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Off-Shell Effects in Knockout Reactions*

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The amplitude for a knockout reaction in the plane-wave impulse approximation is derived from the Faddeev equations. It factorizes into the product of an off-shell two-body t matrix times the bound-state wave function in momentum space. In this paper results are derived for the $(p, 2p)$ experiment from a realistic potential and compared with those arising from some current approximations. Distortion effects are ignored in order to isolate the off-shell effects. We find that for incident protons with lab energy below 200 MeV using an off-shell amplitude can change the cross section significantly as compared with the on-shell approximations in current use. The effect increases with the binding energy of the struck proton and with the recoil momentum. This indicates that $(p, 2p)$ reactions below 200 MeV are sensitive to the off-shell behavior of the two-body amplitude and that they should not be used to obtain nuclear information using on-shell approximations.

I. INTRODUCTION

Interest in knockout reactions which proceed via quasielastic scattering stems mainly from the two distinct kinds of information which may be extracted from them: (1) information about off-shell scattering amplitudes between the projectile and the struck particle, and (2) information about the wave function of the struck particle. Especially in $(p, 2p)$ reactions, most of the emphasis to date has been on the second type. In this paper, we focus on the off-shell information. Our understanding of these effects then allows us to discuss ambiguities in the extraction of the form factors and spectroscopic factors arising from off-shell properties.

Off-shell information may be obtained from many-body systems. It has been found that calculations of the density and binding energy of nuclear matter[1] and the low-energy three-nucleon properties[2] (H^3 binding energy and the doublet n-D scattering length) are very sensitive to the details of the two-nucleon force. Since these systems contain both protons and neutrons, their properties depend on both the $T=0$ and $T=1$ forces and are especially sensitive to the off-shell properties of both the $^1S_0$ and $^3S_1$ states and on the ratio of central to tensor forces. Because the nucleons interact over long periods of time and at low energies, the exclusion principle is important and must be taken into account in deriving the effective interactions.[3] These facts make the extraction of off-shell information from the properties of bound many-body systems quite difficult.

Nuclear reactions provide an additional degree of freedom compared with bound states. Since continuum wave functions have flux at infinity, the investigator may choose a particular set of boundary conditions by specifying the incident particle, target, the energy of the reaction, etc. Comparisons

between the same reaction at different energies can provide sensitive probes of very specific aspects of a nuclear system. In such a comparison the complexities of the nuclear structure may be the same in both cases. This would permit empirical information from one experiment to be used in analysis of another. Thus, the second experiment may depend primarily on the specific property of interest.

We propose to study the $(p, 2p)$ reaction with regard to determining off-shell information about the $T=1$ part of the two-nucleon scattering amplitude. We treat knockout as a three-body reaction via iteration of the Faddeev equations. In the three-body model the target is treated as a two-body bound state with a given wave function. In reality, the structure of the target is much more complicated than this and must be described in terms of parentage expansions.[4] Also, one must include distortion effects, multiple-scattering corrections, and inelastic effects.[5] In this work we ignore all these corrections and examine the plane-wave impulse approximation in order to determine if there are regions of phase space which are sensitive to the off-shell behavior of the proton-proton amplitude.

In most of the work on knockout, on-shell approximations are used for the t matrix. In order to determine whether or not it is necessary to go off shell or whether an on-shell approximation suffices, we have compared the half-shell t matrix elements as prescribed by a one-term impulse approximation from the Faddeev equations with the various on-shell approximations in current use.

Since we find that the on-shell approximations are good at sufficiently high energies, we presume that the bound-state wave function is known, being obtainable from high energy $(p, 2p)$ or $(e, e'p)$ experiments or from a theoretical calculation. (In fact, however, there are always difficulties in the extraction of the wave functions.) The dependence of the $(p, 2p)$ cross section on the fact that the amplitude is off shell is then displayed. We find that for the coplanar symmetric $(p, 2p)$ experiment below about 200 MeV, none of the on-shell approximations tested is completely satisfactory. The disagreement is worse for more deeply bound states. At these energies it is essential to include the effects of refraction before attempting to extract off-shell information from experiment. These distortion effects may reduce the sensitivity of the experiment to off-shell behavior somewhat, but we do not expect qualitative changes in our conclusions.

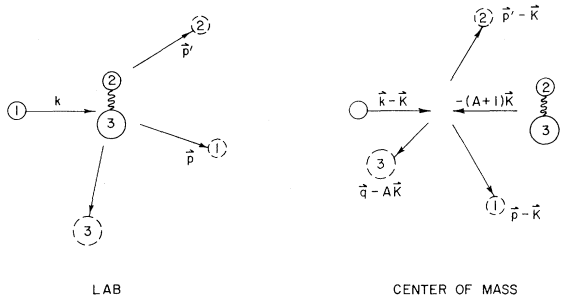

FIG. 1. Kinematics for the equal-mass knockout reaction in the lab and c.m. frames. Particles 1 and 2 are taken to have a mass m while particle 3 has mass Am. $\vec{K}$ is the velocity of the c.m. in the lab times m and equals $\vec{k}/(A+2)$.

In Sec. II we use the Faddeev equation and derive the plane-wave impulse approximation for equal-mass knockout reactions [e.g. $(p, 2p)$] to determine the appropriate half-shell scattering amplitude. The case of unequal masses is relegated to the Appendix. In Sec. III we discuss the kinematics of the coplanar symmetric experiment. The approximations in current use are defined in Sec. IV. We specialize to the proton-proton case and, using the Reid soft-core potential as a model for the interaction, compare the half-shell p-p cross section with those of the various approximations in Sec. V. The effect of these divers approaches on representative $(p, 2p)$ reactions are presented and discussed in Sec. VI. We present our conclusions in Sec. VII.

II. IMPULSE APPROXIMATION

In this section we derive the plane-wave impulse approximation (PWIA) for the knockout reaction in which the incident and the struck particle both have masses equal to m. The core is taken to have a mass Am. The more general case is considered in the Appendix. The equal-mass case will be referred to as the "$(p, 2p)$ case" although the equations are also correct for other cases such as $(\alpha, 2\alpha)$.

We describe the reaction by means of a three-body model. The residual nucleus is treated as an inert core which binds a proton in the initial state of the system. If the wave function of the bound state in momentum space is taken to be ψ, then the scattering amplitude in the c.m. frame is (see Fig. 1)

$$M=\int d\vec{k}'\, d\vec{q}\langle \vec{p}-\vec{K}, \vec{p}'-\vec{K}, \vec{q}-A\vec{K}\,|\,T(E)\,|\vec{k}-\vec{K}, \vec{k}'-\vec{K}, -\vec{k}'-A\vec{K}\rangle\psi(\vec{k}'), \tag{1}$$

where $T(E)$ is the three-body scattering amplitude. In Eq. (1) plane-wave states are used for particles 1, 2, and 3, respectively. The momentum of the incident proton in the lab is $\vec{k}$, and the momenta of the two outgoing protons are $\vec{p}$ and $\vec{p}'$. Since the recoil momentum $\vec{q}$ is not measured, we integrate over it. It is determined from momentum conservation to be

$$\vec{q}=\vec{k}-\vec{p}-\vec{p}'.$$

The energy of the system in the center of mass is

$$E=\frac{k^2}{2m}+Q-\frac{k^2}{2m(A+2)},$$

where Q is the binding energy of the state ψ.

In this three-body model the multiple-scattering series can be obtained by iterating the Faddeev equations.[6] In Faddeev theory, T is broken up into three amplitudes T^i, which satisfy the equations

$$T(E)=\sum_{i=1}^{3} T^i(E), \tag{2}$$

$$T^i(E)=t_i(E)+t_i(E)G_0(E)\sum_{j\neq i} T^j(E).$$

The amplitudes t_i are three-body scattering amplitudes in which the particle i is taken to be a noninteracting spectator to the interaction of the other two. G_0 is the free propagator for the three particles.

The plane-wave impulse approximation consists of approximating the full amplitude by the term corresponding to a single scattering of the two protons. Using the labeling of Fig. 1 we take

$$T\approx t_3. \tag{3}$$

The three-body operator t_3 can be related to the two-body scattering amplitude. Let $\vec{p}_3$ be the relative momentum of particles 1 and 2, $\vec{q}_3$ the momentum of the center of mass of 1 and 2 relative to 3, and $\vec{P}$ the total momentum of the system. Then if the particles 1, 2, 3, have momenta $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$,

$$\vec{p}_3=(m_2\vec{k}_1-m_1\vec{k}_2)/(m_1+m_2), \tag{4}$$

$$\vec{q}_3=\frac{1}{M}[(m_1+m_2)\vec{k}_3-m_3(\vec{k}_1+\vec{k}_2)],$$

$$\vec{P}=\vec{k}_1+\vec{k}_2+\vec{k}_3.$$

We define reduced masses $[(ijk)=(123)]$:

$$\mu_i=m_j m_k/(m_j+m_k),$$

$$M_i=m_i(m_j+m_k)/M, \tag{5}$$

$$M=m_1+m_2+m_3.$$

The Lippman-Schwinger equation for t_3 is

$$\langle\vec{p}_3\vec{q}_3\vec{P}\,|t_3(E)\,|\vec{p}_3'\vec{q}_3'\vec{P}'\rangle=\langle\vec{p}_3\vec{q}_3\vec{P}\,|V_{12}\,|\vec{p}_3'\vec{q}_3'\vec{P}'\rangle+\int d\vec{p}_3''\,d\vec{q}_3''\,d\vec{P}''\,\frac{\langle\vec{p}_3\vec{q}_3\vec{P}\,|V_{12}\,|\vec{p}_3''\vec{q}_3''\vec{P}''\rangle\langle\vec{p}_3''\vec{q}_3''\vec{P}''\,|t_3(E)\,|\vec{p}_3'\vec{q}_3'\vec{P}'\rangle}{E-p_3''^2/2\mu_3-q_3''^2/2M_3-P''^2/2M}. \tag{6}$$

Since the interaction potential depends only on the coordinates conjugate to p_3, one obtains

$$\langle\vec{p}_3\vec{q}_3\vec{P}\,|t_3(E)\,|\vec{p}_3'\vec{q}_3'\vec{P}'\rangle=\delta(\vec{P}-\vec{P}')\delta(\vec{q}_3-\vec{q}_3')\left\langle\vec{p}_3\left|t_3\left(E-\frac{q_3^2}{2M_3}-\frac{P^2}{2M}\right)\right|\vec{p}_3'\right\rangle. \tag{7}$$

This says that only the relative momenta of particles 1 and 2 can be changed by a collision between them. Their total momentum and the total momentum of the system remain unchanged. It also says that in the center of mass the energy relevant for the collision is the total energy minus the energy of the spectator relative to the center of mass of the pair.

Putting the Eqs. (3) and (7) into Eq. (1) gives the impulse approximation for the transition matrix element:

$$M=\langle\vec{p}_{on}\,|t(p_{on}^2/2\mu_3)\,|\vec{p}_{off}\rangle\psi(\vec{k}'), \tag{8}$$

where we have used the notation

$$\vec{p}_{on}=\tfrac{1}{2}(\vec{p}-\vec{p}'), \tag{9a}$$

$$\vec{p}_{off}=\tfrac{1}{2}(\vec{k}-\vec{k}'), \tag{9b}$$

$$\vec{k}'=\vec{p}+\vec{p}'-\vec{k}=-\vec{q}. \tag{9c}$$

The energy appearing in the t matrix is the relative kinetic energy of two protons with relative momenta $\vec{p}_{on}$. Here, t is half off the energy shell (*half shell*), since the magnitude of $\vec{p}_{off}$ is not necessarily the same as that of $\vec{p}_{on}$.

The choice of a final half-shell t matrix element is physically well motivated. For a particle to be on shell[7] its energy and momentum must be related by the free-particle relation $E(k)=k^2/2m$. In the initial state only the incoming particle is on shell. Since the bound proton is interacting with the core, it has potential as well as kinetic energy and is therefore off shell. In the final state both protons are asymptotically free so they are both on shell. If they are pictured as scattering with each other once and then propagating freely with no additional scatterings, the final scattering must occur on shell with the final relative energy.

This argument does not hold in a distorted-wave model. Since the p-p knockout scattering is not the final scattering taking place, the p-p amplitude involved will be fully off shell. Treatment of this case is reserved for a later work. In the present paper we restrict our analysis to the plane-wave impulse approximation in order to isolate the off-shell effects from those of distortion.

III. KINEMATICS

We now consider the coplanar symmetric experiment (Fig. 2). This requires that both protons come out on opposite sides of the beam at equal angles to it and at equal energies. This restriction produces a considerable simplification in the theoretical analysis. Many of the experiments have been carried out in this geometry.[8]

Following the notation of Eqs. (9a)–(9c), in this case we find

$$p_{on} = p \sin\theta , \tag{10a}$$

$$p_{off} = k - p\cos\theta , \tag{10b}$$

$$k' = 2p\cos\theta - k . \tag{10c}$$

Note that the vectors $\vec{p}_{off}$ and $\vec{k}'$ are along the z axis (beam direction), while $\vec{p}_{on}$ is at right angles to it. This means the t matrix element in Eq. (8) describes a 90° scattering in the center of mass of the two protons.

The quantity $p = |\vec{p}|$ can be determined from energy conservation:

$$\frac{k^2}{2m} + Q = \frac{p^2}{2m} + \frac{p'^2}{2m} + \frac{k'^2}{2mA} . \tag{11}$$

For the remainder of the paper we will make the heavy-core approximation, $1/A = 0$.

Setting $p = p'$ this approximation gives

$$p = (\tfrac{1}{2}k^2 + mQ)^{1/2} .$$

With these equations we may show explicitly what the variation of k', p_{off}, and p_{on} is in any particular experiment. They are shown as a function of θ, parametrically for different values of the incident proton's energy in Figs. 3 and 4.

In Fig. 4, note that the value of k' goes through zero fairly rapidly as the angle varies around 45°. The wave function in momentum space will therefore be swept out on either side of 45°. If one is in the energy region where the p-p cross section is approximately constant, the $(p, 2p)$ cross section will be proportional to the square of the wave function. This is the basic idea of the experiment. Note that at lower energies the range of variation of k' is considerably reduced as is its rate of variation.

In Fig. 3 the difference between the on- and off-shell momenta becomes clearly visible. The difference is largest at forward angles. As the angle between the two outgoing momenta becomes smaller their relative momentum p_{on} goes to zero. For a nonzero incident energy, the initial relative momentum p_{off} can never vanish, owing to the binding energy of the struck proton.

We can make some explicit statements about how far off shell the amplitude goes at high energies. Using Eq. (9c) to eliminate p' from Eqs. (9a), (9b), and (11), we obtain

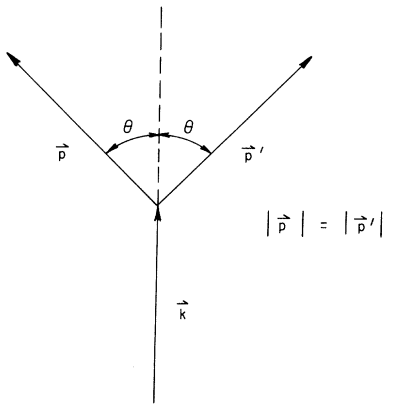

FIG. 2. Coplanar symmetric geometry (equal-mass case).

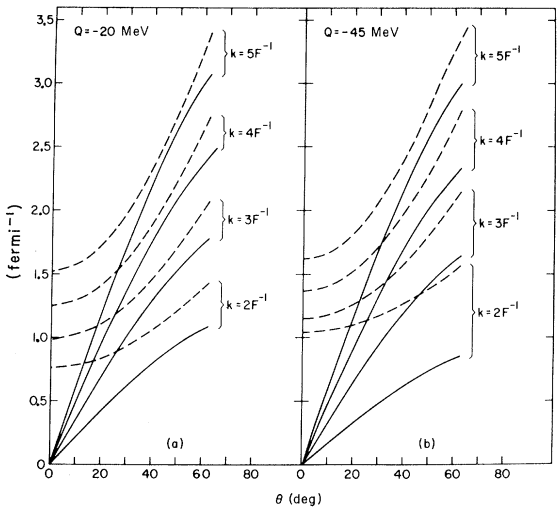

FIG. 3. Values of p_{on} and p_{off} versus θ for different values of k for the coplanar symmetric case. k and θ are as defined in Fig. 2. Q is the binding energy of the struck particle. The value of p_{on} is given by the solid line (—), p_{off} by the dashed line (----).

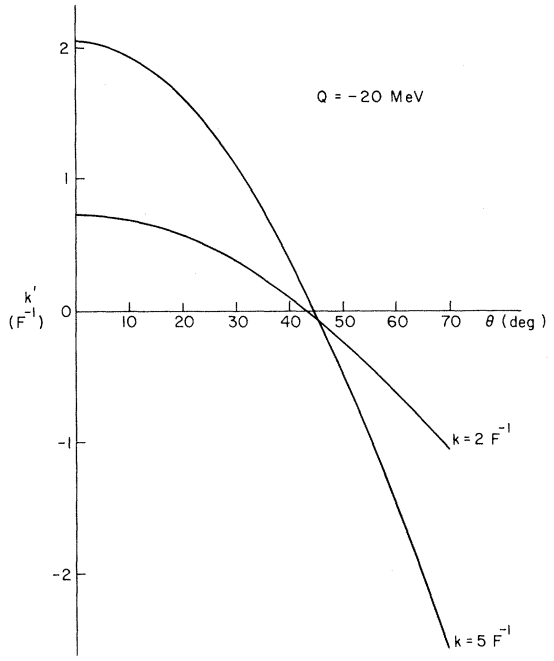

FIG. 4. Values of k' for some typical coplanar symmetric cases.

$$\mathcal{E}(p_{\text{off}}) - \mathcal{E}(p_{\text{on}}) = \tfrac{1}{2}\mathcal{E}(k') - Q\,,$$
$$p_{\text{off}}^2 - p_{\text{on}}^2 = \tfrac{1}{2}k'^2 - mQ\,, \tag{12}$$

where

$$\mathcal{E}(q) = q^2/2\mu_3$$

is the c.m. energy of a pair of protons with relative momentum q. Thus the difference between the on- and off-shell energies depends only on the binding energy and the momentum of the struck nucleon in the nucleus. Increasing either of these quantities increases the distance off shell.

Using Eqs. (10a), (10b) we can get an explicit expression for the difference of the magnitudes of the on- and off-shell momenta. These equations give

$$p_{\text{off}} - p_{\text{on}} = k - p(\sin\theta + \cos\theta) \tag{13}$$

which is always positive. This quantity has a minimum (for fixed k) at $\theta = 45°$. If we expand the above as a function of $1/k$ for fixed θ, we find

$$p_{\text{off}} - p_{\text{on}} = k\left(1 - \frac{\sin\theta + \cos\theta}{\sqrt{2}}\right) + O\left(\frac{1}{k}\right).$$

The coefficent of k does not vanish at fixed θ, so as k goes to infinity the difference function also becomes large. However, as k increases, the image of the wave function is compressed into a smaller angular interval about 45° (see Fig. 4). If we look at fixed k', i.e., at a particular point on the bound-state wave function, θ becomes a function of k and k'; explicitly

$$\cos\theta = \frac{1}{\sqrt{2}}\left(1 + \frac{2mQ}{k^2}\right)^{-1/2}\left(1 + \frac{k'}{k}\right),$$
$$\sin\theta = \frac{1}{\sqrt{2}}\left(1 + \frac{2mQ}{k^2}\right)^{-1/2}\left(1 - \frac{2k'}{k} + \frac{4mQ - k'^2}{k^2}\right)^{1/2}.$$

Substituting these into Eq. (13) and expanding for fixed k', we find the $O(k)$ and $O(1)$ terms cancel leaving

$$p_{\text{off}} - p_{\text{on}} = \frac{1}{k}\left(\frac{k'^2}{2} - mQ\right) + O\left(\frac{1}{k^2}\right) = O\left(\frac{1}{k}\right).$$

Thus, if we are investigating the bound-state wave function, the difference between the off-shell and on-shell momenta vanishes as $1/k$. Consequently, at high energies the off-shell ambiguity vanishes and the wave function can be determined uniquely.

IV. APPROXIMATE METHODS

A number of different techniques have been used to approximate the half-shell t matrix in Eq. (8). Most of these replace the off-shell amplitude by some on-shell amplitude obtained from two-body scattering experiments. We will use the notation $t(E, \theta)$ for the on-shell two-body scattering amplitude at a c.m. energy E and a c.m. scattering angle θ.

One common prescription[9] is to take E to be the final relative energy of the two protons and θ to be the angle between p_{on} and p_{off}. (In the coplanar symmetric case θ is 90°.) This gives

$$\langle \vec{p}_{\text{on}} | t(\mathcal{E}(p_{\text{on}})) | \vec{p}_{\text{off}} \rangle \approx t(\mathcal{E}(p_{\text{on}}), \theta(\vec{p}_{\text{on}}, \vec{p}_{\text{off}}))\,. \tag{14}$$

This will be a good approximation if the half-shell amplitude varies slowly as a function of p_{off}. We will refer to this method as the *final-energy prescription.*

Alternatively, E can be taken to be the initial relative energy with the scattering angle as in the final-energy prescription.[10] This gives

$$\langle \vec{p}_{\text{on}} | t(\mathcal{E}(p_{\text{on}})) | \vec{p}_{\text{off}} \rangle \approx t(\mathcal{E}(p_{\text{off}}), \theta(\vec{p}_{\text{on}}, \vec{p}_{\text{off}}))\,. \tag{15}$$

This is the *initial-energy prescription.*

Another method relating the off-shell t to an on-shell t is to assume that the amplitude depends only on an energy and a momentum transfer.[11] The separation of the two-proton vertex and a particular set of Galileo invariant variables are shown in Fig. 5. Other choices have been considered by Bonbright.[12] The energy and momentum transfer are chosen so as not to involve the leg corresponding to the bound proton, since it is off its mass shell. The assumption is that permitting this leg

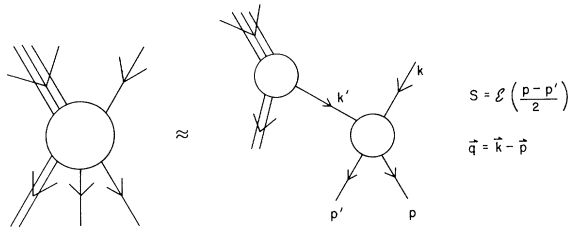

FIG. 5. Diagram for the knockout reaction in the impulse approximation. A set of invariant variables is indicated.

to go off its mass shell by a small amount does not affect the amplitude. This procedure need not always yield an on-shell t matrix which can be obtained from a two-body experiment, since the two-body reaction satisfies the inequality

$$q^2/4mS = \cos\bar{\theta} < 1,$$

which is not required in the knockout process. We will refer to this as the *final-energy-and-momentum-transfer prescription*. Explicitly, we wish to choose a momentum q and an angle $\bar{\theta}$ such that

$$q^2 = [\tfrac{1}{2}(\vec{p} - \vec{p}')]^2 = p_{on}^2$$

and

$$2q \sin\tfrac{1}{2}\bar{\theta} = |\vec{k} - \vec{p}|.$$

In the coplanar symmetric case

$$|\vec{k} - \vec{p}|^2 = k^2 + p^2 - 2kp\cos\theta = p_{on}^2 + p_{off}^2 \qquad (16)$$

[see Fig. 2 and Eq. (10)]. Solving for $\sin\bar{\theta}$ gives

$$\sin^2\tfrac{1}{2}\bar{\theta} = \tfrac{1}{4}(1 + p_{off}^2/p_{on}^2)$$

or

$$\bar{\theta} = 2\sin^{-1}\tfrac{1}{2}(1 + p_{off}^2/p_{on}^2)^{1/2}.$$

In order for a solution to exist we must have

$$\tfrac{1}{2}(1 + p_{off}^2/p_{on}^2)^{1/2} < 1$$

or

$$p_{off}^2 < 3p_{on}^2.$$

This is not satisfied for all of the permissible phase space even in the coplanar symmetric $(p, 2p)$ reaction (see Fig. 3). In order to use this prescription for the entire range of experimentally observed variables, one would have to analytically continue the on-shell amplitude to unphysical values of $\cos\bar{\theta}$. Where the prescription is permissible one obtains

$$\langle \vec{p}_{on} | t(\mathcal{E}(p_{on})) | \vec{p}_{off} \rangle \approx t(\mathcal{E}(p_{on}), \bar{\theta}).$$

Another type of approximation, the *pseudopotential method*, was introduced by Lim and McCarthy.[13] A potential is chosen such that its Born approximation fits some subset of the on-shell data. This potential is then used in Born approximation in Eq. (8) and in distorted-wave calculations. A local energy-independent pseudopotential is chosen to fit the scattering data at 90°. In the coplanar symmetric case this is actually another method for obtaining an on-shell prescription. Because this is not the usual description of the pseudopotential method, we will derive it in some detail.

Lim and McCarthy assume their potential is central and includes exchange forces. The cross section for $(p, 2p)$ is given by Eqs. (7) and (8) of Ref. 13 to be

$$\frac{d^3\sigma}{d\Omega_L d\Omega_R dE_L} \propto \tfrac{3}{4}A_{11}^2 |\langle X_L X_R - X_R X_L | t | X_0\psi\rangle|^2 + \tfrac{1}{4}A_{10}^2 |\langle X_L X_R + X_R X_L | t | X_0\psi\rangle|^2. \qquad (17)$$

The first term is triplet scattering and the second is singlet. X_L and X_R are the wave functions of the outgoing left and right protons, respectively. The incident p has wave function X_0 and ψ is the bound-state wave function. Since the force is assumed to be central, the singlet and triplet wave functions are the same. The coefficients A_{11} and A_{10} are linear combinations of the exchange parameters (W, M, B, H) (see *Introduction to Nuclear Theory*, p. 82).[14] If t is taken to be a local function $t(\vec{r}_1 - \vec{r}_2)$, and X_L, X_R, X_0 are replaced by plane waves with momenta as in Fig. 1 then

$$\langle X_L X_R | t | X_0\psi\rangle = \tilde{t}(\vec{p} - \vec{k})\tilde{\psi}(\vec{p} + \vec{p}' - \vec{k}),$$
$$\langle X_R X_L | t | X_0\psi\rangle = \tilde{t}(\vec{p}' - \vec{k})\tilde{\psi}(\vec{p} + \vec{p}' - \vec{k}),$$

where the tilde indicates the Fourier transform. For coplanar symmetric scattering one has

$$|\vec{p} - \vec{k}| = |\vec{p}' - \vec{k}|.$$

The triplet scattering therefore vanishes leaving

$$\frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE_1} \propto A_{10}^2 |\tilde{\psi}(\vec{k}')\tilde{t}(\vec{p} - \vec{k})|^2.$$

The function $t(r)$ is chosen to fit the 90° on-shell scattering data. To get the on-shell cross section ψ is replaced by a plane wave in Eq. (17) giving (Ref. 14, p. 83)

$$\frac{d\sigma}{d\Omega} = \left(\frac{\mu}{2\pi}\right)^2 (3A_{11}^2 |\langle X_L' X_R' - X_R' X_L' | t | X_L X_R\rangle|^2 + A_{10}^2 |\langle X_L' X_R' + X_R' X_L' | t | X_L X_R\rangle|^2).$$

Going to the c.m. frame and letting the initial momenta be $\vec{k}$ and $-\vec{k}$ and the final momenta $\vec{k}'$ and $-\vec{k}'$ gives

$$\langle X_L' X_R' | t | X_L X_R \rangle = \tilde{t}(\vec{k} - \vec{k}'),$$
$$\langle X_R' X_L' | t | X_L X_R \rangle = \tilde{t}(\vec{k} + \vec{k}').$$

At 90° the triplet scattering again vanishes giving

$$\frac{d\sigma}{d\Omega} = \left(\frac{\mu}{2\pi}\right)^2 A_{10}^2 |\tilde{t}(\vec{k} - \vec{k}')|^2$$

because at 90°

$$|\vec{k} - \vec{k}'| = |\vec{k} + \vec{k}'|.$$

Thus for any momentum of magnitude q, $t(q)$ may be related to an on-shell amplitude at 90°, in particular

$$\tilde{t}(q) = t(\mathcal{E}(q), 90°).$$

Using Eq. (16), the locality of the t matrix implies that the pseudopotential prescription is equivalent to

$$\langle \vec{p}_{on} | t(\mathcal{E}(p_{on})) | \vec{p}_{off} \rangle \approx t\left(\frac{\mathcal{E}(p_{on}) + \mathcal{E}(p_{off})}{2}, 90°\right) \tag{18}$$

when p_{off} and p_{on} are orthogonal.

A detailed consideration of the angular distributions given by the pseudopotential is necessary when refraction effects are included. This is somewhat more complicated than the 90° case due to the nonvanishing of the triplet scattering. Since we are only considering the coplanar symmetric case in the PWIA, only the 90° scattering is relevant. Because the Lim-McCarthy pseudopotential does not, in fact, fit the 90° on-shell cross section very well, we will use the above on-shell prescription rather than their pseudopotential. We refer to this as the *average-energy prescription*.

V. *p-p* CROSS SECTION

For the purpose of numerical examples we now restrict our attention to the proton-proton interaction and the effects of the divers approximations on $(p, 2p)$ cross sections. In this section we will compare the half-shell t matrices generated by the Reid soft-core potential[15] with those generated by the on-shell approximations discussed in the previous section. The on-shell information is also generated by this potential. This gives us a realistic model of the two-proton interaction in which we may evaluate the approximations.

The half-shell t matrices are evaluated using the routine TMAT. The relative p-p scattering wave function is calculated in a partial-wave basis using the Cowell algorithm[16] with a Runge-Kutta starting routine. The Coulomb force is omitted and the wave functions are matched to the phase-shifted free wave functions at a radius 8.57 F. In the calculation reported here, a mesh size of 0.071 F is used in the Cowell routine. The results are insensitive (less than 1% variation) to mesh sizes from 0.02 to 0.1 F and matching radii from 5 to 15 F.

Once the wave functions are obtained in coordinate space, t matrices are obtained by Fourier-transforming the potential times the wave function according to the equation

$$\langle k' | t_l(\mathcal{E}(k)) | k \rangle = \langle k' | V_l | \psi_l^{(+)}(k) \rangle,$$

where

$$\psi_l^{(+)}(k, r) \xrightarrow[r \to \infty]{} e^{i\delta_l} [\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr)].$$

The integration is performed via a 240-point mesh in a Simpson's-rule routine.

The value of the on-shell t matrix, calculated by Fourier-transforming the wave function, gives a value of the phase shift via

$$\langle k | t_l(\mathcal{E}(k)) | k \rangle = -e^{i\delta_l} \frac{\sin\delta_l}{k}. \tag{19}$$

The phase shift calculated this way agrees with the phase shift calculated from matching in the differential equation to better than 0.5% in most cases, the agreement being best for the large phase shifts. The t matrices were calculated at discrete values of k and k'. Intermediate values were obtained using a four-point Lagrange interpolation. For the $^1S_0$ state a grid size of 0.15 F^{-1} was used while for the other states (which are not as rapidly varying) a grid of 0.3 F^{-1} was used. The interpolation routine agrees with the direct calculation of TMAT to 0.1%.

The only relevant coupled state is the $J = 2$ state ($^3P_2$-$^3F_2$). For this state the 2×2-component wave function $\psi_{ll'}$ is calculated by a coupled Cowell routine. The 2×2 t matrix,

$$\langle k' | t_{ll'}^{JS}(\mathcal{E}(k)) | k \rangle = \sum_{l'} \int dr\, r j_l(k'r) V_{ll''}(r) \psi_{l''l'}^{JS}(k, r),$$

is calculated as in the uncoupled case.

To calculate the cross section we have retained only the $l = l' = 1$ component of this t matrix. Partial waves above $l = 2$ were dropped. This seems reasonable, since all energies calculated are below $k = 2.8$ F^{-1} ($E_{lab} \sim 650$ MeV), and the energies relevant to the experiment are all below $k = 2$ F^{-1} ($E_{lab} \sim 350$ MeV). Recent phase-shift analyses[17] indicate that, in magnitude, the $^3F_2$ phase shift is less than 10° at 700 MeV and less than 2° below 400 MeV.

However, when one goes far off shell the above statements may fail. The higher partial waves can contribute significantly off shell if the cross section is small. [For example, the dip in Fig. 7(a) may be modified upon the inclusion of the $^3F_2$ state.]

With these approximations the half-shell cross section in momentum space is given by[18]

$$^1\left(\frac{d\sigma}{d\Omega}\right) \propto (|t^0_{00}|^2 + 5|t^2_{20}|^2) + P_2(\theta)\left[\tfrac{50}{7}|t^2_{20}|^2 + 10\mathrm{Re}(t^0_{00}t^2_{20}{}^*)\right] + P_4(\theta)\left(\tfrac{90}{7}|t^2_{20}|^2\right),$$

$$^3\left(\frac{d\sigma}{d\Omega}\right) \propto (|t^0_{11}|^2 + 3|t^1_{11}|^2 + 5|t^2_{11}|^2) + P_2(\theta)\left[\tfrac{3}{2}|t^1_{11}|^2 + \tfrac{7}{2}|t^2_{11}|^2 + 4\mathrm{Re}(t^0_{11}t^2_{11}{}^*) + 9\mathrm{Re}(t^1_{11}t^2_{11}{}^*)\right], \quad (20)$$

where we have used the notation t^J_{LS} and the normalization of Eq. (19). The coupled state is normalized so that it goes over to the above when the tensor force goes to zero. At $\theta = 90°$ Eq. (20) becomes

$$^1\left(\frac{d\sigma}{d\Omega}\right)_{90°} \propto |t^0_{00}|^2 + \tfrac{25}{4}|t^2_{20}|^2 - 5\mathrm{Re}(t^0_{00}\,t^2_{20}{}^*),$$

$$^3\left(\frac{d\sigma}{d\Omega}\right)_{90°} \propto |t^0_{11}|^2 + \tfrac{9}{4}|t^1_{11}|^2 + \tfrac{13}{4}|t^2_{11}|^2 - 2\mathrm{Re}(t^0_{11}t^2_{11}{}^*) - \tfrac{9}{2}\mathrm{Re}(t^1_{11}t^2_{11}{}^*).$$

Note that there are now contributions from the triplet states, since in general $L \cdot S$ forces are permitted.

The half-shell cross sections for the Reid soft-core potential are plotted in Fig. 6 in mb/sr. On-shell points are marked by a heavy dot.

At low energies the amplitude is strongly peaked for low values of p_{off} and has a dip. This is due to the behavior of the $^1S_0$ state which dominates at low energy. The scattering amplitude in this state has a zero as a function of p_{off} for values of p_{on} below 1.5 F^{-1}. For higher p_{on} the amplitude is dominated by P and D waves which vanish for $p_{\mathrm{off}} = 0$ and have a peak. In the transition region the cross section is fairly flat. For both p_{on} and p_{off} greater than 1.5 F^{-1} and less than 3 F^{-1}, the cross section is very slowly varying around 4 mb/sr.

In Fig. 7 the half-shell cross sections are compared with the various on-shell approximations. We have chosen to display these results as a function of p_{off} parametrically with p_{on}. In such a plot the final energy prescription is a constant (independent of p_{off}), and the initial energy prescription is a curve which is the same on all the graphs (independent of p_{on}). If the curves had been plotted as a function of p_{on} parametrically with p_{off} these would have been switched.

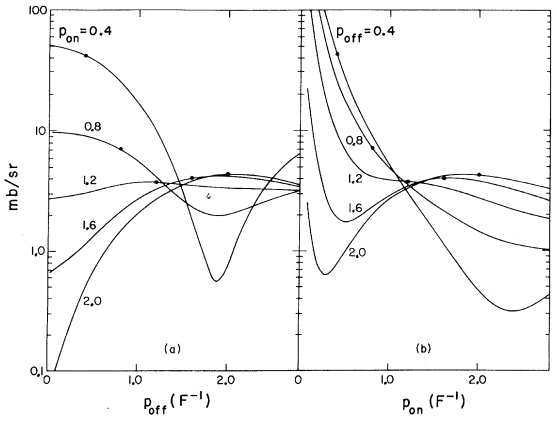

FIG. 6. Off-shell p-p cross sections at 90° in the c.m. $|\langle p_{\mathrm{on}}, 90° | t(p^2_{\mathrm{on}}/2\mu_3) | p_{\mathrm{off}}, 0° \rangle|^2$, as given by the Reid soft-core potential. The on-shell points are marked by a heavy dot. In (a) the cross sections are plotted as a function of p_{off} parametrically with p_{on}. In (b) the roles of p_{off} and p_{on} are reversed.

From Fig. (3) we recall the relevant off-shell regions for the coplanar symmetric $(p, 2p)$ experiment. For low or intermediate energies ($k < 4$ F^{-1}) $p_{\mathrm{off}} \sim 1.5$-2 F^{-1} while p_{on} is always less than p_{off} and can be zero. Thus for Fig. 7 the relevant region is to the right of the on-shell point up to about 2 F^{-1}. These regions are indicated on the figures by the solid bars.

One result is clear from these curves: The initial- or average-energy prescriptions give much better approximations to the exact half-shell cross section than does the final-energy prescription. The average-energy prescription does slightly better than the initial energy. For both p_{on} and p_{off} above 1.2 F^{-1} all the prescriptions become approximately equal.

This agrees with the results of Jain[11] which include distortion. He finds that at energies between 100 and 460 MeV, the initial-energy prescription is superior to the final-energy prescription. This supports our hypothesis that the correct t matrix for the impulse approximation is the half-shell amplitude.

These statements are specific to the proton-proton interaction and should not be taken to imply similar statements for other interactions. For example, for the $(\alpha, 2\alpha)$ reaction there are indications that the final-energy approximation is preferable to the initial.[19] Qualitatively the results for the p-p interaction can be explained from the behavior of the cross section. For high energy the p-p cross sections are virtually constant. As the energy is lowered below 1 F^{-1}, the cross section begins to rise rapidly. Letting one of the momenta go off shell to a larger value reduces the cross section again. The most appropriate energy is therefore somewhat higher than the final relative energy.

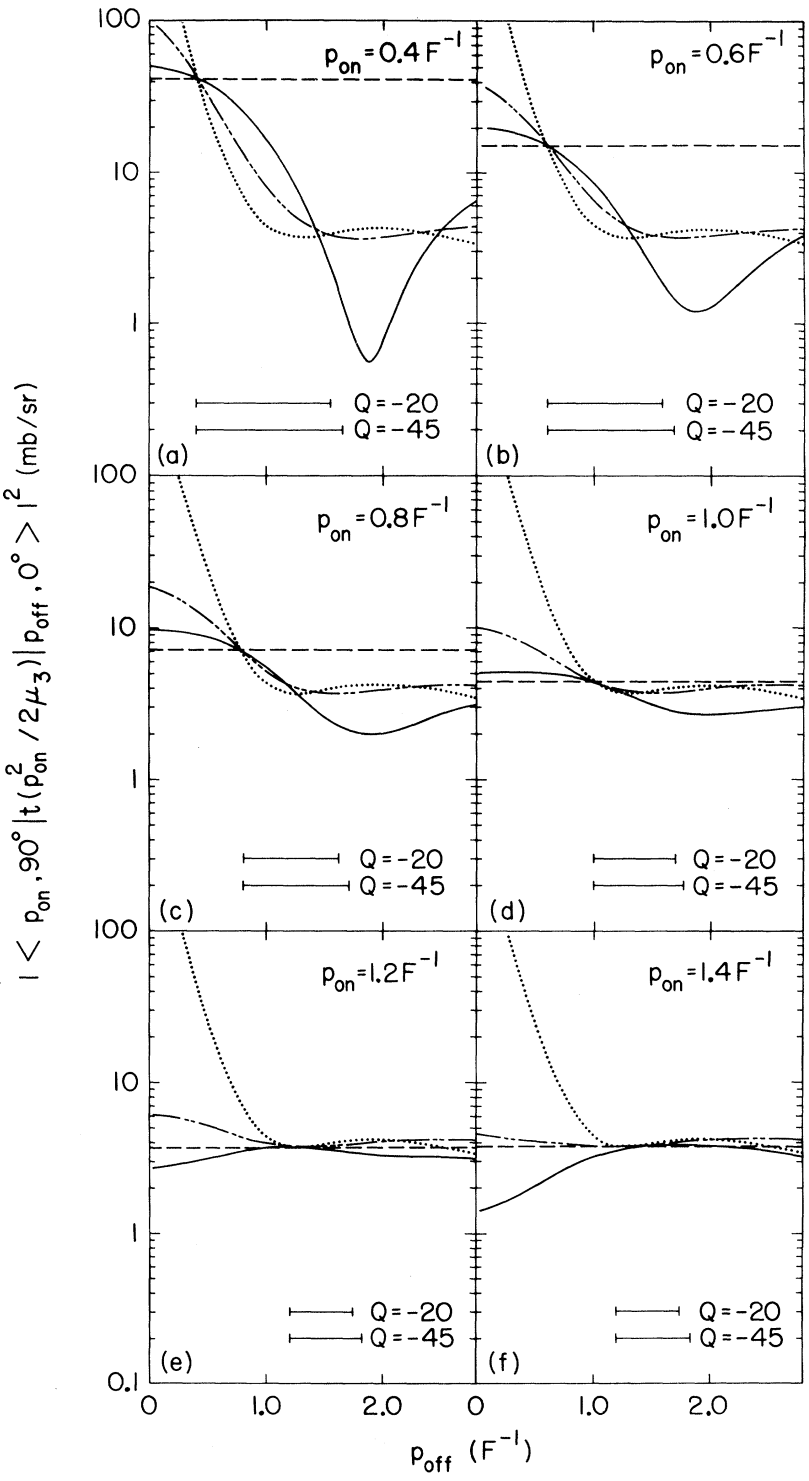

FIG. 7. Comparison of on-shell approximations with the half-shell cross section plotted as a function of p_{off} for several values of p_{on}. The half-shell cross section is shown as a solid line (—), the final-energy prescription as a dotted line (••••), and the average-energy prescription as dot-dash line (-•••-•••-•••). The solid bars under the curves indicate the regions relevant to the $(p, 2p)$ reaction in the PWIA for $k < 4$ F^{-1} for binding energies of 20 and 45 MeV.

The energy-and-momentum-transfer procedure is not displayed. Since, for most of the energies discussed the p-p cross section is virtually independent of angle, the prescription is identical with the final-energy prescription.[20]

VI. $(p, 2p)$ CROSS SECTIONS

The effects of using the various on-shell approximations on the $(p, 2p)$ cross sections are shown in Fig. 8. Cross sections for both $1s$ and $1p$ wave functions are shown.

The wave functions are chosen to be single-particle states in an oscillator well. The parameters of the well are taken to be appropriate for a light nucleus in the region of the sd shell. Since the results of this paper consist of comparisons of different approximations for t on the $(p, 2p)$ cross section, the particular wave function chosen makes no difference in our conclusions. The main results are all contained in Fig. 7. Only the ratios of the cross sections are being investigated in Figs. 8 and 9. It is for this reason that we felt it unnecessary to use more realistic wave functions. Explicitly we take

$$\nu = \tfrac{1}{2} M\omega, \quad \omega = 13.4 \text{ MeV},$$

$$\psi_{1s}(q) = e^{-q^2/4\nu},$$

$$\psi_{1p}(q) = q\, e^{-q^2/4\nu}.$$

The normalization of the wave function and phase factors have been ignored. The cross sections displayed are simply the square of the matrix ele-

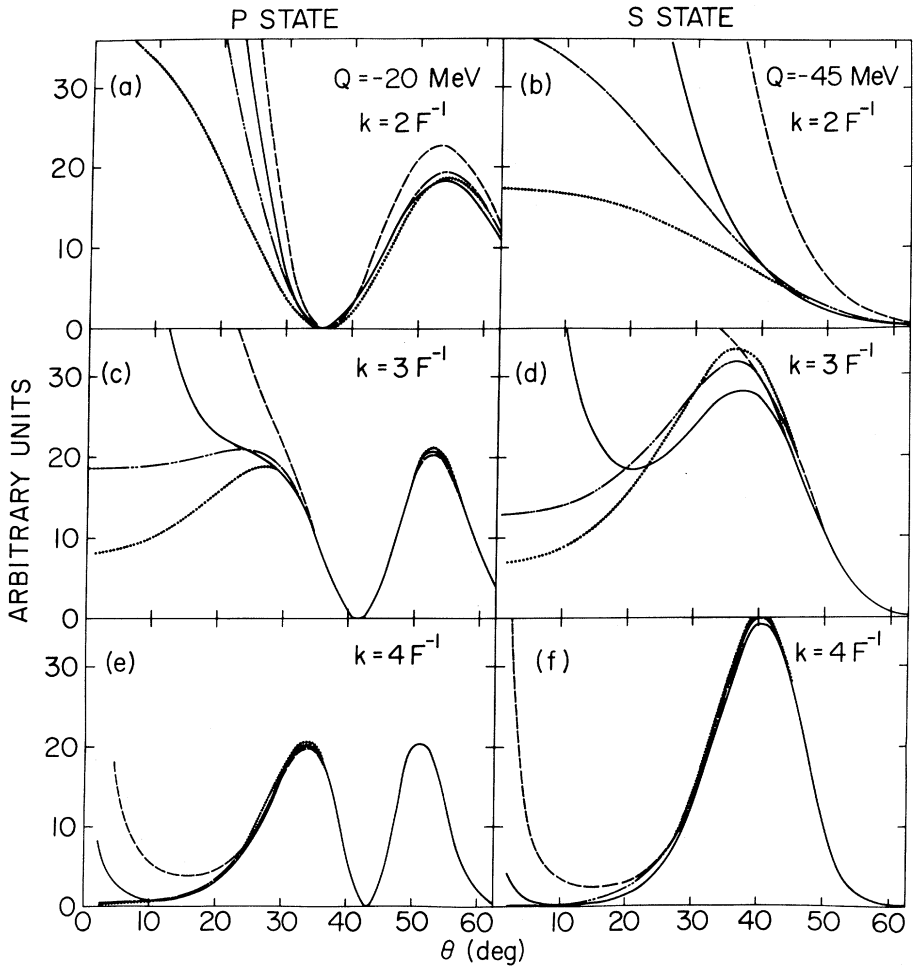

FIG. 8. Comparison of coplanar symmetric $(p, 2p)$ cross sections in the various approximate PWIA's as a function of the energy of the incident proton. The curves are labeled as in Fig. 7. Curves (a), (c), and (e) refer to knocking out a p-state proton whose wave function in momentum space is taken to be $qe^{-1.55q^2}$ and whose binding energy is taken to be 20 MeV. Curves (b), (d), and (f) refer to knocking out an s-state proton whose wave function in momentum space is taken to be $e^{-1.55q^2}$ and whose binding energy is taken to be 45 MeV. Phase-space factors and normalization constants have been omitted.

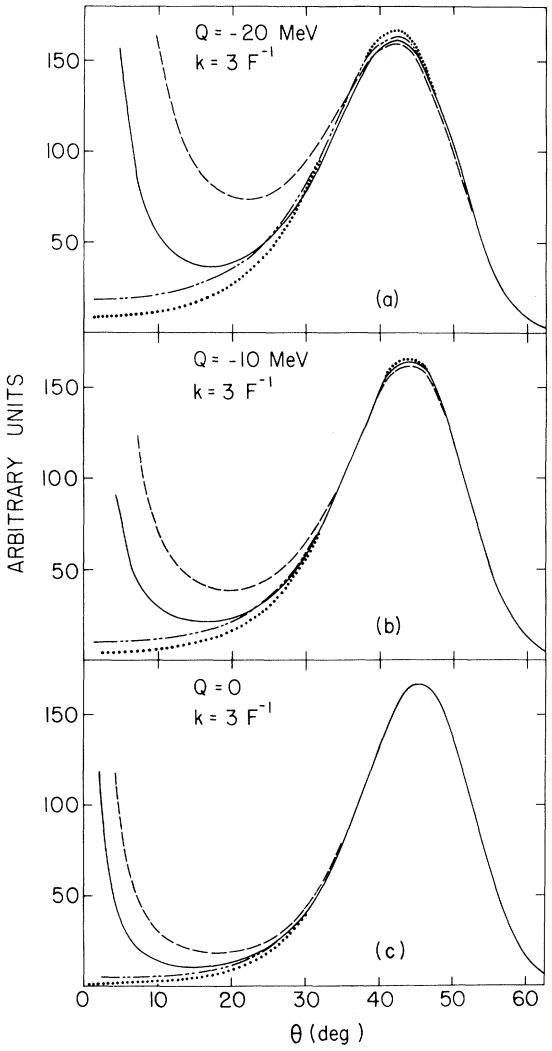

FIG. 9. Comparison of coplanar symmetric $(p, 2p)$ cross sections in the various approximate PWIA's as a function of the binding energy of the struck particle. The curves are labeled as in Fig. 7. In all the curves the bound-state wave function is taken to be the same as the s-state in Fig. 8, the momentum of the incident particle is taken to be $k = 3$ F^{-1}. Note that Fig. 8(d) is also relevant for this sequence.

ment in Eq. (1).

The results displayed in Fig. 8 are as follows: as the incident energy increases, the on-shell approximations show increasing agreement with the half-shell result and with each other. At 320 MeV ($k = 4$ F^{-1}) the agreement is excellent over most of the wave function. At lower energies, the differences become more marked. At 180 MeV ($k = 3$ F^{-1}) the final-energy prescription is quite bad below 45°. The initial- and average-energy prescriptions work fairly well down to about 20°. At 80 MeV ($k = 2$ F^{-1}), the final-energy prescription is bad for all the angles shown while the initial- and average-energy prescriptions work above about 35°.

Figures 8(d) and 9 display the effect of decreasing the binding energy for fixed incident energy. In all these cases the wave function used is the same. The binding energy is only allowed to affect the kinematics. As is indicated by Eq. (12), as Q decreases so does the off-shell effect. Again, the initial- and average-energy prescription agree with the half-shell prescription to smaller angles (farther off shell) than does the final-energy prescription for all curves.

VII. SUMMARY

Based on a study of the impulse approximation coming from one term of the Faddeev equations and using the Reid soft-core potential as a realistic model for the p-p interaction, we find that $(p, 2p)$ reactions above 300 MeV are not sensitive to the off-shell behavior of this interaction. All of the on-shell prescriptions studied give equivalent results. This reaction, therefore, can be used in this energy region to extract reliable information about the nuclear wave functions.

Conversely, the reaction below 200 MeV is sensitive to the off-shell behavior of the p-p interaction and is therefore best suited to the study of this interaction. Moreover, the extraction of nuclear information becomes tenuous if on-shell approximations are used. If one must use an on-shell approximation, the initial- and average-energy prescriptions appear to be the best. In this energy region, refraction is important and the coupling between distortion and the off-shell effects needs further careful consideration.

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We would like to thank Professor H. G. Pugh and Professor Daphne Jackson for helpful discussions and critical readings of the manuscript. In addition, we are grateful to Dr. N. R. Yoder for suggestions and assistance with the numerical aspects of this work.

APPENDIX

In this Appendix we display the PWIA equations for the case when all three particles have different masses. The lab momenta, incoming and outgoing, are taken to be as in Fig. 1(a). The velocity of the center

of mass in the lab is $\vec{k}/M$, so Eq. (1) becomes

$$M=\int d\vec{k}'\,d\vec{q}\left\langle \vec{p}-\frac{m_1}{M}\vec{k},\vec{p}'-\frac{m_2}{M}\vec{k},\vec{q}-\frac{m_3}{M}\vec{k}\left|T(E)\right|\frac{m_2+m_3}{M}\vec{k},\vec{k}'-\frac{m_2}{M}\vec{k},-\vec{k}'-\frac{m_3}{M}\vec{k}\right\rangle\psi(\vec{k}') \tag{A1}$$

with

$$\vec{q}=\vec{k}-\vec{p}-\vec{p}',$$

as before.

Definitions in Eqs. (4) and (5) still hold for arbitrary masses as do Eqs. (6) and (7). Using the momenta in (A1) and inserting (3) and (7) gives

$$M=\langle\vec{p}_{on}\,|t(p_{on}^2/2\mu_3)\,|\vec{p}_{off}\rangle\,\psi(\vec{k}'), \tag{A2}$$

as before, except now

$$\begin{aligned}\vec{p}_{on}&=(m_1\vec{p}'-m_2\vec{p})/(m_1+m_2),\\ \vec{p}_{off}&=(m_1\vec{k}'-m_2\vec{k})/(m_1+m_2),\\ \vec{k}'&=\vec{p}+\vec{p}'-\vec{k}.\end{aligned} \tag{A3}$$

The equation for energy conservation [Eq. (11)] becomes

$$\frac{k^2}{2m_1}+Q=\frac{p^2}{2m_1}+\frac{p'^2}{2m_2}+\frac{k'^2}{2m_3}.$$

Putting in $\vec{k}'$ in terms of $\vec{k}$, $\vec{p}$, and $\vec{p}'$ and defining

θ = angle between $\vec{p}$ and $\vec{k}$,

θ' = angle between $\vec{p}'$ and $\vec{k}$,

ϕ = angle between $\vec{p}$ and $\vec{p}'$

gives

$$\frac{k^2}{2}\,\frac{m_3-m_1}{m_1m_3}+Q=\frac{p^2}{2}\,\frac{m_3+m_1}{m_1m_3}+\frac{p'^2}{2}\,\frac{m_3+m_2}{m_2m_3}-\frac{1}{m_3}\left[pk\cos\theta+p'k\cos\theta'-pp'\cos\phi\right]. \tag{A4}$$

Since the equation is quadratic, there are generally two values of p for any given p'. Equation (A4) is the equation of a curve which gives the locus of kinematically allowed points in the p-p' plane. (For examples of some curves, see Ref. 13.)

We now return to the equal-mass coplanar symmetric case and use (A4) to determine the corrections to the kinematics due to the fact that $A\neq\infty$. From Eqs. (A2) and (A3) it is clear that the same PWIA equations hold as given in the text. Only the value of p determined from Eq. (11) will differ.

Upon setting $m_1=m_2=m$, $m_3=Am$, $\theta=\theta'$, $\phi=2\theta$, and $p=p'$, Eq. (A4) gives

$$p^2\left(1+\frac{2\cos^2\theta}{A}\right)-2p\left(\frac{k\cos\theta}{A}\right)-\left[\frac{k^2}{2}\left(1-\frac{1}{A}\right)+mQ\right]=0.$$

This equation has the solutions

$$p=\left[1+\frac{2\cos^2\theta}{A}\right]^{-1}\left(\frac{k\cos\theta}{A}+\left\{\frac{k^2\cos^2\theta}{A^2}+\left(1+\frac{2\cos^2\theta}{A}\right)\left[\frac{k^2}{2}\left(1-\frac{1}{A}\right)+mQ\right]\right\}^{1/2}\right).$$

Since the value of the radical is larger than the term preceding it within the large parentheses the condition that p be positive requires the choice of the + sign.

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Deuteron Scattering from a Polarized ^{3}He Gas Target*

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Scattering asymmetries in the angular range $78° \leq \theta_{c.m.} \leq 127°$ have been measured for 9.9- and 11.9-MeV (lab energy) unpolarized deuterons incident on a polarized ^{3}He gas target.

INTRODUCTION

A polarized ^{3}He gas target has been developed which is suitable for investigation of scattering and nuclear reactions; nuclear polarization was achieved by the optical-pumping techniques developed by Colegrave, Schearer, and Walters[1](CSW).

POLARIZATION MEASUREMENT AND ERRORS

Several papers[1-5] have discussed the relationship between optical signals generated in the pumping process and the magnitude of the nuclear polarization. Equation (9) from Ref. 1 (slightly rearranged) is

$$\frac{\Delta I}{I} = \frac{P}{3+P^2}\left[\,6-2P+3(1-P)^2\,\frac{a-b-c}{a+b+c}\,\right], \quad (1)$$

where the ratio $(a-b-c)/(a+b+c) \equiv k$ depends on the relative illumination of the $F=\frac{1}{2}$ and $F=\frac{3}{2}$ levels. Assuming equal illumination, CSW took $k=-\frac{1}{6}$. Greenhow[2] showed that the two levels were not equally illuminated and suggested a formula derivable from $k=\frac{1}{2}$. Klinger[3] and Hauer and Klinger[4] (HK) measured directly the nuclear polarization (P) and the optical signal $\Delta I/I$. Equation (1) can be written

$$P = \frac{\Delta I}{I} \times \tfrac{1}{2} \times \frac{1+\frac{1}{3}P^2}{1+\frac{1}{2}k-P(k+\frac{1}{3})+k\frac{1}{2}P^2} = \alpha\frac{\Delta I}{I}\,. \quad (2)$$