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Macroscopic Self-Consistency of the Collective Model of Deformed Nuclei*

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It has been suggested, in effect, that in a deformed nucleus the number of protons (or the amount of charge) which are carried along by the rotational motion may, on the average, be approximated as the product of the atomic number Z and the deformation parameter δ. The theoretical justification of this suggestion is discussed, and a more accurate expression is obtained. Assuming that the interaction of the rotational motion with an external magnetic field is entirely due to the current associated with the amount of charge following the rotational motion, on the average, and also in accordance with the cranking approximation, a macroscopic expression for the rotational gyromagnetic ratio g_R is derived. This expression, supplemented by the usual macroscopic formula for the intrinsic quadrupole moment Q_0, may constitute a macroscopic self-consistency relation among the collective parameters in a rotational band, namely, Q_0, g_R, and the moment of inertia J. The values of g_R calculated from the experimental values of J and Q_0 are tabulated for the ground-state rotational bands of both even-even and odd-mass nuclei. The aforementioned macroscopic self-consistency is then tested by comparing these calculated values of g_R with both empirical values and previous microscopic calculations. According to the present approach, the well-known lowering of g_R from the usual fluid-model value is mainly due to the limitation on the number of protons which can follow the rotational motion, on the average. It is not clear, however, whether there is any direct connection between this limitation and the pairing interaction which seems to play a rather essential role in the current microscopic calculations. This puzzling situation is further illustrated by considering the moments of inertia of even-even nuclei.

I. INTRODUCTION

Rotational band structure has been experimentally established in the excitation spectra of nuclei in several regions of the Periodic Table. According to the collective model of Bohr and Mottelson,[1] the static and dynamic properties of each rotational band (with band-head angular momentum not equal to $\frac{1}{2}$) can be characterized by four parameters, namely, the moment of inertia J, the intrinsic quadrupole moment Q_0, the rotational gyromagnetic ratio g_R, and the intrinsic gyromagnetic ratio g_K, which can be related to certain measurable quantities such as $E2$ and $M1$ transition probabilities and the level spacings in the rotational band.

These characteristic parameters may also be calculated theoretically in terms of the intrinsic properties of the nucleus. Such a microscopic approach has had considerable success in recent years, particularly in the calculation of the moments of inertia and rotational gyromagnetic ratios of even-even nuclei.[2,3]

Although a microscopic approach has the advantage of testing the nuclear model in detail, we attempt to establish, in this paper, certain macroscopic relations among these parameters which may be directly verified with the experimental values. Such macroscopic relations, if they are valid, are certainly useful in the analysis of the experimental results but may also be of interest in evaluating the microscopic theories.

In Nilsson's formalism of the intrinsic motion,[4,5] the single-particle wave function depends critically on the deformation parameter δ which characterizes the deviation of the average potential from spherical symmetry. Although δ may be theoretically estimated as the eccentricity of the deformed potential which minimizes the total nuclear energy,[4,5] it is usually obtained from the experimental value of Q_0 through the following macroscopic relation[1,6,7]

$$Q_0 = (9/5\pi)^{1/2} Z R_0^2 \beta (1 + 0.16\beta + \cdots), \qquad (1)$$

where Z is the atomic number, R_0 is the mean radius of the charge distribution, and β is related to δ by the equation $\delta = \frac{3}{2}(5/4\pi)^{1/2}\beta$ (see below; this δ is not exactly the same δ as defined in Refs. 4 and 5, and, strictly speaking, δ or β should be regarded as the deformation parameter for the charge distribution). It is thus clear that this macroscopic relation is actually one of the empirical foundations in the current microscopic calculations.

In the following section (Sec. II), one additional macroscopic relation [Eq. (14)] is derived, which expresses g_R as a function of Z, R_0, J, δ, and the mass of the proton M_p. Therefore, based on the experimental values of J and g_R, δ may also be calculated. However, if self-consistency is ensured, this calculated value of δ should agree with the value derived from the experimental value of Q_0. In other words, Eq. (1) supplemented by the macroscopic relation derived in the following section may constitute a macroscopic self-consistency relation among the parameters mentioned above, except for the intrinsic gyromagnetic ratio g_K.

It may be added, parenthetically, that macroscopic self-consistency relations in the sense indicated above are actually not uncommon in physics. For instance, the Wiedemann-Franz law[8] is obviously an equation of this nature. Although microscopic theories of thermal and electric conductivities of metal with various degrees of sophistication do exist, the virtue of this remarkable macroscopic equation can hardly be overlooked.

A basic difference between the roles played by J and g_R should be noted at this point: While the former is essentially a characteristic of the rotational motion itself, the latter specifies the strength of the interaction of the rotational motion with external magnetic field. It has not so far been possible, however, to write down this interaction from first principles without relying on some intuitive model or concept. In the microscopic approach, the interaction of the rotational motion with external magnetic field is usually taken into account in an indirect way by calculating the mean value of the vector sum of the magnetic moments of the nucleons in the cranking approximation.[3,6,9] In the present paper, we take a macroscopic but more direct point of view by assuming that the interaction of the rotational motion with an external magnetic field is entirely due to the current associated with the amount of charge carried along on the average by the rotational motion. This point of view is apparently consistent with the fluid-model approach in which g_R is simply the effective charge of the rotational flow or, in other words, the ratio of the number of protons (Z') to the total number of nucleons (A') following the rotational motion on the average. [The interpretation of the usual fluid-model value, $g_R = Z/A$ (here A is the mass number), is not unique. It seems equally acceptable to regard Z/A as the rotational gyromagnetic ratio of a uniformly charged rigid rotator in which all nucleons are rigidly carried along by the rotational motion and the charge distribution coincides with the mass distribution.]

It will be clear later that for uniform charge distribution Z' can be directly related to the deformation parameter δ although the ratio Z'/A' may not be specified without ambiguity. Based on this result and also in accordance with the cranking approximation, an effective magnetic-moment operator associated with the rotational motion is proposed in Sec. II, from which the aforementioned macroscopic expression of g_R can easily be obtained.

The values of g_R calculated from the experimenal values of J and Q_0 are tabulated for the ground-state rotational bands of both even-even and odd-mass nuclei. The consistency between Eq. (1) and this macroscopic expression of g_R is then tested by comparing the calculated values of g_R with the empirical values.

Following arguments similar to those presented in Sec. II, the moments of inertia of even-even nuclei are briefly discussed in Sec. III.

II. ROTATIONAL GYROMAGNETIC RATIO

It has been shown on the basis of empirical analyses and theoretical calculations that under favorable conditions the equilibrium shape of a deformed nucleus (and the shape of the average potential) may be approximated as a prolate spheroid, and the low-lying states may be described as rotational states of this spheroid which rotates about an axis perpendicular to its symmetry axis (see below).[1,4-7]

The rotation of the nuclear system as a whole certainly carries along a certain amount of matter and charge which in turn depends on the nature of the nuclear matter. As a first approximation, the rotating nucleus may be regarded as a uniformly charged rigid rotator of the shape of a prolate spheroid (in the microscopic approach this amounts to neglecting the pair correlation of the nucleonic motion).[6] Then, letting $\vec{\Omega}$ be the angular velocity of rotation, the velocity of points inside the spheroid due to the rotational motion is $\vec{\Omega}\times\vec{r}$ (here $\vec{r}$ is the radius vector in a space-fixed coordinate system with its origin at the center of the spheroid), and, if the interaction of the rotational motion with external magnetic field is entirely due to the charge current caused by the rotation of the spheroid, the magnetic moment associated with the rotational motion is then given by $(Ze/2cV)\int \vec{r}\times(\vec{\Omega}\times\vec{r})d\tau$; here e is the charge of proton, c is the speed of light, and the integral is taken over the volume V of the spheroid. Since the charge distribution is assumed to be uniform, the normalized integral $(Z/V)\int \vec{r}\times(\vec{\Omega}\times\vec{r})d\tau$ may be replaced by the summation

$$\sum_{i=1}^{Z}\vec{r}_i\times(\vec{\Omega}\times\vec{r}_i),$$

which is to be regarded as an operator after the rotational motion is quantized; here $\vec{r}_i$ is the radius vector of the ith proton.

A rotating nucleus does not behave, however, like a rigid rotator, as manifested by the magnitude of the moment of inertia. Therefore, as a better approximation, we assume that only a limited number of protons Z' (or a limited amount of charge $Z'e$) can follow the rotational motion on the average. Then, based on this assumption, the magnetic moment associated with the rotational motion may be effectively taken as follows:

$$\vec{\mu}_{\text{rot}}=\frac{e}{2c}\frac{Z'}{Z}\sum_i \vec{r}_i\times(\vec{\Omega}\times\vec{r}_i). \tag{2}$$

The physical meaning of Z' will be discussed in detail later.

In the above discussion, we only attempt to make Eq. (2) more plausible and we make no pretense of taking the general argument as a rigorous microscopic justification. Therefore, we shall regard Eq. (2) as a phenomenological model which may only be judged by its implications at this stage.

Now, in accordance with the cranking approximation, the quantization of the rotational motion may be achieved by setting[10]

$$J\vec{\Omega}=\hbar\vec{R}, \tag{3}$$

where $\vec{R}$ is the angular momentum operator associated with the rotational motion.

Thus, from Eq. (2) and Eq. (3), we have the following rotational magnetic moment operator in units of the nuclear magneton:

$$\vec{\mu}_{\text{rot}}=\frac{M_p}{J}\frac{Z'}{Z}\sum_i \vec{r}_i\times(\vec{R}\times\vec{r}_i). \tag{4}$$

Since $\vec{R}$ is related to the rotation of the whole system, it is reasonable to assume that $\vec{r}_i$ and $\vec{R}$ commute (that is, $\vec{r}_i$ commutes with $\vec{I}-\vec{j}$; here $\vec{I}$ is the total angular momentum and $\vec{j}$ the total intrinsic angular momentum).

Once the rotational magnetic-moment operator is given, the corresponding gyromagnetic ratio can simply be obtained by calculating the expectation value of the z component of this magnetic moment operator with suitable wave function. We may recall that, according to the collective model, the total magnetic-moment operator can be separated into two independent parts, one due to the intrinsic motion and the other due to the rotational motion.[4] Therefore, we can calculate the expectation value of one part without affecting the other.

For an average potential with symmetries of a spheroid, the properly symmetrized wave function may be written as follows[1,11]:

$$\Psi_{IMK}=\left(\frac{2I+1}{16\pi^2}\right)^{1/2}\left[D^I_{MK}\psi_K+(-1)^{I+K}D^I_{M-K}\psi_{-K}\right]. \tag{5}$$

Here M and K are the projections of $\vec{I}$ along the space-fixed z axis and the nuclear symmetry axis, respectively, D^I_{MK} are the rotation matrices, and ψ_K is the intrinsic wave function.

Thus, by using Eq. (4) and Eq. (5), we obtain

$$\begin{aligned}\mu_{\text{rot}}=&\left[I-\frac{K^2}{I+1}\left[1+(2I+1)(-1)^{I+1/2}b_0\,\delta_{K\,1/2}\right]\right]\\&\times\frac{M_p}{J}\frac{Z'}{Z}\langle\psi_K|\sum_i r_i^2|\psi_K\rangle\\&-\frac{M_p}{J}\frac{Z'}{Z}\langle\Psi_{IIK}|\sum_i z_i(\vec{r}_i\cdot\vec{R})|\Psi_{IIK}\rangle,\end{aligned} \tag{6}$$

where b_0 is the decoupling factor for $K=\frac{1}{2}$. The expression in the square brackets in the first term is well known.[4,11] The physical meaning of the second term will be discussed below.

In the body-fixed coordinate system, the second term on the right-hand side of Eq. (6) reduces to the following form:

$$+\frac{M_p}{J}\frac{Z'}{Z}\langle\Psi_{IIK}|\sum_\nu\sum_i D^1_{0\nu}r'_{i\nu}(r'_{i+}R_-+r'_{i-}R_+)|\Psi_{IIK}\rangle. \tag{6a}$$

Here $r'_{i0}=z'_i$, $r'_{i\pm}=\mp(x'_i\pm iy'_i)/\sqrt{2}$, $R_\pm=\pm(R_{x'}\pm iR_{y'})/\sqrt{2}$, and (x',y',z') are referred to the body-fixed coordinate system. According to the fluid model, the moment of inertia about the nuclear symmetry axis z' is strictly zero.[12] Although the fluid model may not be realistic, there is experimental evidence that there is no quasirigid rotation about a symmetry axis; this evidence is the absence of rotational states in spherical nuclei, at least for low excitations.[12] It is thus not unreasonable to assume that the rotational angular momentum $\vec{R}$ is perpendicular to the nuclear symmetry axis z' and consequently the states $R_{\pm}\Psi_{IMK}$ have no physical meaning, or, in other words, they are spurious states so long as the average potential remains axially symmetric. Therefore, the second term on the right-hand side of Eq. (6) will be discarded, although it may not vanish identically if the conventional calculation is strictly followed.

From another point of view, the dropping of the second term on the right-hand side of Eq. (6) amounts essentially to neglecting the mixing of states belonging to competing average potentials of different symmetries (axially symmetric and asymmetric). Apparently, such a type of mixing of states is expected to be negligible in the region of "permanently" deformed nuclei.

By comparing Eq. (6) (without the second term) with the usual formula for the magnetic moment[4,11] and assuming uniform charge distribution, we have

$$g_R = \frac{M_p}{J}\frac{Z'}{Z}\langle\psi_K|\sum_i r_i^2|\psi_K\rangle = \frac{3M_p Z' R_0^{\,2}}{5J}, \qquad (7)$$

which is almost as simple as the fluid-model expression. In obtaining Eq. (7), the usual convention $\langle r_i^2\rangle = \frac{3}{5}R_0^{\,2}$ has been employed.

We see from Eq. (7) that g_R depends critically on the number of protons Z' (or the amount of charge $Z'e$) which follow the rotational motion on the average. The role of J in the denominator should also be noted. It turns out that the odd-even difference of g_R (see Tabel I and Table II) is mainly due to the odd-even difference of J.

Now, we come to the problem of determining Z'. Let us first assume that the neutron group and the proton group in a nucleus are incapable of exchanging angular momentum[13] and consequently we may talk of an effective number of protons following the rotational motion on the average without considering the exact distribution of the neutrons. From the experimental evidence that spherical nuclei do not possess a rotational spectrum (at least for low-lying states), it seems reasonable to assume that Z' should be related to the deformed part of the charge distribution. Then, by assuming uniform charge distribution, Z' may simply be taken as the product of Z and v/V; here V is the total volume of the deformed nucleus of the shape of a prolate spheroid and v the volume which is outside the sphere of radius equal to the semiminor axis of the spheroid and centered at the center of the spheroid. Thus, letting a and b be the semimajor and semiminor axes of the prolate spheroid, we obtain[14]

$$Z' = (a-b)\,Z/a\,. \qquad (8)$$

It should be pointed out that the use of Z' in Eq. (2) is equivalent to saying that the protons which do not follow the rotational motion on the average are assumed to contribute to the current density only a term which is curl free. This, in turn, will contribute to the magnetic moment only through a surface term which would vanish for spherical nuclei. Since the protons which on the average are inside the sphere of radius equal to the semiminor axis of the deformed nucleus are taken to be the protons which do not follow the rotational motion on the average, it is implicitly assumed that the "internal" protons (or nucleons) are ignorant of the deformed surface, and this means that some interactions with the "outer" nucleons are being ignored. This assumption is apparently consistent with the fluid model in which the fluid flow is assumed totally irrotational. On the other hand, this assumption may also not be unrealistic from an intuitive point of view, because we are talking about the average effect and not about the motion of the individual nucleon.

In order to express Eq. (8) in terms of δ, we may solve the following two equations[15]:

$$a - b = R_0\,\delta\,, \qquad (9)$$

$$ab^2 = R_0^{\,3}\,, \qquad (10)$$

where the first equation is simply a definition and the second equation is the condition of constant volume.

The two cubic equations for a and b derived from Eq. (9) and Eq. (10) can easily be solved, and the results are

$$a = R_0(1 + \tfrac{2}{3}\delta + \tfrac{1}{9}\delta^2 - \tfrac{2}{81}\delta^3 + \cdots)\,, \qquad (11)$$

$$b = R_0(1 - \tfrac{1}{3}\delta + \tfrac{1}{9}\delta^2 - \tfrac{2}{81}\delta^3 + \cdots)\,. \qquad (12)$$

These are the only real roots in the range of values of δ that we are interested in.

It is apparent that Eq. (9) is exactly satisfied by a and b given in Eq. (11) and Eq. (12), respectively, but Eq. (10) is satisfied only to the first order of δ. This indicates that R_0 should be a function of δ. However, it can easily be shown that R_0 can be treated as a constant to the first order of δ, inclusive. [A similar situation is noted in Nilsson's oscillator potential, the mean angular frequency of which is also δ dependent. Equation (11) and Eq. (12) are apparently consistent with Nilsson's original parametrization of the oscillator potential.][4]

Thus, substituting Eq. (11) and Eq. (12) into Eq. (8), we have

$$Z' = Z\,\delta(1 - \tfrac{2}{3}\delta + \tfrac{1}{3}\delta^2 + \cdots)\,, \qquad (13)$$

which may be accurate up to any order of δ that we may wish to evaluate because Eq. (8) is independent of R_0. On the other hand, after inserting

Eq. (13) into Eq. (7), we obtain

$$g_R = \frac{3}{5J} Z M_p R_0^2 \delta(1 - \tfrac{2}{3}\delta + \cdots), \qquad (14)$$

in which only second-order-of-δ accuracy can be expected if R_0 is to be treated as a constant.

The remarkable feature of Eq. (1) [or Eq. (15), see below] and Eq. (14) is that they contain all the relevant collective parameters of the collective model (that is, Q_0, J, and g_R) and are interrelated by the deformation parameter δ. Therefore, if these two equations are valid, they apparently constitute a self-consistency relation among the parameters involved.

The remaining problem is now to determine the relation between δ defined above and the deformation parameters used previously. These parameters often cause confusion if we are not aware of their different definitions.

It is easy to show that the intrinsic quadrupole moment of a uniformly charged prolate spheroid is given by

$$Q_0 = \tfrac{2}{5} Z(a^2 - b^2) = \tfrac{4}{5} Z R_0^2 \delta(1 + \tfrac{1}{6}\delta + \cdots), \qquad (15)$$

which is identical to Eq. (1) if we set

$$\delta = \tfrac{3}{2}(5/4\pi)^{1/2}\beta . \qquad (16)$$

This is the relation between δ and β mentioned previously. The equality sign in Eq. (16) should be emphasized. The relation between δ and other deformation parameters can be obtained in a similar way, and, in particular, we have $\delta = \epsilon[1 + \frac{1}{3}\epsilon + O(\epsilon^2)]$; here ϵ is defined in Appendix A of Ref. 4. It should be noted that the empirical data compiled in Ref. 5 are actually for ϵ, although the same symbol δ was employed.

The consistency between Eq. (1) [or Eq. (15)] and Eq. (14) can obviously be tested in several ways. However, in view of the fact that the experimental accuracy of g_R is not yet comparable with those of Q_0 and J, the most convenient way is probably to calculate g_R from the experimental values of Q_0 and J and compare it with the experimental value. The calculated values of g_R for the ground-state rotational bands are listed in Table I for even-even nuclei and in Table II for odd-mass nuclei.

There are still considerable inconsistencies in the experimental data of g_R reported from various laboratories, which in some cases render the comparison between the theoretical values and the experimental data almost meaningless. Therefore, for the sake of clarity, the experimental values listed in Table I for even-even nuclei are arbitrarily selected in favor of present calculations. The empirical values[16,17] listed in Table II for odd-mass nuclei are also not up to date, but they serve the purpose of comparison with present calculations rather well. (For more complete compilation of the experimental data of g_R, see Prior, Boehm, and Nilsson[18] and Grodzins.[19])

It seems apparent that the calculated values of g_R for even-even nuclei agree with the experimental values reasonably well, and their general compatibility with the microscopic calculations of Nilsson and Prior[3] may not be considered as accidental.

Except for a few cases of odd-proton nuclei, the calculated values of g_R for odd-mass nuclei are also compatible with the microscopic calculations of Grin' and Pavlichenkov[9] and are in fairly good agreement with the empirical values (also see the following paper).[20] It is of particular interest to note that the remarkably low values of g_R for Dy^{161} and Er^{167} seem to be well confirmed experimentally.[18,19] The odd-even difference of g_R can easily be seen by comparing Table I and Table II, and, according to the present calculation, this odd-even difference is mainly due to the odd-even difference of J. The even-Z–odd-Z difference is, however, not so pronounced as previously claimed.[9,16–18]

A phenomenological model investigated by Greiner[21] is also in rough accord with present results. This model leads to a nearly constant value $g_R \simeq 0.33$ for even-even nuclei in the region of rare earths.

In Ref. 18, Prior and his coworkers have recalculated g_R following procedures similar to those employed by Nilsson and Prior[3] and Grin' and Pavlichenkov.[9] These new microscopic results are on the average higher than the old microscopic values as listed in Table I and Table II and are therefore less compatible with present calculations. It is to be noted, however, that the microscopic results for even-even nuclei based on the pairing-plus-quadrupole model[22] are consistently lower than the present calculations.

The reasonable success of the present approach is, of course, tempered by the large discrepancies which appear in several cases of odd-proton nuclei. The explanation of these large discrepancies is probably beyond the scope of the present macroscopic approach.

Nevertheless, there exists no exact theory at this time which can make accurate predictions in all cases, and the experimental uncertainty of g_R is still considerable. More significant, perhaps, is the fact that a single macroscopic expression such as Eq. (14) can accomplish at least as much as the microscopic calculations for both even-even and odd-mass nuclei. Therefore, in view of these facts, the basic idea of the present approach may still deserve careful inspection despite the aforementioned shortcoming. The discussion of the moments of inertia of even-even nuclei presented in the following section (Sec. III) may give further illustration of this point.

The reliability of Eq. (14) may be further tested by analyzing the magnetic dipole transition probabilities of odd-mass nuclei. The results of this analysis will be presented in the following paper.[20] This analysis shows that the accuracy of the predictions of Eq. (14) appears to be much better than the accuracy of the microscopic calculations in the region of rare earths except for the aforementioned few cases of odd-proton nuclei.

III. GENERAL DISCUSSION

The extensive microscopic calculation of the moment of inertia has been based on the cranking-model formula applied in the quasiparticle formalism, including pair correlation of the nucleonic motion.[2,3,6,18] The effect of pairing interaction is essentially to reduce the moment of inertia from the rigid-body value [Eq. (18)]. On the other hand,

TABLE I. Rotational gyromagnetic ratios of deformed even-even nuclei. The first column lists the nuclei for which rotational gyromagnetic ratios in the ground-state rotational bands have been calculated according to Eq. (14). Column two gives the deformation parameter δ [see the discussion immediately following Eq. (16)] and column three the energy of the lowest (2+) rotational excitation; these values are taken from Ref. 2 and Ref. 7. Theoretical values of g_R according to Eq. (14) are listed in column four and the theoretical values of Nilsson and Prior (Ref. 3) are listed in column five. The last column gives the references for the experimental data of g_R which are listed in column six. $R_0 = 1.2A^{1/3}$ F.

Nucleus	δ	$3\hbar^2/J$ (keV)	g_R (theor.) Eq. (14)	(Ref. 3)	g_R (exp.)	Ref.
$_{60}Nd^{150}$	0.24	130	0.30	...	0.310±0.021	a
Sm^{152}	0.27	122	0.33	0.341	0.35 ±0.03	b
$^{62}Sm^{154}$	0.31	83	0.25	0.295	0.288±0.029	c
$_{64}Gd^{154}$	0.28	123	0.36	0.367	0.367±0.03	b,d
Gd^{156}	0.39	89	0.33	0.333	0.32 ±0.03	b
Gd^{158}	0.44	79	0.32	0.319	0.315±0.025	c
Gd^{160}	0.45	76	0.31	0.307	0.303±0.026	c
$_{66}Dy^{160}$	0.33	86	0.30	0.318	...	
Dy^{162}	0.34	82	0.30	0.311	...	
Dy^{164}	0.39	73	0.28	0.304	...	
$_{68}Er^{164}$	0.31	90	0.31	0.306	...	
Er^{166}	0.31	80	0.28	0.303	0.305±0.015	e
Er^{168}	0.31	80	0.29	0.309	...	
Er^{170}	0.31	79	0.28	0.296	...	
$_{70}Yb^{170}$	0.28	84	0.28	0.313	...	
Yb^{172}	0.29	78	0.27	0.308	0.279±0.014	f
Yb^{174}	0.29	76	0.27	0.305	0.247±0.013	f
Yb^{176}	0.29	82	0.29	0.324	0.299±0.015	f
$_{72}Hf^{176}$	0.28	89	0.31	0.245		
Hf^{178}	0.29	91	0.34	0.245	0.356±0.035	g
Hf^{180}	0.26	93	0.32	0.280	0.313±0.035	h
$_{74}W^{182}$	0.25	100	0.34	0.231	0.336±0.044	i
W^{184}	0.23	112	0.37	0.284	0.38 ±0.05	j
W^{186}	0.23	124	0.41	0.352	0.34 ±0.03	k
$_{76}Os^{186}$	0.19	137	0.39	...	0.32 ±0.015	k
Os^{188}	0.17	155	0.36	...	0.310±0.027	k
$_{90}Th^{232}$	0.24	52	0.24	0.26	...	
$_{92}U^{238}$	0.27	44	0.24	0.25	...	

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the limitation on the number of protons which can follow the rotational motion on the average has also the effect of reducing the contribution of the protons to the rotational gyromagnetic ratio. It is not clear, however, whether there is any direct connection between this limitation and the pairing interaction. This puzzling situation may be further demonstrated by considering the moments of inertia of even-even nuclei in which the spins of the nucleons are essentially paired off, and the moment of inertia may simply be described in terms of the mass flow associated with the amount of matter following the rotational motion on the average.

To be more accurate, we may have to introduce another deformation parameter δ_0 which characterizes the deviation of the mass distribution from spherical symmetry.[21] By setting the angular momentum associated with the rotational motion equal to $(M'A'/A)\sum_i \vec{r}_i \times(\vec{\Omega}\times\vec{r}_i)$ and following a derivation similar to that presented in Sec. II, we obtain

$$J=\tfrac{3}{5}AM'R'^2\delta_0(1-\tfrac{2}{3}\delta_0+\cdots), \tag{17}$$

where M' is the average nucleonic mass and R' the mean radius of the mass distribution.

In terms of the moment of inertia of a rigid body conventionally defined by

$$J_{\rm rig}=\tfrac{2}{5}AM'R_0^2(1+\tfrac{1}{3}\delta+\cdots), \tag{18}$$

we have

$$\frac{J}{J_{\rm rig}}\simeq\tfrac{3}{2}\delta_0\,\frac{1-\frac{2}{3}\delta_0}{1+\frac{1}{3}\delta}, \tag{19}$$

TABLE II. Rotational gyromagnetic ratios of deformed odd-mass nuclei. The first column lists the nuclei for which rotational gyromagnetic ratios in the ground-state rotational bands have been calculated according to Eq. (14). Column two and column three give the deformation parameter δ and the moment of inertia J, respectively; these values are taken from Ref. 5 [see the discussion immediately following Eq. (16)]. Theoretical values of g_R according to Eq. (14) are listed in column four, and the theoretical values of Grin' and Pavlichenkov are listed in column five. Column six and column seven list two sets of empirical values of g_R taken from Ref. 16 and Ref. 17, respectively. $R_0=1.2A^{1/3}$ F.

Nucleus	δ	$3\hbar^2/J$ (keV)	g_R (theor.) Eq. (14)	(Ref. 9)	g_R (exp.) (Ref. 16)	(Ref. 17)
$_{13}Al^{25}$	0.39	1380	0.33	···	···	···
$_{63}Eu^{153}$	0.33	71	0.24	0.44	0.452 ± 0.012	0.475 ± 0.008
$_{64}Gd^{155}$	0.34	72	0.26	0.27	0.34 ± 0.07	0.31 ± 0.04
Gd^{157}	0.34	66	0.24	0.28	0.22 ± 0.06	0.27 ± 0.03
$_{65}Tb^{159}$	0.34	70	0.26	0.37	0.24 ± 0.09	0.35 ± 0.14
$_{66}Dy^{161}$	0.33	38	0.14	0.09	0.25 ± 0.10	0.26 ± 0.11
Dy^{163}	0.33	63	0.23	···	0.243 ± 0.021	0.27 ± 0.25
$_{67}Ho^{165}$	0.33	63	0.24	0.53	0.30 ± 0.07	0.44 ± 0.15
$_{68}Er^{167}$	0.32	52	0.19	0.11	0.124 ± 0.047	0.153 ± 0.020
$_{69}Tm^{169}$	0.32	74	0.28	···	0.38 ± 0.12	···
Tm^{171}	0.31	72	0.27	···	···	···
$_{70}Yb^{171}$	0.31	73	0.27	···	···	···
Yb^{173}	0.31	73	0.28	0.28	0.20 ± 0.09	0.28 ± 0.04
$_{71}Lu^{175}$	0.31	76	0.29	···	···	···
Lu^{177}	0.28	79	0.29	···	···	···
$_{72}Hf^{177}$	0.29	75	0.28	0.25	0.215 ± 0.04	0.17 ± 0.01
Hf^{179}	0.28	66	0.25	0.10	0.203 ± 0.034	0.258 ± 0.016
$_{73}Ta^{181}$	0.25	91	0.31	0.22	0.327 ± 0.009	0.319 ± 0.008
$_{74}W^{183}$	0.22	95.1	0.30	···	···	···
$_{75}Re^{183}$	0.22	98	0.32	···	···	···
Re^{185}	0.20	107	0.32	0.29	0.413 ± 0.042	0.420 ± 0.043
Re^{187}	0.20	115	0.35	0.37	0.413 ± 0.043	0.377 ± 0.044
$_{90}Th^{231}$	0.25	36	0.18	···	···	···
$_{91}Pa^{233}$	0.26	36	0.19	···	···	···
$_{92}U^{233}$	0.26	34	0.18	···	···	···
U^{235}	0.26	31	0.16	···	···	···
$_{93}Np^{237}$	0.27	28	0.15	···	···	···
Np^{239}	0.28	27	0.15	···	···	···
$_{94}Pu^{239}$	0.28	37	0.22	···	···	···
Pu^{241}	0.29	38	0.24	···	···	···
$_{95}Am^{241}$	0.29	36	0.23	···	···	···
Am^{243}	0.29	36	0.23	···	···	···
$_{96}Cm^{245}$	0.28	37	0.23	···	···	···

in which the difference between R' and R_0 has been ignored (it seems more appropriate to use R' instead of R_0 in the definition of $J_{\rm rig}$). Equation (18) can easily be derived from Eq. (18) of Ref. 6 by using Eq. (16), or directly from the formula $J_{\rm rig} = AM'(a^2+b^2)/5$ and Eqs. (11) and (12).

It has been suggested that the distributions of protons and neutrons do not exactly coincide, and their corresponding deformation parameters δ_p (essentially δ in this paper) and δ_n are related by the following expression[21]:

$$\delta_n/\delta_p = (G_p/G_n)^{1/2}, \tag{20}$$

where G_p and G_n are the pairing parameters for protons and neutrons, respectively.

Empirical analysis of the odd-even mass difference indicates that G_p is in general larger than G_n.[3] Therefore, δ_n (and consequently δ_0) should be larger than δ. Although the exact relation between δ_0 and δ is not yet known, we may phenomenologically set $\delta_0 \simeq \delta(1+k\delta)$ and see whether one value of k can be found so that Eq. (19) fits the experimental data over a wide range. It is found that, for $k=\frac{1}{3}$, we have

$$J/J_{\rm rig} \simeq \tfrac{3}{2}\delta(1-\tfrac{2}{3}\delta), \tag{21}$$

which is indeed in fair agreement with the experimental data in the region of rare earths (for easy check, see Fig. 10–22, p. 280, Ref. 12).

It seems clear from the above discussion that according to the present point of view the reduction of the moment of inertia (of an even-even nucleus) from the rigid-body value is mainly due to the limitation on the number of nucleons which can follow the rotational motion on the average. We may argue that the pairing interaction opposes the deformation caused by the lower-multipole interactions and therefore acts in the same direction as limiting the number of nucleons which can follow the rotational motion on the average. This argument may not be consistent, however, with current microscopic calculations in which the deformation parameter is specified before the pairing interaction is taken into account.

In view of the above puzzling situation, a few remarks about the quasiparticle formalism are in order. The quasiparticle formalism in the nuclear case is mathematically patterned after the theories of superconductivity and superfluids. In spite of the usefulness of this mathematical scheme and its literal interpretation advanced by many investigators, the question of whether nuclear matter is indeed in a superconducting or superfluid state in the physical sense may not be considered as settled. However, in view of the mathematical similarity, the quasiparticle formalism in the nuclear case certainly inherits some of the macroscopic implications; notably, the decrease of the moment of inertia of a superfluid confined in a rotating container.[23] Statistically speaking, the decrease of moment of inertia of a superfluid is due to the fact that the superfluid component cannot follow the rotation of the container. Thus, again in the sense of similarity, the decrease of moment of inertia in the nuclear case may be equivalently described as the limitation on the number of nucleons which can follow the rotational motion on the average. From this macroscopic point of view, the present approach may not be entirely incompatible with the quasiparticle formalism, although the apparent conflict between their basic assumptions still remains to be resolved.

The present approach is essentially phenomenological in character. Nevertheless, the regularities revealed in this paper should probably not be overlooked even from the microscopic point of view.

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Magnetic Dipole Transition Probabilities of Deformed Odd-Mass Nuclei*

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Based on the ground-state magnetic moments, the magnetic dipole transition probabilities in the ground-state rotational bands of deformed odd-mass nuclei are analyzed assuming that the rotational gyromagnetic ratios are given by a theoretical expression derived previously.

According to the collective model of Bohr and Mottelson,[1,2] the magnetic properties of the ground-state rotational band (with ground-state angular momentum $I_0 \neq \frac{1}{2}$) of a deformed odd-mass nucleus are characterized by the intrinsic gyromagnetic ratio g_K and the rotational gyromagnetic ratio g_R. These two parameters are derived primarily from the ground-state magnetic moment μ_0 and the magnetic dipole transition probability $B(M1;\, I_i \to I_f)$ between any two states in the rotational band and with angular momenta I_i and I_f, respectively, through the following model-dependent relations[2,3]:

$$\mu_0 = \frac{I_0}{I_0+1} g_R + \frac{I_0^2}{I_0+1} g_K \,, \tag{1}$$

$$M_0 = I_0(g_K - g_R) \,, \tag{2}$$

$$M_0 = \pm\left[\frac{4\pi B(M1;I_i \to I_f)}{3\langle I_i 1 I_0 0 | I_i 1 I_f 0\rangle^2}\right]^{1/2} , \tag{3}$$

where $\langle I_i 1 I_0 0 | I_i 1 I_f 0\rangle$ is a vector-addition coefficient. The sign of M_0 in Eq. (3) is determined by

$$\text{sgn}\epsilon = \text{sgn}\,\frac{g_K - g_R}{Q_0} , \tag{4}$$

where Q_0 is the intrinsic quadrupole moment and ϵ the ratio between the electric quadrupole and magnetic dipole matrix elements for the transition.

The values of g_K and g_R derived from Eqs. (1)–(4) based on the experimental values of μ_0 and M_0 serve two practical purposes. Firstly, they may be used in predicting the magnetic moments of the excited states and the probabilities for other magnetic dipole transitions in the ground-state rotational band. These predictions are, so far, in fairly good agreement with measurements and consequently lend strong support to the collective model.[4,5] Secondly, they may be taken as empirical values for testing any theoretical calculations of g_K and g_R. (Here we reserve the term empirical value for any quantity not directly measured but semiempirically calculated.) These theoretical calculations are more or less independent of Eqs. (1)–(4) and therefore may add insight in understanding the nuclear structure.

Theoretical calculations of g_R for odd-mass nuclei have been considered by several investigators.[6–8] However, in this paper, we are particularly interested in the following macroscopic expression[8]:

$$g_R = \frac{3}{5J} Z M_p R_0^2 \delta(1 - \tfrac{2}{3}\delta + \cdots) , \tag{5}$$

where Z is the atomic number, M_p the mass of the proton, R_0 the mean radius of the charge distribution, J the moment of inertia, and δ the deformation parameter. The remarkable feature of this expression is that g_R, J, and Q_0 are interrelated in a self-consistent way.[8] Furthermore, this expression is supposed to be valid for both even-even and odd-mass nuclei.[8]