{\mu\beta}(P)\big|_{P^2=-M^2}=[h_3(-M^2)-M\,h_4(-M^2)]\frac{1}{i}\sigma^{\mu\beta}\,. \tag{117}$$

The most general form of the vertex function, without imposing Ward's identity, is[22]

$$\begin{aligned}\tilde{\Gamma}^\mu(P,Q)=&\hat{F}_1\gamma^\mu+\frac{2}{i}\hat{F}_2\sigma^{\mu\nu}Q_\nu+\frac{2}{i}\hat{F}_3Q^\mu+\frac{1}{i}\hat{F}_4[\gamma(P+Q)\gamma^\mu-\gamma^\mu\gamma(P-Q)]+2\hat{F}_5[\gamma(P+Q)\sigma^{\mu\nu}Q_\nu-\sigma^{\mu\nu}Q_\nu\gamma(P-Q)]\\&+4\hat{F}_6\gamma QQ^\mu+\hat{F}_7[\gamma(P+Q)\gamma^\mu+\gamma^\mu\gamma(P-Q)]+\frac{2}{i}\hat{F}_8[\gamma(P+Q)\sigma^{\mu\nu}Q_\nu+\sigma^{\mu\nu}Q_\nu\gamma(P-Q)]+\frac{4}{i}\hat{F}_9\gamma PQ^\mu\\&+\hat{F}_{10}\gamma(P+Q)\gamma^\mu\gamma(P-Q)+\frac{2}{i}\hat{F}_{11}\gamma(P+Q)\sigma^{\mu\nu}Q_\nu\gamma(P-Q)+\frac{2}{i}\hat{F}_{12}\gamma(P+Q)\gamma(P-Q)Q^\mu,\end{aligned} \tag{118}$$

where $\hat{F}_i=\hat{F}_i(p^2,q^2,(p-q)^2)$ is usually expressed in the original momentum variables. Taking the gradient of (118) with respect to Q_β, letting $Q_\beta\to 0$ and going to the mass shell results in

$$\begin{aligned}\Gamma^{\mu\beta}(P)=\partial_Q^\beta\tilde{\Gamma}^\mu(P,Q)\Big|_{\substack{Q=0\\P^2=-M^2}}&=\frac{2}{i}\sigma^{\mu\beta}[\hat{F}_2(-M^2,-M^2,0)-\hat{F}_7(-M^2,-M^2,0)-2M\hat{F}_8(-M^2,-M^2,0)\\&\quad+M\hat{F}_{10}(-M^2,-M^2,0)+M^2\hat{F}_{11}(-M^2,-M^2,0)]\\&=\frac{2}{i}\sigma^{\mu\beta}\left[\frac{\mu}{2M}-\hat{F}_7(-M^2,-M^2,0)+M\hat{F}_{10}(-M^2,-M^2,0)\right],\end{aligned} \tag{119}$$

since the magnetic moment is given by

$$\mu=F_2(0)=2M[\hat{F}_2(-M^2,-M^2,0)-2M\hat{F}_8(-M^2,-M^2,0)+M^2\hat{F}_{11}(-M^2,-M^2,0)]\,.$$

Comparing (117) and (119), we see that

$$\mu=M[h_3(-M^2)-Mh_4(-M^2)]+2M[\hat{F}_7(-M^2,-M^2,0)-M\hat{F}_{10}(-M^2,-M^2,0)]\,. \tag{120}$$

In taking the gradient of the vertex function before going to the mass shell, we have obtained two terms in (119) which do not contribute to the magnetic moment. Thus, to extract the magnetic moment from the functions h_3 and h_4, we must calculate $\hat{F}_7$ and $\hat{F}_{10}$ on the mass shell at zero momentum transfer as shown in (120).

$\hat{F}_7$ and $\hat{F}_{10}$ can be obtained from the Ward identity. Taking the gradient of (109) with respect to Q_β and letting $Q\to 0$ yields

$$\tilde{\Gamma}^\mu(P,Q)\big|_{Q=0}=\partial^\mu G^{-1}(P)=2[A'(P^2)\gamma P+B'(P^2)]P^\mu+A(P^2)\gamma^\mu, \tag{121}$$

since

$$G^{-1}(P)=A(P^2)\gamma P+B(P^2)\,,$$

where

$$A(P^2)=Z_2+\int d\kappa\,\frac{T_R(\kappa)}{\kappa^2+P^2},$$

$$B(P^2)=Z_2M_0-\int d\kappa\,\frac{T_R(\kappa)}{\kappa^2+P^2}.$$

From (118), we see that left-hand side of (121) can be written

$$\tilde{\Gamma}^\mu(P, Q)\Big|_{Q=0} = F_1(p^2, p^2, 0) - 2F_7(p^2, p^2, 0)P^\mu + F_{10}(p^2, p^2, 0)[P^2\gamma^\mu - 2\gamma P\, P^\mu]. \tag{122}$$

By equating (121) and (122), taking some traces, and going to the mass shell, we obtain the desired result

$$MF_{10}(-M^2, -M^2, 0) - F_7(-M^2, -M^2, 0) = \int d\kappa \frac{T_R(\kappa)}{(\kappa - M)(\kappa^2 - M^2)}. \tag{123}$$

Equations (116), (115), (120), and (123) provide a prescription for calculating the magnetic moments of the nucleon.

D. Magnetic-Moment Results for π-Meson Coupling

The spectral representations of the Green's function (59) and of the inverse Green's function (58) were substituted into (116c) and (116e) and the k^4 integrations performed using the standard Feynman methods. The integral equations (116a) were solved numerically by iteration on an IBM 7094 computer. The equations were first solved in the "pole approximation," for which only the pole term in the Green's function was used. The equations were then solved again using the entire Green's function (and entire inverse Green's function).

The fourth and fifth columns of Table II list the magnetic moments both in the "pole approximation" and when using the complete Green's function. The close agreement between the predicted isovector moment and the experimental value is clearly somewhat fortuitious.

At this point, it is of interest to compare our results with the considerably simpler first-order iteration of (102). This iteration is completely equivalent to what is normally called first-order perturbation theory[23] if only the pole contribution of the Green's function is included. When the entire spectral function $A_R(\kappa)$ is included, the first iteration does not satisfy the Ward identity. If, however, the first iteration of (102) is linearized in the spectral function, i.e., if for terms which are quadratic in the spectral function, we make the replacement

$$A_R(\kappa)A_R(\kappa') \to A_R(\kappa)\delta(\kappa - \kappa'),$$

the result satisfies the Ward identity. Column 3 of Table II displays the first iteration results which satisfy the Ward identity, and column 4 displays the results which do not. Column 2 reproduces the results given in Ref. 23.

The differences between the pole contributions and the total contributions are very close in the case of the exact solutions, columns 5 and 6, to what they are for the first-order predictions which do not satisfy the Ward identity. This is an example of how, when working in a crude approximation, a solution which satisfies certain general conditions such as the Ward identity is *not necessarily* closer to the exact solution (which satisfies these conditions) than is a solution which does not.

The poor agreement between the predictions of (116) and experiment in the isoscalar case can be partly understood as follows: Dispersion theory and G-parity conservation imply that the lowest mass process which contributes to the isovector component of the magnetic moment is the one in which two intermediate-state pions couple the nucleon and photon; whereas for the isoscalar component, the lowest mass process is the three-pion intermediate state coupling the nucleon and photon. In our theory, however, the three-pion intermediate state is not possible, and therefore the lowest

TABLE II. Comparison of theoretical predictions and experimental values of the isoscalar and isovector components of the magnetic moment of the nucleon.

	Magnetic moment in nuclear magnetons							
		First iteration			Full solution to integral equation		Full solution including the ω meson	
	Experimental	Pole only	Full $A(\kappa)$. Satisfies the Ward identity	Full $A(\kappa)$. Does not satisfy the Ward identity	Pole only	Full $A(\kappa)$	Pole only	Full $A(\kappa)$
Isoscalar $\frac{1}{2}(\mu_p + \mu_n)$	−0.06	−1.64	−1.81	−1.53	−1.52	−1.40	−1.85	−1.71
Isovector $\frac{1}{2}(\mu_p - \mu_n)$	1.85	2.16	2.34	2.25	1.76	1.86	1.82	1.90

mass process contributing to the isoscalar magnetic moment is the nucleon-antinucleon intermediate state. Since we do not take into account the important three-pion intermediate state, it is perhaps not surprising that our isoscalar prediction is in poor agreement with experiment. Other people, for example Drell and Pagels,[24] have found that in order to obtain a reasonably small value for the isoscalar magnetic moment, the nucleon-antinucleon intermediate state must either be greatly supressed or neglected entirely.

E. Inclusion of the ω Meson

The ω meson is easily included in the calculation of electromagnetic properties for the nucleon. In addition to the field equations (8), we must include the equation describing the coupling of the ω mesons to the nucleon current,

$$D_0^{-1}(x_1, x_2)\varphi^\mu(x_2) = -g_{0\omega}\bar{\psi}(x_1)\gamma^\mu\psi(x_1)\,, \tag{124}$$

where

$$D_0^{-1}(x_1, x_2) = (-\partial_1^2 + m_{0\omega}{}^2)\delta(x_1 - x_2)\,.$$

The gauge invariant replacement $p_\mu \to p_\mu - qA_\mu$, does not affect (124), since the ω meson is uncharged. The contribution of the ω meson to the variation of the "potential" with respect to the electromagnetic field must therefore vanish

$$\frac{\delta\langle 14|v|31'\rangle}{\delta A_\mu(2)} = 0\,.$$

The only contribution of the ω meson to the integral equation for the vertex function is through the term

$$\frac{i}{e}\langle 14|v|31'\rangle_\omega \frac{\delta G(34^+)}{\delta A_\mu(2)} = i\langle 14|v|31'\rangle_\omega G(35)\Gamma^\mu(56;2)G(64^+)\,, \tag{125}$$

where

$$\langle 12|v|34\rangle_\omega = -g_{0\omega}{}^2\gamma^\mu_{\zeta_1\zeta_3}\delta(x_1 - x_3)(\gamma_\mu)_{\zeta_2\zeta_4} \times\delta(x_2 - x_4)D_0(x_1 - x_2)\,.$$

Since the ω is an isoscalar meson, $\langle v\rangle_\omega$ is diagonal in isospin space, and therefore (125) contributes equally to the isoscalar and isovector integral equations.

The derivation of the integral equations for the gradient of the vertex functions proceeds as before. The result is

$$\Gamma_i^{\mu\beta}(P) = \text{RHS of (111)} + \frac{g_\omega{}^2}{i}\int_{k^4} D_0(P-k)\gamma^\alpha\begin{bmatrix}\text{Bracketed quantity}\\ \text{in (111)}\end{bmatrix}\gamma_\alpha\,. \tag{126}$$

Substituting (114) and (115) into (126) and taking the imaginary part yields three coupled integral equations for the spectral functions $g_i(t)$ when the ω mesons are included:

$$g_i(t) = I_i(t) + \sum_{j=2}^{4}\int ds\, K_i^j(t,s)g_j(s) \qquad i = 2, 3, 4\,, \tag{127a}$$

where

$$I_i(t) = I_i^\pi(t) + I_i^\omega(t) = \frac{1}{\pi}\text{Im}[\tilde{I}_i^\pi(-t) + \tilde{I}_i^\omega(-t)]\,, \tag{127b}$$

$$\sum_i \tilde{I}_i^\omega(P^2)\Omega_i^{\mu\beta}(P) = \frac{g_\omega{}^2}{i}\int_{k^4} D_0(P-k)\gamma^\alpha\begin{bmatrix}\text{Bracketed quantity}\\ \text{in (116c)}\end{bmatrix}\gamma_\alpha\,, \tag{127c}$$

$$K_i^j(t,s) = \frac{\text{Im}}{\pi}\int_{k^4}\frac{\tilde{H}_{\pi i}^j(-t,k) + \tilde{H}_{\omega i}(-t,k)}{s + k^2 - i\epsilon}\,, \tag{127d}$$

$$\sum_{i,j=2}^{4}\Omega_i^{\mu\beta}(P)\int_{k^4}\tilde{H}_{\omega i}^j(P^2,k)h_j(k^2) = \frac{g_\omega{}^2}{i}\int_{k^4}D_0(P-k)\gamma^\alpha G(k)\Gamma^{\mu\beta}(k)G(k)\gamma_\alpha\,. \tag{127e}$$

$\tilde{I}_i^\pi$ and $H_{\pi i}^j$ are defined in (116c) and (116e).

Equations (127) were solved in the same manner as were (116), as described in Sec. D. Columns 7 or 8 of Table II lists the magnetic moments both in the pole approximation and when using the complete Green's function. We note that inclusion of ω mesons brings us no closer to the experimental result.

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†Present address: Department of Physics and Astronomy, and Center for Theoretical Physics, University of Maryland, College Park, Maryland.

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Nuclear-Scattering Problem in the Generalized Hartree-Fock Approximation*

Louis S. Celenza
Physics Department, Brooklyn College of the City University of New York, Brooklyn, New York 11210
(Received 18 February 1970)

The generalized Hartree-Fock approximation of Kerman and Klein is extended to include continuum states of a collective nature. The equations for scattering in the random-phase approximation are rederived.

I. INTRODUCTION

Theories of nuclear scattering which proceed from an atomistic viewpoint have recently been developed.[1,2] These owe their derivations to the techniques of nuclear many-body perturbation theory. The equations of the random-phase approximation (RPA) or quasiboson approximation also have been employed in the context of the nuclear-scattering problem.[3–5] Furthermore, calculations in the continuum using the RPA have been performed, and the analytic properties of the predicted S matrix have been examined.[6,7] The essential feature of these methods is that a unified picture is obtained which predicts both the bound and resonant states, and allows the calculation of an S matrix.