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Off-Energy-Shell T Matrix for Local Potentials with Singular Core Interactions. II. Tensor and Coulomb Forces*

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A previous derivation of an integral equation, which gives the complete off-energy-shell t matrix for two-body scattering via local potentials with singular core interactions, is generalized to include the effects of tensor and Coulomb forces. The core interaction is described by the boundary-condition model (specified energy-independent logarithmic derivative of the radial wave function at the core radius). The formalism is now being applied to the calculation of trinucleon bound-state and scattering parameters using the Faddeev formalism and realistic nucleon-nucleon forces.

I. INTRODUCTION

In a previous analysis (referred to hereafter as KTI)[1], we have derived and numerically studied an explicit integral equation which gives the complete off-shell t matrix for two-body elastic scattering via a local potential with a singular core interaction. The boundary-condition model[2] (specified energy-independent logarithmic derivative of the radial wave function at the core radius) was used to describe the core interaction. The hard-core interaction is, of course, a special case of the boundary-condition model (BCM).

For simplicity, we restricted the treatment in KTI to the case of no coupling between states of different orbital angular momentum (i.e., no tensor coupling) and no long-range Coulomb forces. In response to a number of inquiries concerning our method, we generalize, in this paper, our previous results to the case of tensor and Coulomb forces.

In Sec. II, we give a simple representation for the BCM core interaction containing tensor contributions, which facilitates the derivation of the integral equation for the off-shell t matrix. In Sec. III, we derive the integral equation for the off-shell t matrix for the case of tensor forces inside and outside the core region. In Sec. IV, the effects of long-range Coulomb forces are incorporated into the integral equation for the off-shell t matrix. Section V contains a brief summary and discussion. The formalism of this paper and KTI is now being applied to the calculation of trinucleon bound-state and scattering parameters using the Faddeev formalism[3] and realistic nucleon-nucleon forces.[4]

II. REPRESENTATION OF THE TENSOR CORE INTERACTION IN THE BCM

It will be convenient to work with two sets of basis states in the center-of-mass system

$$|rlsjj_z\rangle\,, \tag{2.1}$$

$$|plsjj_z\rangle\,. \tag{2.2}$$

The states (2.1) are simultaneous eigenstates of $r=|\vec{r}|$, where $\vec{r}$ is the relative displacement of the two particles; $(\vec{l})^2=(\vec{r}\times\vec{p})^2$, where $\vec{p}$ is the relative momentum of the two particles; $(\vec{s})^2=(\vec{s}_1+\vec{s}_2)^2$, where $\vec{s}_1$ and $\vec{s}_2$ are the particle spin operators; $(\vec{j})^2=(\vec{l}+\vec{s})^2$; and j_z, with associated quantum numbers r, $l(l+1)$, $s(s+1)$, $j(j+1)$, and j_z, respectively. We use natural units ($\hbar=c=1$) and, for simplicity, use the same symbol for an operator and its corresponding quantum number. The states (2.2) are simultaneous eigenstates of $H_0=(\vec{p})^2/2\mu$, the free-particle kinetic energy, where μ is the reduced mass; $(\vec{l})^2$; $(\vec{s})^2$; $(\vec{j})^2$; and j_z. The phases of the states (2.1) and (2.2) are chosen so that the time-reversal operation T acts according to $T|\ldots j, j_z\rangle=(-)^{j+j_z}|\ldots j, -j_z\rangle$.

The closure relations for the states (2.1) and (2.2) are

$$\sum_{lsjj_z}\int_0^\infty dr\, r^2|rlsjj_z\rangle\langle rlsjj_z|=1\,, \tag{2.3}$$

and

$$\frac{2}{\pi}\sum_{lsjj_z}\int_0^\infty p^2dp\,|plsjj_z\rangle\langle plsjj_z|=1\,. \tag{2.4}$$

The transformation from one basis to the other is given by

$$\langle rl's'j'j_z'\,|plsjj_z\rangle=j_l(pr)\delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{j_z'j_z}\,. \tag{2.5}$$

Since we are mainly interested in applications to nucleon systems, we will assume that $(\vec{j})^2$, j_z, and $(\vec{s})^2$ are conserved in two-body scattering and will usually suppress the quantum-number labels j, j_z, and s.

$$|rlsjj_z\rangle\to|rl\rangle\,, \tag{2.6}$$

$$|plsjj_z\rangle\to|pl\rangle\,, \tag{2.7}$$

$$\langle rl'|pl\rangle = \delta_{l'l}\, j(pr)\,, \tag{2.8}$$

$(sjj_z$ fixed).

Furthermore, we will restrict our attention to scattering involving tensor-coupled states with $s=1$, $j\neq 0$, $l=j\pm 1$, since we have already treated scattering for the case of uncoupled partial-wave states in KTI.

In the following analysis, the states $|rl_\pm\rangle$ and $|pl_\pm\rangle$ represent the fully designated states $|r\, l{=}j\pm 1\; s{=}1jj_z\rangle$ and $|p\, l{=}j\pm 1\; s{=}1jj_z\rangle$, respectively. If the nucleon-nucleon potential has the form

$$V(\vec{r},\vec{p},\vec{s}_1,\vec{s}_2) = V_C(r) + 4V_s(r)\vec{s}_1\cdot\vec{s}_2 + 4V_T(r)[3\vec{s}_1\cdot\hat{r}\,\vec{s}_2\cdot\hat{r} - \vec{s}_1\cdot\vec{s}_2] + V_{ls}(r)\vec{l}\cdot\vec{s}\,, \tag{2.9}$$

then

$$V|rl_\pm\rangle = V^\pm(r)|rl_\pm\rangle + V^T(r)|rl_\mp\rangle\,, \tag{2.10}$$

where

$$V^+(r) = V_C(r) - \frac{2(j+2)}{2j+1}V_T(r) + V_s(r) - (j+2)V_{ls}(r)\,, \tag{2.11}$$

$$V^-(r) = V_C(r) - \frac{2(j-1)}{2j+1}V_T(r) + V_s(r) + (j-1)V_{ls}(r)\,, \tag{2.12}$$

and

$$-V^T(r) = 6\frac{[j(j+1)]^{1/2}}{2j+1}V_T(r)\,. \tag{2.13}$$

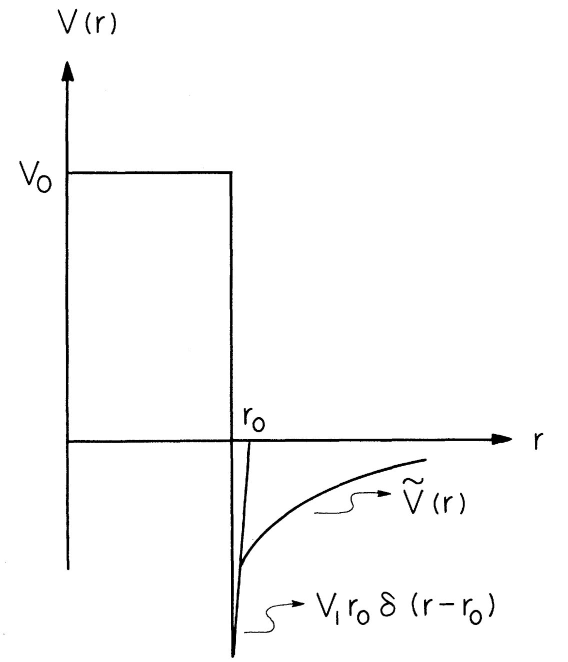

FIG. 1. Local potential forms for $V^\pm$, V^T which give the BCM in the limit described in the sentence containing (2.30)–(2.32).

The off-shell t matrix is related to the state $|qpl\rangle$, which satisfies the off-shell Lippmann-Schwinger equation

$$|qpl\rangle = |pl\rangle + \frac{1}{(q^2/2\mu) - H_0}V|qpl\rangle\,. \tag{2.14}$$

Taking the scalar product of (2.14) with $\langle rl_\pm|$ and using the closure relations (2.3) and (2.4), we find

$$\begin{aligned}\langle rl_\pm|qpl\rangle &= j_{l_\pm}(pr)\delta_{l_\pm l} + \frac{2}{\pi}\int_0^\infty k^2dk\frac{\langle rl_\pm|kl_\pm\rangle\langle kl_\pm|V|qpl\rangle}{(q^2-k^2)/2\mu}\\ &= j_{l_\pm}\delta_{ll_\pm} - \int_0^\infty dr'(r')^2\frac{2}{\pi}\int_0^\infty\frac{k^2dk\, j_{l_\pm}(kr)j_{l_\pm}(kr')}{k^2-q^2}[U^\pm(r')\langle r'l_\pm|qkl\rangle + U^T(r')\langle r'l_\mp|qkl\rangle]\\ &= j_{l_\pm}(kr)\delta_{ll_\pm} - \frac{1}{r}\int_0^\infty dr'G_q^\pm(r|r')[U^\pm(r')r'\langle r'l_\pm|qkl\rangle + U^T(r')r'\langle r'l^\pm|qkl\rangle]\,,\end{aligned} \tag{2.15}$$

where

$$U = 2\mu V\,, \tag{2.16}$$

and

$$G_q^\pm(r|r') = \frac{2}{\pi}\int_0^\infty\frac{k^2dk}{k^2-q^2}rr'j_{l_\pm}(kr)j_{l_\pm}(kr')\,. \tag{2.17}$$

Since

$$\left(\frac{d^2}{dr^2} + q^2 - \frac{l_\pm(l_\pm+1)}{y^2}\right)G_q^\pm(r|r') = -\delta(r-r')\,, \tag{2.18}$$

we have

$$\left(\frac{d^2}{dr^2}+q^2-\frac{l_\pm(l_\pm+1)}{r^2}\right)r\langle rl_\pm|qpl\rangle=(q^2-p^2)rj_{l_\pm}(pr)\delta_{l_\pm l}+U^\pm(r)r\langle rl^\pm|qpl\rangle+U^\mp(r)r\langle rl_\mp|qpl\rangle\ . \tag{2.19}$$

From (2.15) and (2.17), it follows that

$$\lim_{r\to 0} r\langle rl_\pm|qpl\rangle=0\,, \tag{2.20}$$

As in KTI, we represent the BCM as an appropriate limit of the local potential given in Fig. 1. Repulsive square barriers of strength $V^\pm$, V^T extend from particle separation $r=0$ to $r=r_0$, δ-function interactions $-V_1^\pm\delta(r-r_0)$, $-V_1^T\delta(r-r_0)$ border the square repulsive barriers, and there are local potentials $\tilde{V}^\pm(r)$, $\tilde{V}^T(r)$ for $r>r_0$. In this section and in Sec. III, we assume that $\tilde{V}^\pm(r)$ and $\tilde{V}^T(r)$ approach zero faster than $1/r$ as $r\to\infty$.

Since we will eventually have $U^\pm$ ($\equiv 2\mu V^\pm$) and U^T ($\equiv 2\mu V^T$) approach ∞, we may assume that

$$U^\pm,\ U^T >> q^2\,,$$

$$U^\pm,\ U^T >> \frac{(j\pm 1)(j\pm 1+1)}{r^2} \quad \text{for } r\neq 0\,. \tag{2.21}$$

An analysis similar to that in KTI then gives the following behavior for $r\langle rl|qpl'\rangle$ in the interval $0<r<r_0^-=r_0-\epsilon$ (ϵ = infinitesimal positive number)

$$[r\langle r|qp\rangle]\equiv\begin{bmatrix} r\langle rl_-|qpl_-\rangle & r\langle rl_+|qpl_-\rangle \\ r\langle rl_-|qpl_+\rangle & r\langle rl_+|qpl_+\rangle\end{bmatrix}\approx(q^2-p^2)[q^2-p^2-U]^{-1}[rj(pr)]$$

$$+\frac{1}{x_1^1x_2^2-x_1^2x_2^1}\begin{bmatrix} x_2^2x_1^1e^{\alpha_1(r-r_0)}-x_1^2x_2^1e^{\alpha_2(r-r_0)} & x_1^1x_1^2(-e^{\alpha_1(r-r_0)}+e^{\alpha_2(r-r_0)}) \\ x_2^1x_2^2(e^{\alpha_1(r-r_0)}-e^{\alpha_2(r-r_0)}) & x_2^2x_1^1e^{\alpha_2(r-r_0)}-x_1^2x_2^1e^{\alpha_1(r-r_0)}\end{bmatrix}[r_0\langle r_0|qp\rangle]\,, \tag{2.22}$$

where

$$[U]=\begin{bmatrix} U^- & U^T \\ U^T & U^+\end{bmatrix}\ , \tag{2.23}$$

$$[q^2-p^2-U]=\begin{bmatrix} q^2-p^2-U^- & U^T \\ U^T & q^2-p^2-U^+\end{bmatrix}, \tag{2.24}$$

$$[rj(pr)]=\begin{bmatrix} rj_{l_-}(pr) & 0 \\ 0 & rj_{l_+}(pr)\end{bmatrix}\ . \tag{2.25}$$

$[\ldots]^{-1}$ denotes matrix inverse and the x_j^i (i, $j=1,2$) and $\alpha_i{}^2$ are, respectively, the eigenvector components and corresponding eigenvalues of the matrix $[U]$.

$$[U]\begin{bmatrix} x_1^i \\ x_2^i\end{bmatrix}=\alpha_i^2\begin{bmatrix} x_1^i \\ x_2^i\end{bmatrix}\ ,\quad i=1,\ 2; \tag{2.26}$$

$$[x_1^i x_2^i]\begin{bmatrix} x_1^j \\ x_2^j\end{bmatrix}=\delta_{ij}\ ; \tag{2.27}$$

$$\det[U-\alpha_i{}^2]=0\ . \tag{2.28}$$

The α_i in (2.22) are to be considered as positive numbers, i.e., $\alpha_i=+\sqrt{\alpha_i{}^2}$. The explicit expressions for the α_i in terms of $U^\pm$ and U^T will not be given here, since they are not necessary for obtaining our final result. The derivative of $\lim_{\epsilon\to 0^+}[r\langle r|q=p+i\epsilon,p\rangle]=[r\langle r|pp\rangle]$ at $r=r_0^+=r_0+0^+$ is found by integrating (2.19) over the δ-function interactions at $r=r_0$.

$$\frac{d}{dr}[r\langle r|q=p,p\rangle]|_{r_0^+}=\begin{bmatrix}\frac{x_2^2x_1^1\alpha_1-x_2^1x_1^2\alpha_2}{x_1^1x_2^2-x_1^2x_2^2}-U_1^-r_0 & \frac{x_2^1x_2^2(\alpha_1-\alpha_2)}{x_1^1x_2^2-x_1^2x_2^2}-U_1^Tr_0\\ \frac{x_2^1x_2^2(\alpha_1-\alpha_2)}{x_1^1x_2^2-x_2^1x_1^2}-U_1^Tr_0 & \frac{x_1^1x_2^2\alpha_2-x_2^1x_1^2\alpha_1}{x_1^1x_2^2-x_1^2x_2^1}-U_1^+r_0\end{bmatrix}[r_0\langle r_0|q=p,p\rangle]\ . \tag{2.29}$$

The BCM corresponds to $U^\pm$, U^T, $U_1^\pm$, $U_1^T\to\infty$ with

$$\frac{x_2^2x_1^1\alpha_1-x_2^1x_1^2\alpha_2}{x_1^1x_2^2-x_1^2x_2^1}-U_1^-r_0\to\text{finite constant}=\frac{1}{r_0}f^j_{j-1}\ , \tag{2.30}$$

$$\frac{x_2^2x_1^1\alpha_2-x_2^1x_1^2\alpha_1}{x_1^1x_2^2-x_1^2x_2^1}-U_1^+r_0\to\text{finite constant}=\frac{1}{r_0}f^j_{j+1}\ , \tag{2.31}$$

$$\frac{x_2^1x_2^2(\alpha_1-\alpha_2)}{x_1^1x_2^2-x_1^2x_2^1}-U_1^Tr_0\to\text{finite constant}=\frac{1}{r_0}f^j\ . \tag{2.32}$$

Note that if U^T, $U_1^T\to0$, $x_j^i\to\delta_{ij}$, $\alpha_1\to\sqrt{U^+}$, $\alpha_2\to\sqrt{U^-}$, $f^j\to0$, and $f^j_{j\pm1}\to r_0(\sqrt{U^\pm}-r_0U_1^\pm)$ in agreement with the result for uncoupled states found in KTI.

III. INTEGRAL EQUATION FOR THE OFF-SHELL t MATRIX IN THE CASE OF TENSOR FORCES

The integral equation (2.15), in the notation introduced in (2.22)–(2.25), becomes

$$[r\langle r|qp\rangle]=[rj(pr)]-\int_0^\infty dr'r'[G_q(r|r')][U(r')][r'\langle r'|qp\rangle]\ , \tag{3.1}$$

where $U/2\mu$ is the total interaction and

$$[G_q(r|r')]=\begin{bmatrix}G_q^-(r|r') & 0\\ 0 & G_q^+(r|r')\end{bmatrix}\ . \tag{3.2}$$

$[U]$ may be written as $[U_{\mathrm{BCM}}]+[\tilde U]$, where $[\tilde U]$, the "outside" potential, vanishes in the core region $(0<r<r_0^+)$, and $[U_{\mathrm{BCM}}]$ is the core interaction discussed in Sec. II. From (2.22)–(2.32), we find that

$$[U_{\mathrm{BCM}}(r)][r\langle r|qp\rangle]=-(q^2-p^2)[rj(pr)]\theta(r_0-r)+\left(\frac{1}{r_0}[f]\delta(r-r_0^-)+\frac{d}{dr}\delta(r-r_0^-)[1]\right)[r_0\langle r_0|qp\rangle]\ , \tag{3.3}$$

where $\theta(r_0-r)$ is the Heaviside unit function and

$$[f]=\begin{bmatrix}f^j_{j-1} & f^j\\ f^j & f^j_{j+1}\end{bmatrix}\ . \tag{3.4}$$

In deriving (3.3), we have used the fact that

$$\alpha^2e^{-\alpha(r-r_0)}\to\alpha\delta(r-r_0^-)+\frac{d}{dr}\delta(r-r_0^-) \tag{3.5}$$

for $r<r_0$ and α very large.

Substituting (3.3) into (3.1) and setting $r=r_0$, we may solve formally for $[r_0\langle r_0|qp\rangle]$. After substituting the expression for $[r_0\langle r_0|qp\rangle]$ into (3.1), we obtain

$$[r\langle r|qp\rangle]=[r\tilde j(pr)]+(q^2-p^2)\int_0^{r_0}dr'[\tilde G_q(r|r')][r'j(pr')]-\int_{r_0}^\infty dr'[\tilde G_q(r|r')][\tilde U(r')][r'\langle r'|qp\rangle]\ , \tag{3.6}$$

where

$$[r\tilde j(pr)]=[rj(pr)]-\left[\frac{1}{r_0}G_q(r|r_0)f-\frac{d}{dr'}G_q(r|r')\Big|_{r'=r_0^-}\right]\left[1+\frac{1}{r_0}G_q(r_0|r_0)f-\frac{d}{dr'}G_q(r_0|r')\Big|_{r'=r_0^-}\right]^{-1}[r_0j(pr_0)]\ , \tag{3.7}$$

and

$$[\tilde{G}_q(r|r')] = [G_q(r|r')] - \left[\frac{1}{r_0} G_q(r|r_0)f - \frac{d}{dr''} G_q(r|r'')\bigg|_{r''=r_0^-}\right]\left[1 + \frac{1}{r_0} G_q(r_0|r_0)f - \frac{d}{dr''} G_q(r_0|r'')\bigg|_{r''=r_0^-}\right]^{-1} [G_q(r_0|r')] \ . \tag{3.8}$$

The off-shell t-matrix operator $t(q)$ satisfies the integral equation

$$t(q) = -U + U \frac{1}{q^2 - 2\mu H_0} t(q) \ , \tag{3.9}$$

and has matrix elements given by

$$[\langle p'|t(q)|p\rangle] = \begin{bmatrix} \langle p'l_-|t(q)|pl_-\rangle & \langle p'l_+|t(q)|pl_-\rangle \\ \langle p'l_-|t(q)|pl_+\rangle & \langle p'l_+|t(q)|pl_+\rangle \end{bmatrix}$$

$$= -[\langle p'|U|qp\rangle] = -\int_0^\infty [\langle p'|r\rangle][\langle r|U|qp\rangle]r^2 dr$$

$$= -\int_0^\infty dr[rj(p'r)][U(r)][r\langle r|qp\rangle] \ , \tag{3.10}$$

where

$$[\langle p'|U|qp\rangle] = \begin{bmatrix} \langle p'l_-|U|qpl_-\rangle & \langle p'l_+|U|qpl_-\rangle \\ \langle p'l_-|U|qpl_+\rangle & \langle p'l_+|U|qpl_+\rangle \end{bmatrix} \ , \tag{3.11}$$

with similar identifications for $[\langle r|U|qp\rangle]$ and $[\langle p'|r\rangle]$. On the energy shell $[q \to q$ (real and positive) $+ i\epsilon$, $\epsilon \to 0^+$, $p' = p = q]$,

$$q[\langle p'=q|t(q+i\epsilon)|p=q\rangle] = (1/2i)[S-1] \ , \tag{3.12}$$

where $[S]$ is a two-by-two unitary and symmetric matrix in the case of the usual strong-interaction symmetries.

Use of the decomposition $U = U_{\rm BCM} + \tilde{U}$, and (3.3) in (3.10) yields

$$[\langle p'|t(q)|p\rangle] = -\int_0^\infty dr[rj(p'r)]\left\{(p^2 - q^2)\theta(r_0 - r)[rj(pr)] + \left(\frac{1}{r_0}[f]\delta(r - r_0) + \frac{d}{dr}\delta(r - r_0^-)[r_0\langle r_0|qp\rangle]\right)\right\}$$

$$- \int_{r_0}^\infty dr[rj(p'r)][\tilde{U}(r')][r\langle r|qp\rangle] \ . \tag{3.13}$$

After inserting the previously determined expression for $[r_0\langle r_0|qp\rangle]$ into (3.13), we have

$$[\langle p'|t(q)|p\rangle] = [\langle p'|t_{\rm BCM}(q)|p\rangle] + [\langle p'|\tilde{t}(q)|p\rangle] \ , \tag{3.14}$$

where $t_{\rm BCM}(q)$ is the "pure" BCM result (for $\tilde{U}=0$).

$$[\langle p'|t_{\rm BCM}(q)|p\rangle] = (q^2 - p^2)\int_0^{r_0} dr[r\bar{\bar{j}}(p'r)][rj(pr)] - \{[r_0 j(p'r_0)] - [r_0\bar{\bar{j}}(p'r_0)]\}[G_q(r_0|r_0)]^{-1}[r_0 j(pr_0)] \ , \tag{3.15}$$

with

$$[r\bar{\bar{j}}(p'r)] = [rj(p'r)] - \left([r_0 j(p'r_0)]\frac{1}{r_0}[f] - \frac{d}{dr'}[r'j(p'r')]\bigg|_{r'=r_0}\right)\left[1 + \frac{1}{r_0} G_q(r_0|r_0)f - \frac{d}{dr'} G_q(r_0|r')\bigg|_{r'=r_0^-}\right]^{-1}[G_q(r_0|r)], \tag{3.16}$$

and $\tilde{t}(q)$ is the contribution to $t(q)$ coming from the "outside" interaction $\tilde{V}$.

$$[\langle p'|\tilde{t}(q)|p\rangle] = -\int_{r_0}^\infty dr[r\bar{\bar{j}}(p'r)][\tilde{U}(r)][r\langle r|qp\rangle] \ . \tag{3.17}$$

Use of (2.17) and (3.6) in (3.17) gives an integral equation for $[\langle p'|\tilde{t}(q)|p\rangle]$.

$$[\langle p'|\tilde{t}(q)|p\rangle] = [\langle p'|U_1(q)|p\rangle] - \frac{2}{\pi}\int_0^\infty \frac{k^2 dk}{q^2 - k^2}[\langle p'|U_2(q)|k\rangle][\langle k|\tilde{t}(q)|p\rangle] \ , \tag{3.18}$$

where

$$[\langle p'|U_1(q)|p\rangle] = -\int_{r_0}^{\infty} dr[r\bar{\bar{j}}(p'r)][\tilde{U}(r)]\{[r\tilde{j}(pr)] + (q^2-p^2)\int_0^{r_0} dr'[\tilde{G}_q(r|r')][r'j(pr')]\} , \tag{3.19}$$

and

$$[\langle p'|U_2(q)|p\rangle] = -\int_{r_0}^{\infty} dr[r\bar{\bar{j}}(p'r)][\tilde{U}(r)][rj(pr)] . \tag{3.20}$$

For the case $f^j = 0$, $[\langle p'|t(q)|p\rangle]$ becomes a diagonal matrix whose elements $\langle p'l_\pm|t(q)|pl_\pm\rangle$ are given in KTI.

In the hard-core limit of the BCM, $[f] \to \infty$, we have

$$[\langle p'|t_{\rm BCM}(q)|p\rangle] \to [\langle p'|t_{\rm HC}(q)|p\rangle] = (q^2-p^2)\int_0^{r_0} dr\{[rj(p'r)] - [r_0 j(p'r_0)][G_q(r_0|r_0)]^{-1}[G_q(r_0|r)]\}[rj(pr)]$$
$$- [r_0 j(p'r_0)][G_q(r_0|r_0)]^{-1}[r_0 j(pr_0)] , \tag{3.21}$$

$$[\langle p'|U_1(q)|p\rangle] \to -\int_{r_0}^{\infty} dr\{[rj(p'r)] - [r_0 j(p'r_0)][G_q(r_0|r_0)]^{-1}[G_q(r_0|r)]\}[\tilde{U}(r)]$$
$$\times\{[rj(pr)] - [G_q(r|r_0)][G_q(r_0|r_0)]^{-1}[r_0 j(pr_0)]\} , \tag{3.22}$$

and

$$[\langle p'|U_2(q)|p\rangle] = -\int_{r_0}^{\infty} dr\{[rj(p'r)] - [r_0 j(p'r_0)][G_q(r_0|r_0)]^{-1}[G_q(r_0|r)]\}[\tilde{U}(r)][rj(pr)] . \tag{3.23}$$

IV. MODIFICATIONS OF THE INTEGRAL EQUATION FOR THE OFF-SHELL t MATRIX, DUE TO LONG-RANGE COULOMB FORCES

The analysis of the previous sections of this paper and of KTI must be modified if the interaction V has a long-range Coulomb contribution. In order to avoid the complications arising from the infinite range of an unscreened Coulomb force,[5] we make the assumption (which is usually physically justified) that for particle separations $r > R$ (where $R \ggg$ range of the nuclear force) the Coulomb force is totally screened.[6-8]

Specifically, we assume that

$$V = V_n + V_C , \tag{4.1}$$

where V_C is a repulsive screened Coulomb interaction with

$$V_C(0 < r < R) = Z_1 Z_2 e^2/r \quad (e = \text{electron charge}),$$
$$V_C(r > R) = 0 , \tag{4.2}$$

and V_n is a short-range nuclear interaction with a singular core behavior described by the BCM.

Let $|plsjj_z\rangle_C^\pm$ be a simultaneous eigenstate of H_C ($\equiv p^2/2\mu + V_C$), $(\vec{l})^2$, $(\vec{s})^2$, $(\vec{j})^2$, and j_z with associated eigenvalues $p^2/2\mu$, $l(l+1)$, $s(s+1)$, $j(j+1)$, and j_z, respectively. The $\pm$ superscript denotes the outgoing-(incoming-) wave asymptotic character of the state in coordinate space. The transformation matrix between these states and the states $|rlsjj_z\rangle$ is

$$\langle rl's'j'j_z'|plsjj_z\rangle_C^\pm = \delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{j_z'j_z} e^{\pm i\gamma_l^C(p)}\bar{F}_l(\eta, pr)/pr ,$$

$$\eta = \mu Z_1 Z_2 e^2/p , \tag{4.3}$$

where

$$\bar{F}_l(\eta, pr)/pr = \cos\gamma_l^C(p) j_l(pr) - \sin\gamma_l^C(p) n_l(pr) \quad \text{for } r > R , \tag{4.4}$$

$$\bar{F}_l(\eta, pr) = N_l^C(p) F_l(\eta, pr) \quad \text{for } 0 < r < R , \tag{4.5}$$

$$\frac{d}{dr}\ln F_l(\eta, pr)\Big|_{r=R} - \frac{1}{R} = \frac{d}{dr}\ln[\cos\gamma_l^C(p)j_l(pr) - \sin\gamma_l^C(p)n_l(p)]\Big|_{r=R}, \tag{4.6}$$

and

$$N_l^C F_l(\eta, pR)/pR = \cos\gamma_l^C(p)j_l(pR) - \sin\gamma_l^C(p)n_l(pR). \tag{4.7}$$

$F_l(\eta, \rho)$ is the regular Coulomb radial wave function[9] which satisfies the equations

$$\left(\frac{d^2}{d\rho^2} + 1 - \frac{l(l+1)}{\rho^2} - \frac{2\eta}{\rho}\right) F_l(\eta, \rho) = 0, \tag{4.8}$$

$$F_l(\eta, \rho) = 0, \tag{4.9}$$

$$F_l(\eta, \rho) \approx \sin[\rho - \tfrac{1}{2}l\pi - \eta\ln 2\rho + \sigma_l(\eta)], \quad \rho >> [l(l+1) + \eta^2]^{1/2} \tag{4.10}$$

with

$$\sigma_l(\eta) = \arg\Gamma(l + 1 + i\eta). \tag{4.11}$$

If $(pR)^2 >> l(l+1) + \eta^2(p)$, it is easily seen that

$$\gamma_l^C(p) = \sigma_l(\eta) - \eta\ln 2pR + O(\eta/2pR), \tag{4.12}$$

$$N_l^C(p) = 1 + O(\eta/2pR). \tag{4.13}$$

The closure relation for the states $|plsjj_z\rangle_C^{\pm}$ is

$$\frac{2}{\pi}\sum_{lsjj_z}\int_0^\infty p^2dp\,|plsjj_z\rangle_C^{\pm}\,{}_C^{\pm}\langle plsjj_z| = 1. \tag{4.14}$$

The use of a screened Coulomb potential allows us to employ the usual relations of formal scattering theory. The off-shell Lippmann-Schwinger equation, with V given by (4.1), is

$$\begin{aligned}|qp\rangle &= |p\rangle + \frac{1}{q^2/2\mu - H_0}(V_n + V_C)|qp\rangle \\ &= |qp\rangle_C + \frac{1}{q^2/2\mu + H_C} V_n|qp\rangle,\end{aligned} \tag{4.15}$$

where $|qp\rangle_C$, the off-shell "pure" Coulomb scattering state, satisfies the equation

$$\begin{aligned}|qp\rangle_C &= |p\rangle + \frac{1}{q^2/2\mu - H_0} V_C|qp\rangle_C \\ &= |p\rangle + \frac{1}{q^2/2\mu - H_C} V_C|p\rangle.\end{aligned} \tag{4.16}$$

The quantum-number labels $lsjj_z$ are omitted in (4.15), (4.16), and will be omitted hereafter.

The off-shell Coulomb t matrix $\langle p'|t_C(q)|p\rangle$ is given by

$$\begin{aligned}\langle p'|t(q)|p\rangle &= -\langle p'|U_C|qp\rangle_C \\ &= -\langle q^*p'|U_C|p\rangle,\end{aligned} \tag{4.17}$$

where q^* is the complex conjugate of q.

Note that

$$\lim_{\epsilon\to 0^\pm}|q = p + i\epsilon, p\rangle_C = |p\rangle_C^{\pm}, \tag{4.18}$$

$$\begin{aligned}-\langle p'|U_C|p\rangle_C^{+} &= \lim_{\epsilon\to 0^+}\langle p'|t_C(q = p + i\epsilon)|p\rangle \\ &= \langle p'|t_C(p)|p\rangle,\end{aligned} \tag{4.19}$$

$$\begin{aligned}-{}_C^{-}\langle p'|U_C|p\rangle &= \lim_{\epsilon\to 0^+}\langle p'|t_C(q = p' + i\epsilon)|p\rangle \\ &= \langle p'|t_C(p')|p\rangle,\end{aligned} \tag{4.20}$$

and

$$-{}_C^{-}\langle p|U_C|p\rangle = -\langle p|U_C|p\rangle_C^{+} = \langle p|t_C(p)|p\rangle$$
$$= \frac{e^{i\gamma^C(p)}\sin\gamma^C(p)}{p}\ , \tag{4.21}$$

where $|p\rangle_C^{\pm}$ and $\gamma^C(p)$ are given by (4.3)–(4.11). In order to obtain the general off-shell Coulomb state $|qp\rangle_C$, we subtract the Lippmann-Schwinger equation for $|p\rangle_C^{+}$,

$$|p\rangle_C^{+} = |p\rangle + \lim_{\epsilon\to 0^+} \frac{1}{p^2/2\mu + i\epsilon - H_C} V_C|p\rangle\ , \tag{4.22}$$

from (4.15) and insert the closure sum (4.14) over incoming-wave states in front of V_C.

$$|qp\rangle_C = |p\rangle_C^{+} + \lim_{\epsilon\to 0^+}\frac{2}{\pi}\int_0^\infty k^2dk\,|k\rangle_C^{-}\ {}_C^{-}\langle k|V_C|p\rangle\left(\frac{1}{q^2/2\mu - k^2/2\mu} - \frac{1}{p^2/2\mu + i\epsilon - k^2/2\mu}\right)$$
$$= |p\rangle_C^{+} + \lim_{\epsilon\to 0^+}\frac{2}{\pi}\int_0^\infty k^2dk\,|k\rangle_C^{-}\ \langle k|t_C(k)|p\rangle\left(\frac{1}{q^2-k^2} - \frac{1}{p^2+i\epsilon-k^2}\right). \tag{4.23}$$

After taking the scalar product of (4.15) with the states $\langle r|$ and using the closure relations (2.3) and (4.14), we find

$$[r\langle r|qp\rangle] = [r\langle r|qp\rangle_C] + \int_0^\infty dr'\frac{2}{\pi}\int_0^\infty \frac{k^2dk}{q^2k^2/2\mu} rr'[\langle r|k\rangle_C^{\pm}][{}_C^{\pm}\langle k|r'\rangle][V_n(r')][\langle r'|qp\rangle]$$
$$= [r\langle r|qp\rangle_C] - \int_0^\infty dr'[G_q^C(r|r')][U_n(r')][r'\langle r'|qp\rangle]\ , \tag{4.24}$$

where

$$[G_q^C(r|r')] = \begin{bmatrix} G_q^{C,-}(r|r') & 0 \\ 0 & G_q^{C,+}(r|r') \end{bmatrix}\ , \tag{4.25}$$

with

$$G_q^{C,\pm}(r|r') = \frac{2}{\pi}\int_0^\infty \frac{dk}{k^2-q^2} F_{l_\pm}(\eta, kr)F_{l_\pm}(\eta, kr') \tag{4.26}$$

for r, $r' \langle R$ and R large enough to make $N_l^C(p)$ in (4.5) very nearly unity. For $q \to q(\text{real and positive}) + i0^+$,

$$G_q^{C,\pm}(r|r') = (1/q)H_{l_\pm}^{(1)}(\eta, qr)F_{l_\pm}(\eta, qr)\ . \tag{4.27}$$

$H_l^{(1)}(\eta,\rho)$ is given by

$$H_l^{(1)}(\eta,\rho) = G_l(\eta,\rho) + iF_l(\eta,\rho)\ ,$$

where $G_l(\eta,\rho)$ is the irregular Coulomb radial wave function, which satisfies (4.8) and behaves asymptotically as

$$G_l(\eta,\rho) \xrightarrow[\rho \gg [l(l+1+\eta^2)]^{1/2}]{} \cos[\rho - \eta\ln 2\rho - \tfrac{1}{2}l\pi + \sigma_l(\eta)]\ . \tag{4.28}$$

For nonphysical values of q, we must use the more general expression

$$G_q^{C,\pm}(r|r') = 2^{2l}(qr_>)^{l+1}(qr_<)^{l+1}e^{-\pi\eta}\frac{\Gamma(l+1+i\eta)\Gamma(l+1-i\eta)}{[(2l+1)!]^2}e^{iqr_>}e^{iqr_<}\,{}_1F_1(l+1+i\eta;\,2l+2;\,-2iqr_<)$$
$$\times 2iW_1(l+1+i\eta;\,2l+2;\,-2iqr_>)\ ,\quad l = l_\pm\ , \tag{4.29}$$

where ${}_1F_1$ and W_1 are defined, e.g., in the book by Mott and Massey.[10]

Using the first form of (4.15), it is easy to see that

$$\left\{\left(\frac{d^2}{dr^2}+q^2-U_C(r)\right)\begin{bmatrix}1&0\\0&1\end{bmatrix}-\frac{1}{r^2}\begin{bmatrix}l_-(l_-+1)&0\\0&l_+(l_++1)\end{bmatrix}\right\}[r\langle r|qp\rangle]=(q^2-p^2)[rj(pr)]+[U_n(r)][r\langle r|qp\rangle]\ ,$$
$$\lim_{r\to 0}[r\langle r|qp\rangle]=0\ . \tag{4.30}$$

If we express U_n as $U_{n,\mathrm{BCM}}+\tilde{U}_n$, an analysis similar to that of Sec. II and III leads to (3.3) with U_{BCM} replaced by $U_{n,\mathrm{BCM}}$, and to (3.6)–(3.8) with $\tilde{U}$ replaced by $\tilde{U}_n$, $[r\tilde{j}(pr)]$ replaced by

$$[r\langle r\tilde{|}qp\rangle_C]=[r\langle r|qp\rangle_C]-\left[\frac{1}{r_0}G_q^C(r|r_0)f-\frac{d}{dr'}G_q^C(r|r')\bigg|_{r'=r_0^-}\right]\left[1+\frac{1}{r_0}G_q^C(r_0|r_0)-\frac{d}{dr'}G_q^C(r|r')\bigg|_{r'=r_0^-}\right]^{-1}[r_0\langle r_0|qp\rangle_C]\ , \tag{4.31}$$

and $[G_q(r|r')]$ replaced by $[G_q^C(r|r')]$.

We now use (4.15) to decompose the off-shell t-matrix element into "nuclear" and "Coulomb" parts.

$$\begin{aligned}[\langle p'|t(q)|p\rangle]&=-[\langle p'|U_C+U_n|qp\rangle]\\ &=-[{}_C\langle q^*p'|U_n|qp\rangle]-[{}_C\langle q^*p'|U_C|p\rangle]\\ &=-[{}_C\langle q^*p'|U_n|qp\rangle]-[\langle p'|U_C|qp\rangle_C]\ .\end{aligned} \tag{4.32}$$

The last term in (4.32) is the off-shell Coulomb scattering amplitude, which may be evaluated by using (4.23).

$$\begin{aligned}[\langle p'|t_C(q)|p\rangle]&=-[\langle p'|U_C|qp\rangle_C]=-[\langle p'|U_C|p\rangle_C^+]-\lim_{\epsilon\to 0^+}\frac{2}{\pi}\int_0^\infty k^2dk[\langle p'|U_C|k\rangle_C^-][\langle k|t_C(k)|p\rangle]\left(\frac{1}{q^2-k^2}-\frac{1}{p^2+i\epsilon-k^2}\right)\\ &=[\langle p'|t_C(p)|p\rangle]+\lim_{\epsilon\to 0^+}\frac{2}{\pi}\int_0^\infty k^2dk[\langle k|t_C(k)|p'\rangle^*][\langle k|t_C(k)|p\rangle]\left(\frac{1}{q^2-k^2}-\frac{1}{p^2+i\epsilon-k^2}\right)\\ &=[\langle p'|t_C(p)|p\rangle]+ip[\langle p|t_C(p)|p'\rangle^*][\langle p|t_C(p)|p\rangle]+\frac{2}{\pi}\int_0^\infty k^2dk[\langle k|t_C(k)|p'\rangle^*][\langle k|t_C(k)|p\rangle]\\ &\quad\times\left(\frac{1}{q^2-k^2}-\mathrm{P}\frac{1}{p^2-k^2}\right)\cdot\end{aligned} \tag{4.33}$$

The P in (4.33) denotes that the principal value of the integral is to be calculated. An analysis similar to that of Sec. III gives

$$\begin{aligned}-[{}_C\langle q^*p'|U_n|qp\rangle]&=-[{}_C\langle q^*p'|U_{n,\mathrm{BCM}}+\tilde{U}_n|qp\rangle]\\ &=[\langle p'|t_n(q)|p\rangle]=[\langle p'|t_{n,\mathrm{BCM}}(q)|p\rangle]+[\langle p'|\tilde{t}_n(q)|p\rangle]\ .\end{aligned} \tag{4.34}$$

$[\langle p'|t_{n,\mathrm{BCM}}(q)|p\rangle]$, the "pure" BCM contribution, is given by

$$[\langle p'|t_{n,\mathrm{BCM}}(q)|p\rangle]=(q^2-p^2)\int_0^{r_0}[r\,{}_C\langle q^*p'\tilde{\tilde{|}}r\rangle][rj(pr)]-\{[r_0\,{}_C\langle q^*p'|r_0\rangle]-[r_0\,{}_C\langle q^*p'\tilde{\tilde{|}}r_0\rangle]\}[G_q^C(r_0|r_0)]^{-1}[r_0\langle r_0|qp\rangle_C]\ , \tag{4.35}$$

where

$$\begin{aligned}[r\,{}_C\langle q^*p'\tilde{\tilde{|}}r\rangle&=[r\,{}_C\langle q^*p'|r\rangle]-\left([r\,{}_C\langle q^*p'|r_0\rangle][f]-\frac{d}{dr'}[r'\,{}_C\langle q^*p'|r'\rangle]\bigg|_{r'=r_0}\right)\\ &\quad\times\left[1+G_q^C(r_0|r_0)\frac{f}{r_0}-\frac{d}{dr'}G_q^C(r_0|r')\bigg|_{r'=r_0^-}\right]^{-1}[G_q^C(r_0|r)]\ ,\end{aligned} \tag{4.36}$$

and $[\langle p'|\tilde{t}_n(q)|p\rangle]$, the contribution due to $\tilde{V}$, is given by

$$[\langle p'|\tilde{t}_n(q)|p\rangle]=-\int_{r_0}^\infty dr[r\,{}_C\langle q^*p'\tilde{\tilde{|}}r\rangle][\tilde{U}_n(r)][r\langle r|qp\rangle]\ . \tag{4.37}$$

In calculating $[\langle p'|\tilde{t}_n(q)|p\rangle]$, we first determine the auxiliary amplitude

$$[\langle p'|\hat{t}_n(q)|p\rangle] = -\int_{r_0}^{\infty} dr[r\, {}_C\langle q^{*}=p'-i0^{+}, p'\tilde{\tilde{|}} r\rangle][\tilde{U}_n(r)][r\langle r|qp\rangle] \,, \tag{4.38}$$

which satisfies the integral equation (3.18) with

$$[\langle p'|U_1(q)|p\rangle] = -\int_{r_0}^{\infty} dr[r\, {}_C^{-}\langle p'\tilde{|} r\rangle][\tilde{U}_n(r)][r\langle r\tilde{|}qp\rangle_C] + (q^2-p^2)\int_0^{r_0} dr'[\tilde{G}_q(r|r')][r'j(pr')] \,, \tag{4.39}$$

where

$$[r\langle r\tilde{|}qp\rangle_C] = [r\langle r|qp\rangle_C] - \left[G_q^C(r|r_0)\frac{f}{r_0} - \frac{d}{dr'}G_q^C(r|r')\bigg|_{r'=r_0}\right]\left[1 + G_q^C(r_0|r_0)\frac{f}{r_0} - \frac{d}{dr'}G_q^C(r_0|r')\bigg|_{r'=r_0}\right]^{-1}[r_0\langle r_0|qp\rangle_C] \,, \tag{4.40}$$

and $[r\, {}_C^{-}\langle p'\tilde{\tilde{|}} r\rangle]$ is given by (4.36) with $[r\, {}_C^{-}\langle p'|r\rangle]$ replacing $[r\, {}_C\langle q^{*}p'|r\rangle]$, and

$$[\langle p'|U_2(q)|k\rangle] = -\int_{r_0}^{\infty} dr[r\, {}_C^{-}\langle p'\tilde{\tilde{|}} r\rangle][\tilde{U}_n(r)][r\langle r|k\rangle_C^{-}] \,. \tag{4.41}$$

$[\langle p'|t_n(q)|p\rangle]$ may then be determined from $[\langle p'|\hat{t}_n(q)|p\rangle]$ by using (4.23) and (4.38).

$$[\langle p'|\tilde{t}_n(q)|p\rangle] = [\langle p'|\hat{t}_n(q)|p\rangle] + \lim_{\epsilon\to 0^{+}}\frac{2}{\pi}\int_0^{\infty} k^2dk[\langle k|t_C(k)|p\rangle^{*}][\langle k|\hat{t}_n(q)|p\rangle]\left(\frac{1}{q^2-k^2} - \frac{1}{p^2+i\epsilon-k^2}\right) \,. \tag{4.42}$$

V. SUMMARY AND DISCUSSION

We have extended the analysis of KTI to the case of tensor and Coulomb forces. The treatment of the case of tensor forces (without long-range Coulomb effects) in Secs. II and III is a straightforward generalization of the treatment of uncoupled states in KTI.

On the other hand, the inclusion of long-range Coulomb forces in the analysis leads to major practical complications, even though we use a screened Coulomb force which allows the use of the ordinary formal theory of scattering. There does not appear to be a simple integral equation which gives the complete off-shell scattering amplitude in terms of the off-shell Coulomb scattering amplitude and eigenstates. One must first determine the auxiliary amplitude $\langle p'|\hat{t}_n(q)|p\rangle$ and then evaluate the complete off-shell t matrix by a one-dimensional quadrature. This procedure must also be used in the case of nonsingular core interactions.

In view of the complications resulting from the decomposition of the off-shell amplitude into a "pure" Coulomb and a "nuclear" part, it may well be more convenient to simply include the screened Coulomb potential in $\tilde{V}$ and to use the formalism of Sec. III.[11]

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