PHYSICAL REVIEW C VOLUME 2, NUMBER 4 OCTOBER 1970

Theory of Pairing Vibrations*

Amit Goswami and Orhan Nalcioğlu

Institute of Theoretical Science and the Department of Physics, University of Oregon, Eugene, Oregon 97403

(Received 18 May 1970)

The bootstrap theory is applied to the case of pairing vibrations; an approximation used by Dang and Klein in their treatment of pairing is shown to be consistent with the bootstrap theory. A modified Tamm-Dancoff approximation is described. The numerical results for $Ni^{58,60,62}$ are compared with the exact solutions and with the results of Dang and Klein.

I.

It is well known that the pairing interaction leads to superconductivity away from the closed shells and to pairing vibrations near closed shells. In the former case, it is treated by the BCS method, which conserves the particle number on the average, and in the later case by the random-phase approximation (RPA). Dang and Klein[1] formulated a self-consistent method which makes use of the states of the even nucleus and the neighboring odd-mass nuclei. In this scheme, the authors attempt to conserve the particle number more exactly and determine the odd and even nuclear properties self-consistently. We have formulated a self-consistent bootstrap theory of vibrations along similar lines and applied it to quadrupole oscillations of heavy spherical nuclei and to the closed shell region.[2] The essence of this method lies in the fact that there exist large field potentials (e.g., in Ref. 2, multipole moments) and with the largeness of the matrix-element argument, we can truncate even-core complete-set expansions. Thus the energies and potentials of even nuclei can be generated self-consistently. The purpose of this present note is twofold, first to establish the relation between the bootstrap theory of vibrations for the pairing interaction and the method of Ref. 1, secondly to develop an extended Tamm-Dancoff approximation (TDA) (RPA) method to deal with the pairing vibrations without self-consistency. The bootstrap theory yields exactly the same equations as Ref. 1 in the case of the pairing interaction, except that we derive an additional equation to get the energies of the even-core states. (Following Ref. 1 we choose these to be the ground states of even nuclei.) Before proceeding any further, let us define the Hamiltonian H and the pair creation operators $A_a{}^\dagger$;

$$H=\sum_\alpha \epsilon_\alpha C_\alpha{}^\dagger C_\alpha - \tfrac{1}{4}|G|\sum_{ab}\sqrt{\Omega_a}\sqrt{\Omega_b}\,A_a{}^\dagger A_b\,, \tag{1}$$

where

$$A_a{}^\dagger=\frac{1}{\sqrt{\Omega_a}}\sum_{m_\alpha} S_\alpha C_\alpha{}^\dagger C_{-\alpha}{}^\dagger\,,$$

and

$$\Omega_a=2j_a+1,\quad S_\alpha=(-)^{j_a-m_\alpha}.$$

In order to derive the energy equation, we take the commutator of the Hamiltonian with a pair creation operator $A_c{}^\dagger$ and take the matrix elements of this commutator between neighboring even nuclear ground states.

$$\begin{aligned}\langle K+N+2|[H,A_c{}^\dagger]|K+N\rangle &= [W(K+N+2)-W(K+N)]\langle K+N+2|A_c{}^\dagger|K+N\rangle\\ &=2\epsilon_c\langle K+N+2|A_c{}^\dagger|K+N\rangle\\ &\quad+|G|\sqrt{\Omega_c}\sum_a\sqrt{\Omega_a}\langle K+N+2|A_a{}^\dagger(\hat N_c-\tfrac12)|K+N\rangle\,,\end{aligned}\tag{2}$$

where $\hat N_c=\sum_{m\gamma}C_\gamma{}^\dagger C_\gamma$ is the number operator, and $|K+N\rangle$ is the ground state of an even $(K+N)$-particle nucleus. If we employ a complete-set expansion in terms of $|\Psi_{K+N}\rangle$ states of an even $K+N$ core in the second term of Eq. (2), we get the following equation

$$\begin{aligned}[W(K+N+2)-W(K+N)]\langle K+N+2|A_c{}^\dagger|K+N\rangle &= 2\epsilon_c\langle K+N+2|A_c{}^\dagger|K+N\rangle+|G|\sqrt{\Omega_c}\sum_{\xi a}\sqrt{\Omega_a}\\ &\quad\times\langle K+N+2|A_a{}^\dagger|\Psi^\xi_{K+N}\rangle\langle\Psi^\xi_{K+N}|\hat N_c-\tfrac12|K+N\rangle\,.\end{aligned}$$

If we only retain $|\Psi^\xi_{K+N}\rangle=|K+N\rangle$ (i.e., the ground state) in the complete-set expansion, we get

$$N_c=\tfrac12\left[1-\frac{\epsilon_c-W/2}{E_c}\right]\,, \tag{3}$$

where $W=W(K+N+2)-W(K+N)=2\lambda$, λ being the usual chemical potential. This, of course, is the BCS number equation. Thus we show that in the limit of extreme truncation (i.e., when the even spectrum is very anharmonic) we get the BCS limit. Incidentally, the numerical results given in Ref. 1 are also obtained with the constraint (3). Thus, the present derivation justifies the use of this constraint.

II.

We shall now show in the weak-coupling limit how a modified version of the TDA (MTDA)[3] can be formulated near closed shells, and we shall also compare this with the results of Ref. 1 and exact solutions for Ni isotopes. In the TDA method, the commutator $[A_a, A_b^\dagger]$ is replaced by its vacuum expectation value, where the vacuum $|K\rangle$ is the doubly-closed-shell nucleus. In contrast to the TDA, we shall evaluate this commutator by taking

$$[A_a, A_b^\dagger] = 2\delta_{ab}[1-2\langle\hat{N}_a\rangle] , \tag{4}$$

where the expectation value $\langle\hat{N}_a\rangle$ is evaluated in the manner described below. We thus correct for the correlations in the even-core states. The next assumption is very much in the spirit of the TDA. Here we assume that the ground state of a $(K+N)$-particle even nucleus can be obtained from the previous $(K+N-2)$-particle nucleus by means of a phonon creation operator $B^\dagger(K+N)$, where

$$B^\dagger(K+N) = \frac{1}{2}\sum_a \frac{X_a(K+N)}{\sqrt{\Omega_a}} A_a^\dagger \tag{5}$$

with

$$X_a(K+N) = \frac{\Omega_a[1-2N_a(K+N-2)]}{2\epsilon_a - [W(K+N)-W(K+N-2)]}\Delta(K+N) \tag{6}$$

and

$$\Delta(K+N) = \tfrac{1}{2}|G|\sum_a X_a(K+N) .$$

Here K is the number of nucleons in the doubly closed shell, and N is an even integer denoting the number of particles outside the doubly closed shell. Thus a $N/2$ phonon state is defined as

$$|K+N\rangle = \beta(K+N)B^\dagger(K+N)\ldots B^\dagger(K+2)|K\rangle , \tag{7}$$

where $\beta(K+N)$ is a normalization constant for the state $|K+N\rangle$. The vacuum for the phonons is defined by

$$B(K+N)|K\rangle = 0 , \tag{8}$$

i.e., we assume shell-model occupation probabilities for the doubly closed shell. In order to determine the energy of the ground state, we sum Eq. (6) over j_a on both sides; this gives us the following MTDA secular equation,

$$\frac{2}{|G|} = \sum_a \frac{\Omega_a[1-2N_a(K+N-2)]}{2\epsilon_a - [W(K+N)-W(K+N-2)]} . \tag{9}$$

We see that in order to calculate the relative energy $W(K+N)-W(K+N-2)$, we need to know the number distribution $N_a(K+N-2)$ from the previous nucleus. This is given as follows,

$$N_a(K+N-2) = \langle K+N-2|\hat{N}_a|K+N-2\rangle . \tag{10}$$

When we substitute the state $|K+N-2\rangle$ from Eq. (7), we obtain the following equation

$$N_a(K+N-2) = \beta^2(K+N-2)\langle K|B(K+2)\cdots B(K+N-2)\hat{N}_a B^\dagger(K+N-2)\cdots B^\dagger(K+2)|K\rangle .$$

The normalization constant $\beta^2(K+N-2)$ is then fixed by demanding the total number $N_T(K+N-2)$ is given correctly by Eq. (10), i.e.,

$$N_T(K+N-2) = \sum_a \Omega_a N_a(K+N-2) . \tag{11}$$

Another quantity that needs to be determined is Δ, the gap parameter. In order to determine this we make use of the approximate boson character of the operators $B^\dagger$.

$$\begin{aligned} 1 &\simeq [B(K+N), B^\dagger(K+N)] \\ &\simeq \langle K+N-2|[B(K+N), B^\dagger(K+N)]|K+N-2\rangle . \end{aligned} \tag{12}$$

This yields the following equation

$$2 = \sum_a \frac{X_a(K+N)}{\Omega_a}[1-2N_a(K+N-2)] , \tag{13}$$

which determines $\Delta(K+N)$.

With the help of Eqs. (9), (11), and (13), we are in a position to calculate the ground-state energies and the number distribution for even nuclei. The numerical results and the conclusion will be presented in the next section.

III.

The numerical results will be presented for Ni isotopes, and compared with the results of Dang and Klein[1] and Kerman, Lawson, and MacFarlane.[4] We use the single-particle energies that were used in Ref. 4 and a coupling constant of $G=0.331$. We shall give the solutions up to three-phonon states, i.e., for $Ni^{58,60,62}$. First we solve Eq. (9) by substi-

TABLE I. Values of the ground-state energy in the model of Ref. 4 (exact solutions), and approximations of Ref. 1 and the present MTDA method.

$K+N$	MTDA	a	b
58	−1.48	−1.48	−1.49
60	−2.06	−2.06	−2.11
62	−1.94	−1.69	−1.75

[a]See Ref. 1. [b]See Ref. 4.

TABLE II. Comparison of single-particle level occupation probabilities for the model of Ref. 4 obtained by the MTDA and Ref. 1.

$K+N$	J_a	MTDA	a
58	$\frac{3}{2}$	0.343	0.319
	$\frac{5}{2}$	0.081	0.094
	$\frac{1}{2}$	0.035	0.041
	$\frac{9}{2}$	0.006	0.007
60	$\frac{3}{2}$	0.533	0.566
	$\frac{5}{2}$	0.241	0.233
	$\frac{1}{2}$	0.108	0.096
	$\frac{9}{2}$	0.020	0.014
62	$\frac{3}{2}$	0.686	0.733
	$\frac{5}{2}$	0.402	0.417
	$\frac{1}{2}$	0.214	0.177
	$\frac{9}{2}$	0.041	0.020

[a]See Ref. 1.

TABLE III. Comparison of the values of gap parameter Δ obtained for the model of Ref. 4 by the MTDA and Ref. 1.

$K+N$	MTDA	a
58	0.87	0.89
60	1.21	1.17
62	1.98	1.31

[a]See Ref. 1.

tuting $N_a(K)=0.0$ and evaluating $W(K+2)-W(K)$. As it is seen from Table I, for up to two phonons this simple method gives excellent results for the energy. Table II gives the number distribution, and the agreement with Ref. 1 is very impressive. Lastly, Table III compares the gap parameters. It is obvious that this MTDA method gives very good results for one- and two-phonon states. It is to be hoped that the two-phonon states in other cases (e.g., octupole vibrations in a closed shell and quadrupole vibrations in heavy spherical nuclei) can be treated by the present method, or else by its RPA generalization (MRPA).

---

*Work supported in part by the U. S. Atomic Energy Commission.

[1]G. D. Dang and A. Klein, Phys. Rev. 143, 735 (1966).

[2]A. Goswami, O. Nalcıoğlu, and A. I. Sherwood, to be published. See also A. M. Dreizler, A. Klein, C.-S. Wu, and G. Do Dang, Phys. Rev. 156, 1169 (1967); G. J. Dreiss, Ph.D. thesis, University of Pennsylvania, 1968 (unpublished).

[3]K. Ikeda, T. Udagawa, and H. Yamaura, Prog. Theor. Phys. (Kyoto) 33, 22 (1965); J. Högaasen-Feldman, Nucl. Phys. 28, 258 (1961); D. R. Bes and R. A. Broglia, Nucl. Phys. 80, 289 (1966).

[4]A. K. Kerman, R. D. Lawson, and M. H. MacFarlane, Phys. Rev. 124, 162 (1961).