PHYSICAL REVIEW C VOLUME 2, NUMBER 4 OCTOBER 1970

Polarization in Elastic Nucleon-Deuteron Scattering*

J. Krauss† and K. L. Kowalski
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
(Received 22 June 1970)

An approximate K-matrix calculation of elastic nucleon-deuteron scattering is carried out including the effects of two-nucleon tensor forces. The results of these computations, which are carried out in the energy range from 11 to 40 MeV, are in reasonable agreement with the experimental nucleon polarizations and differential cross sections.

I. INTRODUCTION

Despite the enormous progress achieved over the past decade in calculating three-nucleon observables, there still remains a large body of unexplained experimental results.[1-7] Outstanding among these data is the nucleon polarization $P(\theta)$ as a function of the c.m. scattering angle θ in elastic nucleon-deuteron scattering. This particular observable is of special interest among many other spin correlation parameters for several reasons, not the least of which is its relatively simple experimental determination. Theoretically, of course, the behavior of $P(\theta)$ is thought to place significant constraints on the structure of the scattering matrix.[8]

It is, perhaps, premature to speculate upon the usefulness of $P(\theta)$ in sorting out the details of the three-nucleon dynamics. Nonetheless, up to the present time no self-contained computations of $P(\theta)$ of any reliability have been carried out, with the possible exception of those at 150 MeV using the impulse approximation.[9-11] On the other hand, phenomenological[12] and semiphenomenological[13,14] considerations of $P(\theta)$ at energies up to 40 MeV have been surprisingly successful and have hinted at the possibility of relatively simple mechanisms being responsible for the observed behavior of $P(\theta)$.

The main reason for the lack of calculations of $P(\theta)$ complementary to those for the differential cross sections[1-6] is the very large number of coupled integral equations which appear upon introducing the necessary noncentral (tensor) N-N forces even via the usual device of separable potentials.[15] Actually, this instance is rather symptomatic of the present state of the three-particle computational art. Namely, the standard, (so called) exact integration procedures present a much too confining framework for executing many of the most interesting three-particle calculations. Most of the usual alternatives, such as variational techniques,[16] or methods using a denumerable, normalizable, complete set of three-particle states[17] appear to be useful mainly at rather low energies and in connection with the bound-state problem.[18]

Recently three approximate methods have been proposed which exploit the simplicities arising from the weak deuteron binding, which are practical for high incident nucleon energies, and whose viability does not depend upon special models of the two-particle interaction.[19-23] All three of these procedures have the common feature of being unitarization techniques via the optical-potential,[19] K-matrix,[20] and N/D[21] formalisms, respectively. The results which have been achieved thus far are very encouraging and suggest that these methods may provide the framework for answering more sophisticated questions in three-nucleon physics, in particular the elucidation of all observed scattering phenomena up to the pion production threshold. Of course, the introduction of an approximate three-particle dynamics is a departure from the predominant style of the recent history of this subject. However, in view of the complexities introduced by "realistic"[24] two-nucleon interactions and high incident energies such methods appear to be unavoidable.

The present investigation consists of the application of the K-matrix method due to Sloan[20] to the computation of $P(\theta)$ and the elastic differential cross section in the energy range 11–23 MeV. This energy range is interesting in two respects. First, the rather extensive experimental exploration of this region shows that the characteristic features of $P(\theta)$ undergo a rather pronounced variation between the extremes of this interval.[25] Thus, this should provide an excellent test of any proposed scheme for computing $P(\theta)$. Second, as we shall comment in detail upon later, these energies are sufficiently high compared with the deuteron binding energy to expect that our approximations to the three-particle dynamics will be justified. Finally, these energies are low enough to permit the incorporation of only the minimum amount of two-nucleon interaction which is consistent with the introduction of tensor forces. This model is also applied for exploratory purposes, without modification, at 40 MeV where, because of the truncations in the two- and three-particle angular momenta, the coherent forward scattering is certainly grossly underestimated.

The remainder of the paper consists of a statement of the principal dynamical approximations used in the K-matrix approach (Sec. II), a discussion of the important features of the computation (Sec. III), and our results and conclusions (Sec. IV).

II. SLOAN APPROXIMATION

We will now outline the Sloan approximation[20] to the integral equations for elastic N-d scattering. A detailed analysis of this scheme using the modified Faddeev equations of Alt, Grassberger, and Sandhas[26] has been given elsewhere.[23] In an effort to elucidate and generalize the essential features of the method, we will proceed here in a slightly different fashion. Coulomb effects will be neglected throughout this work.

If in the Heisenberg picture[27] we separate the scattering operator S into no-scattering and scattering parts in the usual fashion,

$$S=1-2\pi iT\ ,$$

then the unitarity of S implies that

$$T-T^{\dagger}=-2\pi i TT^{\dagger}=-2\pi i T^{\dagger}T. \tag{2.1}$$

Let us introduce a K operator as the solution of the equations

$$\begin{aligned} K_p &= T+i\pi TPK_p\ ,\\ &= T+i\pi K_pPT\ , \end{aligned} \tag{2.2}$$

where P is a projection operator.[28] Unitarity [Eqs. (2.1)] now implies that

$$K_p-K_p^{\dagger}=-2\pi i K_pQK_p^{\dagger}=-2\pi i K_p^{\dagger}QK_p\ , \tag{2.3}$$

where

$$Q=1-P\ .$$

In the case of interest to us, namely elastic N-d scattering, we take[29]

$$P=\sum|\mathrm{in};N,d\rangle\langle\mathrm{in};N,d|$$

so that

$$Q=\sum|\mathrm{in};3N\rangle\langle\mathrm{in};3N|$$

below the threshold for pion production. Then we observe from (2.3) that below the deuteron breakup threshold PK_pP is Hermitian, that above this threshold

$$\Delta\equiv(2i)^{-1}P(K_p-K_p^{\dagger})P$$

is negative semidefinite, and that QK_pQ satisfies a unitarity relation identical to that satisfied by a $3\to3$ transition operator in which production $(2\to3)$ is forbidden. Also, we infer from Eq. (2.2) that the two-particle disconnected parts of QK_pQ and QTQ are necessarily identical.

In essence, the procedure suggested by Sloan consists of choosing a model for K_p which is consistent with the constraints imposed by unitarity.[19,20,23] Namely, above the breakup threshold Δ must be negative definite, and below threshold K_p should be Hermitian.

Given such a model K_p, the elastic N-d scattering amplitude can be determined from the completely on-shell integral equation

$$PTP=PK_pP-i\pi(PK_pP)(PTP)\ , \tag{2.4}$$

which can be solved trivially by a decomposition into partial waves. It is easy to demonstrate from the preceding comments concerning unitarity that the resultant scattering amplitude will satisfy unitarity exactly below the breakup threshold and be consistent with it above threshold.[19,20,23] We note that the determination of the remaining matrix elements of T require the solution of no other integral equations. For example, the breakup amplitudes follow directly from

$$QTP=(QK_pP)(1-i\pi PTP).$$

The choice of the approximate K_p and ensuing computation are, of course, most expeditiously done in the interaction picture. We specify the connection

$$\langle\psi_\beta^{(\pm)}(E_\beta)|\,T\,|\psi_\alpha^{(\pm)}(E_\alpha)\rangle = \delta(E_\beta-E_\alpha)\langle\phi_\beta(E_\beta)|\,U_{\beta\alpha}\,|\,\phi_\alpha(E_\alpha)\rangle\ , \tag{2.5}$$

between the two forms of scattering operators for the general situation in which any pair of particles can form a bound state. In Eq. (2.5) $|\phi_\alpha\rangle$, $\alpha=1,2,3$ refers to a noninteracting two-particle state comprised of particle α moving freely and a bound state of the other two; $\alpha=0$ corresponds to a three-particle plane wave. The $|\psi_\alpha^{(\pm)}\rangle$ refer to the in (+) and out (−) states corresponding to channel α. If we let $\overline{U}_{\beta\alpha}$ denote the interaction-picture operators corresponding to K_p, then in the three-particle c.m. frame and for equal-mass (m) particles Eq. (2.4) becomes $(\alpha,\beta\neq0)$

$$U_{\beta\alpha}(\vec{q}_\beta|\vec{q}_\alpha)=\overline{U}_{\beta\alpha}(\vec{q}_\beta|\vec{q}_\alpha) - \frac{i\pi}{2}\frac{4m}{3}\sum_{\gamma=1}^{3} q_\gamma \int d\Omega_{\vec{q}_\gamma}\overline{U}_{\beta\gamma}(\vec{q}_\beta|\vec{q}_\gamma)U_{\gamma\alpha}(\vec{q}_\gamma|\vec{q}_\alpha), \tag{2.6}$$

where

$$U_{\beta\alpha}(\vec{q}_\beta|\vec{q}_\alpha)=\langle \Phi_\beta(E_\beta,\vec{q}_\beta)|U_{\beta\alpha}|\phi_\alpha(E_\alpha,\vec{q}_\alpha)\rangle$$

with a similar definition for $\overline{U}_{\beta\alpha}(\vec{q}_\beta|\vec{q}_\alpha)$.[30] The momentum of particle α with respect to the c.m. of the other two particles is denoted by $\vec{q}_\alpha$. Since everything is on-shell we have

$$E_\beta=E_\alpha=(3/4m)\vec{q}_\gamma^{\,2}+\epsilon_\gamma\equiv W$$

for all $\alpha,\beta,\gamma\neq 0$, where $\epsilon_\gamma<0$ is the binding energy of the bound pair in channel γ. (We assume only one bound state per particle.)

Equations (2.6) reduce to a single equation if all particles are identical, a situation which obtains in N-d scattering providing we adopt an iso-spin convention. Specifically, if we assume that the two-particle bound states are properly symmetrized and adhere to a cyclic ordering convention in the particle indices, then the elastic N-d amplitude is given by ($\alpha,\beta\neq 0$)

$$U(\vec{q}'|\vec{q})=U_{\alpha\alpha}(\vec{q}'|\vec{q})+2U_{\beta\alpha}(\vec{q}'|\vec{q}),\quad \beta\neq\alpha, \tag{2.7}$$

with $|\vec{q}'|=|\vec{q}|$, independently of α and β.[31] Similar remarks apply to $\overline{U}$. Then Eqs. (2.6) reduce to

$$U(\vec{q}'|\vec{q})=\overline{U}(\vec{q}'|\vec{q})-\frac{i\pi}{2}\frac{4m}{3}q\int d\Omega_{\vec{q}''}\overline{U}(\vec{q}'|\vec{q}'')U(\vec{q}''|\vec{q}), \tag{2.8}$$

where $\overline{U}(\vec{q}'|\vec{q})$ is defined in a manner analogous to definition (2.7).

The simplest choice for $\overline{U}$ which satisfies all the unitarity constraints imposed previously and, in addition, the condition that the two-particle disconnected parts of QTQ and QK_pQ be identical is

$$\overline{U}_{\beta\alpha}=\overline{\delta}_{\beta\alpha}(W-H_0)+\sum_{\gamma=1}^{3}\overline{\delta}_{\beta\gamma}\overline{t}_\gamma\overline{\delta}_{\gamma\alpha},\quad (\text{all }\alpha,\beta), \tag{2.9}$$

with

$$\overline{\delta}_{\beta\alpha}=1-\delta_{\beta\alpha},$$

where H_0 is the three-particle kinetic energy, and $\overline{t}_\gamma$ is that part of the two-particle transition operator in the three-particle Hilbert space for the scattering of particles, $\sigma,\tau\neq\gamma$ with the Dirac-δ-function contribution of the bound-state pole term removed.[20,23] The first term in (2.9) gives rise to the usual pickup or one-nucleon exchange term while the terms involving $\overline{t}_\gamma$ are simple impulse-graph terms. The detailed structure of the matrix elements of the quantities in Eq. (2.9) and their computation will be considered in the next section.

Equation (2.9) constitutes the principal dynamical approximation in this paper. Since it does represent a kind of impulse approximation, we would not expect it to be very good at energies on the order of the deuteron binding energy. An outstanding problem is to construct a more sophisticated approximation for $\overline{U}$ which satisfies the constraints imposed above and yet still does not involve the complication of solving a true three-body integral equation.

III. SOLUTION OF THE K-MATRIX EQUATIONS

We will next discuss some of the particular features of our utilization of Eqs. (2.8) and (2.9). There are two essential aspects to this procedure, namely the calculation of the source term (2.9) and the solution of (2.8) via a partial-wave decomposition using this input.

A. Source Terms

The spin, isospin, and kinematical structure of the matrix elements of (2.9) with respect to deuteron-plus-free-nucleon states has been studied extensively.[10,32–34] Nonetheless, the present notation as well as our eventual computation of the impulse terms differs sufficiently from Ref. 10 to warrant writing out the contribution to $\overline{U}(\vec{q}'|\vec{q})$ explicitly.

The contribution of the nucleon exchange terms in (2.9) to $\overline{U}(\vec{q}'|\vec{q})$ is[35]

$$\left[|\epsilon|+\frac{\hbar^2}{m}(\vec{q}'+\tfrac{1}{2}\vec{q})^2\right]\phi^\dagger(\tfrac{1}{2}\vec{q}'+\vec{q})\,\mathcal{S}\,\phi(\vec{q}'+\tfrac{1}{2}\vec{q}),$$

where $\mathcal{S}$ is a spin exchange operator involving the projectile and one of the target particles. The portion of $\overline{U}$ arising from the impulse terms is

$$\int d\vec{q}''\phi^\dagger(\vec{q}''+\tfrac{1}{2}\vec{q}')\times\left\langle\vec{q}'+\tfrac{1}{2}\vec{q}''\middle|t\left(W-\frac{3}{4m}\vec{q}''^{2}\right)\middle|\vec{q}+\tfrac{1}{2}\vec{q}''\right\rangle\phi(\vec{q}''+\tfrac{1}{2}\vec{q}).$$

Here t denotes

$$t=\tfrac{3}{2}(1-\mathcal{P})t^{(1)}+\tfrac{1}{2}(1+\mathcal{P})t^{(0)},$$

where $\mathcal{P}$ is a spin and space exchange operator upon the same two particles as $\mathcal{S}$; $t^{(0)}$ and $t^{(1)}$ represent the isoscalar and isotriplet components of the two-nucleon transition operator, respectively.

It is evident that the essential practicability of the method of Sec. II does not depend upon the structure of the two-nucleon interaction which is employed to compute the impulse terms.[36] Nonetheless, we assumed, primarily as a matter of mathematical convenience, the simplest separable Yamaguchi forms[37] for the S-wave singlet and S- and D-wave triplet N-N partial-wave states. This is the minimal two-nucleon input consistent with the introduction of a tensor force; it is by no means evident that it is consistent to omit the P-wave states at the energies we are considering. The justification for including only the ones we do is the familiar concept of the pole dominance of off-shell amplitudes.[38]

Considerable controversy still exists concerning the properties of the low-energy N-N system such as the percentage of deuteron D state.[8,39-42] In view of the exploratory nature of the present investigation, we do not propose to enter into this discussion at the present time, but it is clear that a most interesting question is the variation of the three-particle spin-correlation parameters, such as the polarization, as a function of the percentage of D state. For the calculations reported in this paper we chose Phillip's[41] potential parameters in the S-wave singlet case corresponding to a scattering length of -20.34 fm and an effective range of 2.50 fm, and in the triplet case the parameters of Brady *et al.*[42] corresponding to a D-state probability of 7%. This last value is in the tradition of the work of Ref. 10 which was carried out with the Yale-potential-generated[43] two-nucleon information. Given this last assignment, the choice of $r_s = 2.50$ fm in the first instance, which is almost certainly too small,[39-41] can be marginally justified by Phillip's plot[39-41] of the calculations of the triton binding energy and of the N-d doublet scattering length for various combinations of N-N parameters.

The most difficult part of the calculation consisted in the evaluation of the matrix elements of the impulse terms in the three-particle spin space which involve three-dimensional integrals over an intermediate target momentum ($\vec{q}\,''$) as coefficients of products of Clebsch-Gordan coefficients in sums over spins.[10,33,34] The portion of this computation, as well as all others in this paper which involved spin sums, were done directly using a subroutine to generate the values of the Clebsch-Gordan coefficients for arbitrary arguments. No attempt was made to simplify these matrix elements by using more sophisticated recoupling techniques. In view of recent developments[15] this procedure was undoubtedly inefficient in its use of computer time.

The three-dimensional integrals were evaluated in the following fashion. The range of integration

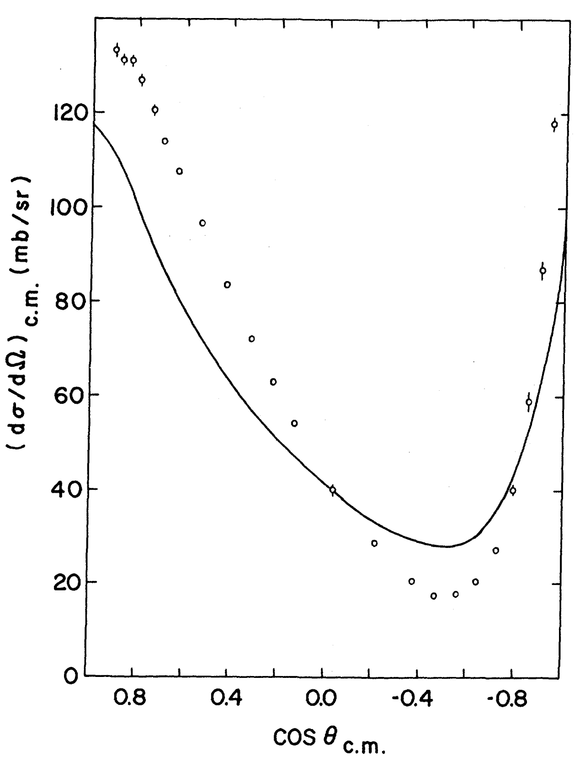

FIG. 1. N-d cross section at 11.0 MeV. The experimental points are taken from Ref. 46 (10.04 MeV).

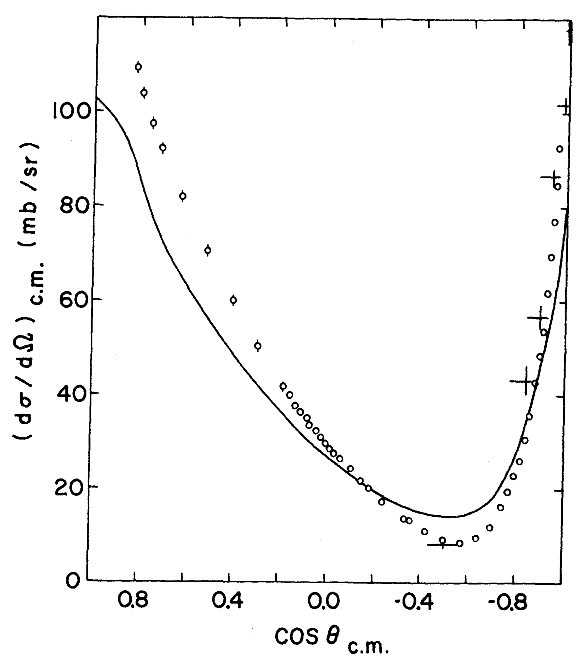

FIG. 2. N-d cross section at 14.4 MeV. The experimental points are taken from Ref. 47 (13.94 MeV, circles) and Ref. 48 (14.1 MeV, crosses).

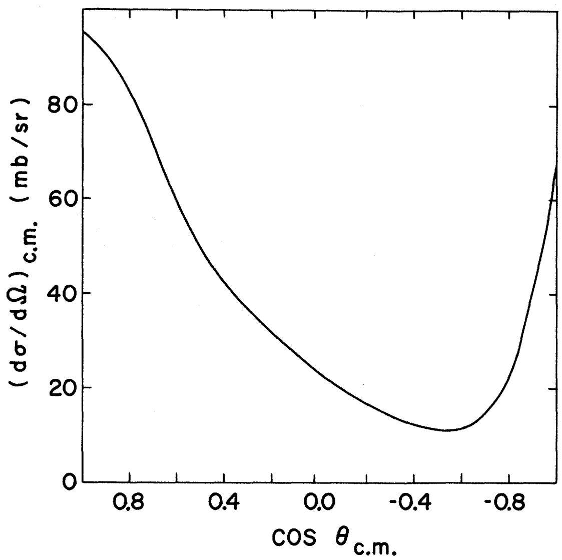

FIG. 3. N-d cross section at 17.5 MeV.

in $|\vec{q}''| = q''$ was subdivided into a region of positive and negative values of the parametric energy E of of the t matrix in the integrand, where

$$E = W - (3/4m)q''^2 .$$

For the isosinglet components the principal-value singularity in q'' arising from the deuteron pole when $E < 0$ was treated by the device of subtracting off the singular part and evaluating it exactly. Schematically, we have for the $E < 0$ part

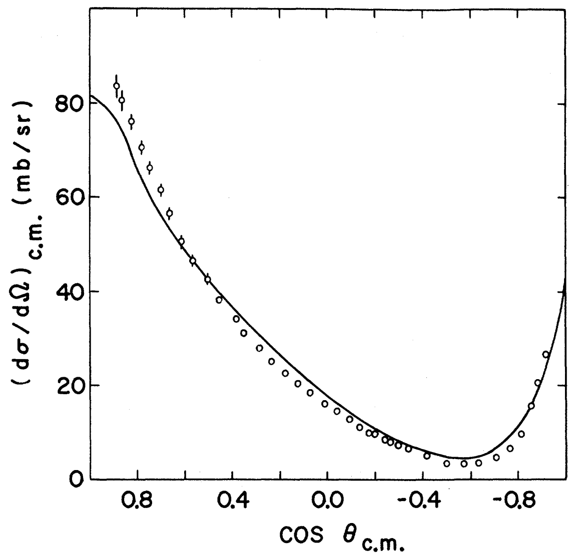

FIG. 5. N-d cross section at 22.7 MeV. The experimental points are taken from Ref. 50 (22.0 MeV).

$$P\int_{-E_c}^{0} dE\, \frac{F(E)}{E-\epsilon} = F(\epsilon)\ln\frac{-\epsilon}{E_c+\epsilon} + \int_{-E_c}^{0} dE\, \frac{F(E)-F(\epsilon)}{E-\epsilon} ,$$

where the cutoff E_c corresponds to $q'' = 2\ \text{fm}^{-1}$. This cutoff in q'' was employed in all cases.

A five-point Gaussian quadrature was used in each of the angular variables. Each of the inte-

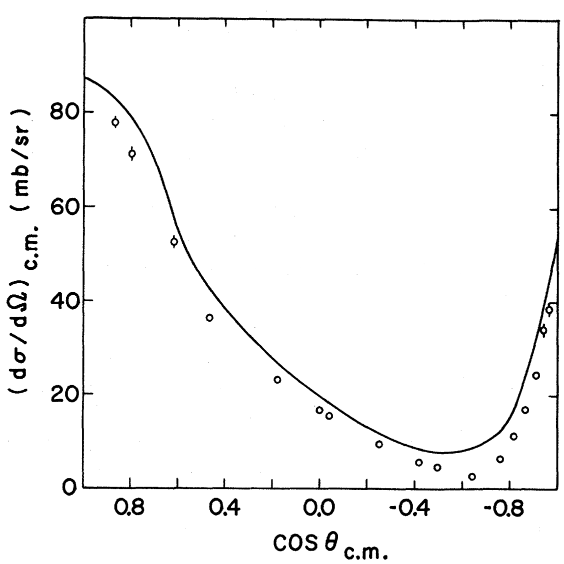

FIG. 4. N-d cross section at 20.15 MeV. The experimental points are taken from Ref. 49 (20.6 MeV).

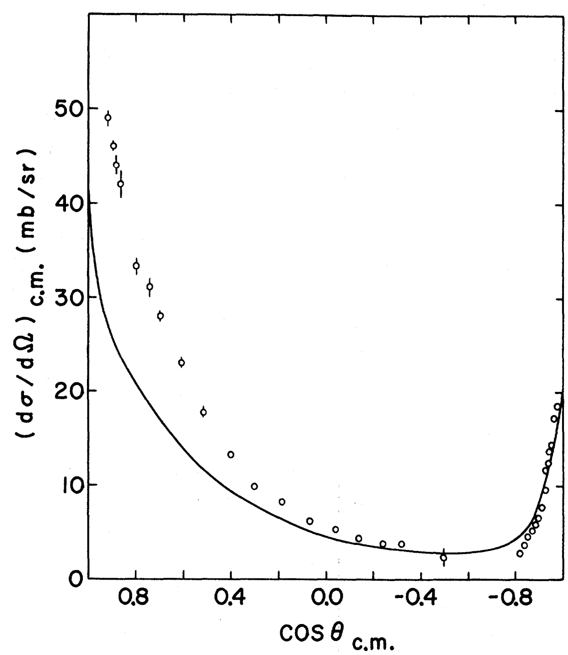

FIG. 6. N-d cross section at 40.0 MeV. The experimental points are taken from Ref. 51.

grals in q'', namely for $E<0$ and $E>0$, was done as a five-point Gaussian quadrature, so this was equivalent to a 10-point mesh in the magnitude. It was found in specific test cases that the values of the matrix elements were stable to within 5% when the mesh was increased.

The calculation of the nucleon exchange terms was straightforward.

B. Partial-Wave Decomposition

Let $|s,m_s\rangle$ denote a three-nucleon spin state appropriate to some definite combination of a deuteron and a free nucleon. We then introduce partial-wave amplitudes $U^J(l',s'|l,s)$ via the usual expansion ($|\vec{q}'|=|\vec{q}|$)

$$\langle s',m_s'|U(\vec{q}'|\vec{q})|s,m_s\rangle=\sum C_{l',s'}(J,M;m_{l'},m_{s'})Y_{l'}^{m_{l'}*}(\vec{q}',\vec{k})U^J(l',s'|l,s)C_{l,s}(J,M;m_l,m_s)Y_l^{m_l*}(\vec{q},\vec{k})\,, \tag{3.1}$$

where the sum is over $J,M,l',m_{l'},\ l,$ and m_l, and $\vec{k}$ defines an axis of quantization.[44]

With a similar definition for $\bar{U}^J(l',s'|l,s)$ Eq. (2.8) can be rewritten in partial-wave form as

$$U^J(l',s'|l,s)=\bar{U}^J(l',s'|l,s)-\frac{i\pi}{2}\frac{4m}{3}q\sum_{l'',s''}\bar{U}^J(l',s'|l'',s'')U^J(l'',s''|l,s)\ , \tag{3.2}$$

where the intermediate sum over the three-particle total spin includes only one of the doublet possibilities. Because of the coupling rules Eqs. (3.2) constitute for a fixed value of J a matrix equation of finite order. As a consequence of parity conservation this order can be reduced further so that for $J=\frac{1}{2}$ we have 2 by 2 matrix equations for each parity sign and for $J\geq\frac{3}{2}$ we have 3 by 3 matrix equations for each parity sign. The symmetry relations which follow from time-reversal invariance do not, of course, reduce the order any further.

The partial-wave amplitudes, $\bar{U}^J(l',s'|l,s)$ of the source terms were computed by inverting the expansion

$$\langle s',m_{s'}|\bar{U}(\vec{q}'|\vec{q})|s,m_s\rangle=\sum Y_{l'}^{m_s-m_{s'}}(\theta,0)\left(\frac{2l+1}{4\pi}\right)^{1/2}\bar{U}^J(l',s'|l,s)C_{l',s'}(J,m_s;m_s-m_{s'},m_s)C_{l,s}(J,m_s;0,m_s)\,, \tag{3.3}$$

which follows from (3.1) by choosing $\vec{k}=\vec{q}$ as well as fixing the scattering plane to correspond to zero azimuthal angle. It is evident from (3.3) that only one angular integration with respect to $\cos\theta=(\vec{q}'\cdot\vec{q})/q^2$ will be involved. This integration was carried out using a five-point Gaussian quadrature with the impulse terms of part A comprising the right-hand side of (3.3).

The remainder of the calculation is trivial and consists of solving Eqs. (3.2) for U^J. We confined ourselves to only $J=\frac{1}{2}$ and $J=\frac{3}{2}$, which corresponds to considering four total angular momentum and parity combinations of Eqs. (3.2).[45]

IV. RESULTS AND CONCLUSIONS

The differential cross sections and (nucleon) polarizations predicted by the model described in the preceding sections are displayed in Figs. 1–6 and 7–12, respectively, along with the relevant data.[46–52] All of the calculated cross sections which are compared with the data have (to varying degrees) too small forward and backward peaks but with the large-angle minimum overestimated.[53] The quality of the fit appears to improve somewhat for $\cos\theta\leq 0$ with increasing energy. Let us now comment on these features in turn with the intent of pointing out the deficiencies of the present as well as previous calculations. We will dwell longest upon the cross sections, since they constitute the only points of comparison with other works.

The depressed forward peak is a persistent feature of virtually all extant three-particle calculations.[54] In a model which possesses an exact three-particle dynamics, and in particular satisfies unitarity exactly, this circumstance can be ascribed to two sources. Namely, there are too few two-nucleon partial waves and/or N-d partial waves included in the computation. The present calculation almost certainly suffers from both of these inadequacies, although the removal of these deficiencies is possible without destroying the essential simplicity of the model. In addition, however, our model lacks exact unitarity and the calculations of Sloan[20] imply that this may be the source of some of the trouble with our results.

The subdued backward peak is probably due to the same combination of effects as in the forward case, with the omission of higher two-nucleon partial-wave states playing a diminished role.[55] This

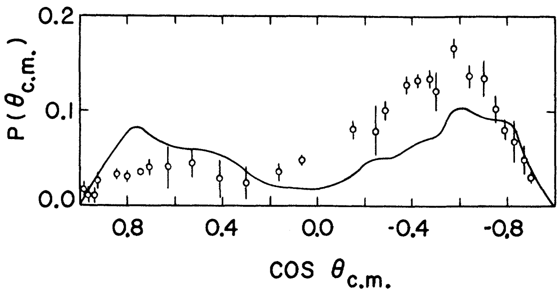

FIG. 7. N-d polarization at 11.0 MeV. The experimental points are taken from Ref. 25.

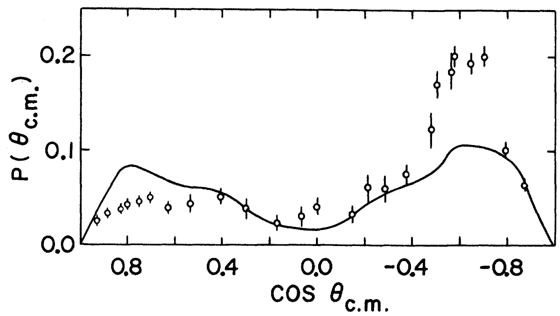

FIG. 8. N-d polarization at 14.4 MeV. The experimental points are taken from Ref. 25.

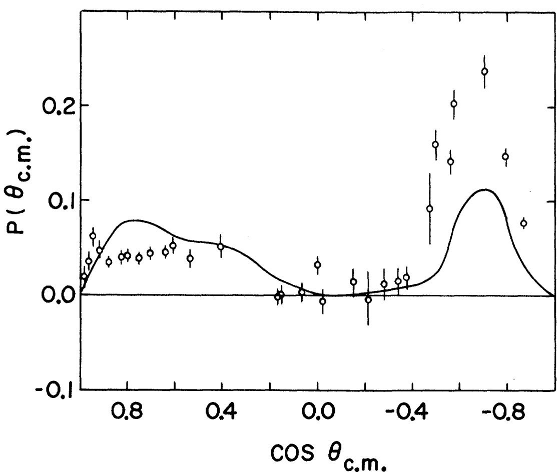

FIG. 9. N-d polarization at 17.5 MeV. The experimental points are taken from Ref. 25.

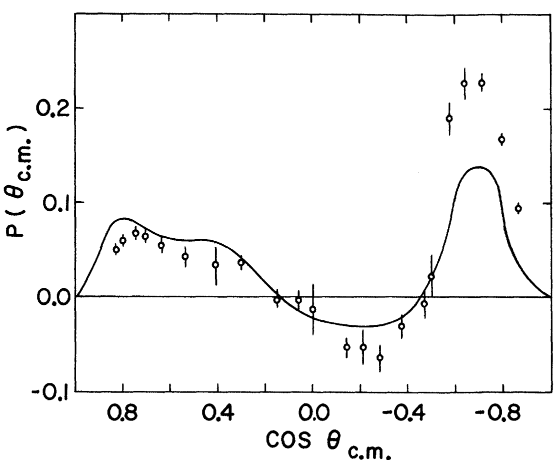

FIG. 10. N-d polarization at 20.15 MeV. The experimental points are taken from Ref. 25.

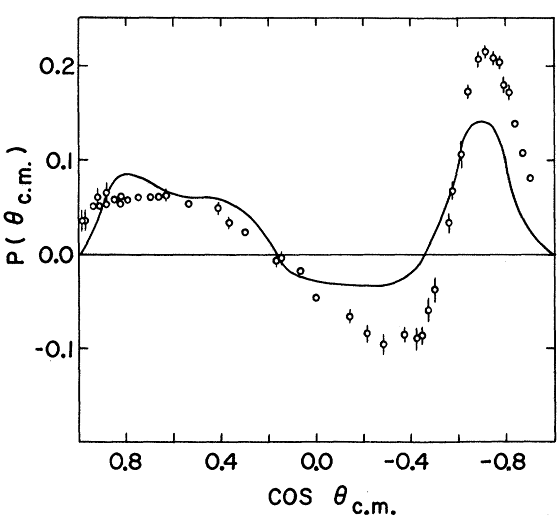

FIG. 11. N-d polarization at 22.7 MeV. The experimental points are taken from Ref. 25.

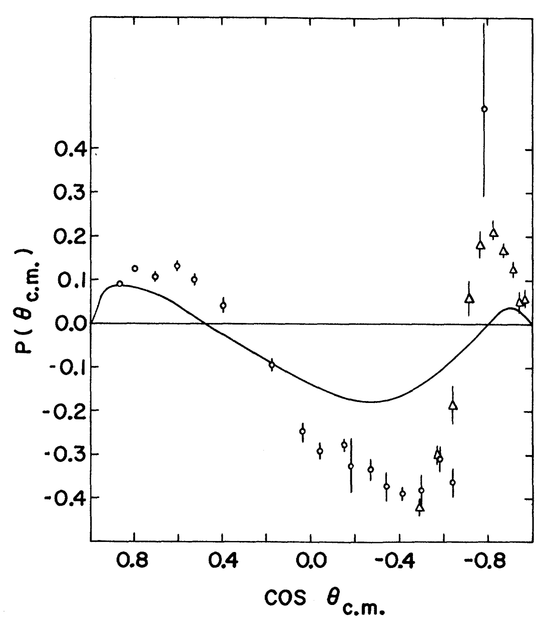

FIG. 12. N-d polarization at 40.0 MeV. The experimental points are taken from Ref. 50 (circles) and Ref. 52 (triangles).

behavior differs markedly from the most recent calculations of Sloan, although he has, presumably, a better accounting of the higher partial waves than we do.

We have no additional comments concerning the region of the minimum, since any deviations here arise from the same sources as those just discussed. Here, however, the dynamical approximations made, rather than the angular momentum truncations, are most influential. Finally, we note that the computed cross section at 40 MeV connects onto the curve calculated in the impulse approximation in Ref. 10 at about 90° to yield, in combination, a rather good fit over the entire angular range in support of the conjecture of Aaron *et al.*[54]

The collection of predicted polarizations from 11 to 22.7 MeV (Figs. 7–11) exhibit a consistent reproduction of all the qualitative features of the rather detailed data of Faivre *et al.*[25] In particular, the evolution of the negative dip is predicted as a function of energy, although its magnitude is eventually underestimated. The failure to match the height of the positive backward peak in $P(\theta)$ is certainly correlated with the overestimate of the large-angle minimum in $d\sigma/d\Omega$. The results at 40 MeV (Fig. 12) for $\cos\theta \leq 0$ indicate that even a fairly accurate cross section is not enough to guarantee a quantitatively accurate $P(\theta)$.

Up to this point we have stressed all the negative aspects of our calculation. Nonetheless, we are forced to extend the remark of Ref. 9 to the set of calculations presented here. Namely, in view of the approximations inherent in our approach and the rather crude representation of the N-N interaction, the agreement of the predictions of the K-matrix model with experiment is perhaps remarkable.

The principal conclusion of this work is the establishment of the K-matrix technique of Sloan as a viable tool for N-d calculations even with the complications of tensor forces. Nevertheless, we do not foresee successive applications of this technique as leading inexorably towards fully accurate quantitative predictions. However, in addition to the obvious refinements which can be made to the calculations described here[56] and the computation of other elastic spin-correlation parameters,[57] the following nontrivial points remain to be explored. First, an eigenphase analysis should be performed on the computed partial-wave amplitudes. Second, the breakup-reaction cross sections and spin-correlation parameters should be calculated. A full and detailed analysis along these lines should provide a fair quantitative outline of all the primary features of moderately high-energy nonrelativistic elastic and inelastic N-d scattering.

---

*Work supported in part by the U. S. Atomic Energy Commission.

†Present address: Bellcomm, Incorporated, Washington, D.C. 20024.

[1]I. Duck, in *Advances in Nuclear Physics,* edited by M. Baranger and E. Vogt (Plenum Press, Inc., New York, 1968), Vol. I, p. 341.

[2]A. Mitra, in *Advances in Nuclear Physics,* edited by M. Baranger and E. Vogt (Plenum Press, Inc., New York, 1969), Vol. III, p. 1.

[3]L. M. Delves and A. C. Phillips, Rev. Mod. Phys. 41, 497 (1969).

[4]R. D. Amado, Ann. Rev. Nucl. Sci. 19, 61 (1969).

[5]*Three-Particle Scattering in Quantum Mechanics,* edited by J. Gillespie and J. Nuttall (W. A. Benjamin, Inc., New York, 1968).

[6]*The Three-Body Problem in Nucleon and Particle Physics,* edited by J. S. C. MacKee and P. M. Rolph (North-Holland Publishing Company, Amsterdam, The Netherlands, 1970). Comprehensive reviews of three-nucleon scattering experiments are contained in the contributions of J. D. Seagrave (p. 41), W. Haeberli (p. 188), and I. Šlaus (p. 337).

[7]K. M. Watson and J. Nuttall, *Topics in Several Particle Dynamics* (Holden-Day, Inc., San Francisco, California, 1967).

[8]H. P. Noyes, in *Proceedings of the International Conference on Polarized Targets and Ion Sources,* (La Documentation Francais, Paris, France, 1967), p. 309. See also H. P. Noyes, Ref. 6, p. 2.

[9]A preliminary report of this work has appeared; J. Krauss and K. L. Kowalski, Phys. Letters 31B, 263 (1970).

[10]H. Kottler and K. L. Kowalski, Phys. Rev. 138, B619 (1965).

[11]P. Benoist-Gueutal and F. Gomez-Gimeno, Phys. Letters 13, 68 (1964); J. Phys. Radium 26, 403 (1965).

[12]J. Hufner and A. de-Shalit, Phys. Letters 15, 52 (1965).

[13]L. M. Delves and P. Brown, Nucl. Phys. 11, 432 (1959); L. M. Delves, *ibid.* 33, 482 (1962).

[14]R. D. Purrington and J. L. Gammel, Phys. Rev. 168, 1174 (1968).

[15]Some simplifications have recently been achieved by I. H. Sloan, Nucl. Phys. A139, 337 (1969). See also M. Stingl and A. S. Rinat-Reiner, to be published.

[16]The three-body literature is rank with articles concerning variational techniques. See Refs. 1–7; particular attention should be devoted to the excellent review article by L. Spruch, Ref. 5, p. 1. For a recent application of this method to scattering see S. C. Pieper, L. Schlessinger, and J. Wright, Phys. Rev. D 1, 1674 (1970).

[17]A. J. Dragt, J. Math. Phys. 6, 533 (1965), and refer-

ences cited therein. See also the contribution of G. Erens, J. L. Visschers, and R. Van Wageningen, Ref. 6, p. 291.

[18]One of the most common reasons for employing these techniques has been the possibility of doing calculations with local two-nucleon interactions. Mention should also be made of another very interesting method for dealing with the Faddeev equations, in this case due to R. A. Malfliet and J. A. Tjon, Nucl. Phys. A127, 161 (1969); Phys. Letters 29B, 391 (1969); 30B, 293 (1963); and to be published.

[19]L. Rosenberg, Phys. Rev. 168, 1756 (1968); R. W. Finkel and L. Rosenberg, Phys. Rev. 168, 1841 (1968).

[20]I. H. Sloan, Phys. Rev. 165, 1587 (1968); Phys. Letters 25B, 84 (1967). Sloan has compared his approximation with exact calculations at 14.1, 50, and 100 MeV. See I. H. Sloan, Phys. Rev. 185, 1361 (1969). The reader should be cautioned that Sloan also suggests, in the first two papers, additional approximations connected with the evaluation of the impulse-graph matrix elements. These are never employed in the present work.

[21]W. Ebenhöh, A. S. Rinat-Reiner, and Y. Avishai, Phys. Letters 29B, 638 (1969); Ann. Phys. (N.Y.) 55, 341 (1969).

[22]A comparative analysis of the methods of Refs. 19 and 20 is contained in Ref. 23.

[23]K. L. Kowalski, Phys. Rev. 188, 2235 (1969).

[24]Precisely what constitutes a "realistic" two-nucleon potential is somewhat a matter of taste. However, there appears to be an almost universal consensus that such a potential should, besides representing the correct deuteron properties and phase shifts up to the inelastic threshold, also generate a scattering amplitude with the analyticity properties we have come to expect from more general characterizations of scattering. Particularly important is smoothness in angular momentum which correlates the scattering in different partial waves; this property is conspicuously absent in most of the conventional nonlocal choices including the one in this paper.

[25]J. C. Faivre, D. Garreta, J. Jungerman, A. Papineau, J. Sura, and A. Tarrats, Nucl. Phys. A127, 169 (1969).

[26]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967).

[27]We recall that the customary transition operators which satisfy the Faddeev (or equivalent) equations are interaction-picture operators and a channel indexing is thereby necessary. However, formal considerations of the consequences of unitarity are most easily expressed in the Heisenberg picture.

[28]The matrix elements of K_p are usually referred to as the reduced K matrix. Cf. R. H. Dalitz and S. F. Tuan, Ann. Phys. (N.Y.) 10, 307 (1960); R. H. Dalitz, Rev. Mod. Phys. 33, 471 (1962). It is important to keep in mind that K_p is not related to the solution of an integral equation with a principal-value singularity in the propagator part of its kernel; nor will it be associated with so-called standing waves as in the two-particle case.

[29]Equivalent expressions for P and Q obtain with "in" replaced by "out."

[30]We have suppressed all spin and isospin dependences in this section.

[31]C. Lovelace, Phys. Rev. 135, B1225 (1964).

[32]In addition to Ref. 10, more detailed expositions can be found in Refs. 33 and 34. Also, Ref. 14 contains explicit expressions for the cross section and polarizations (both spinor and tensor) in elastic N-d scattering in addition to an extensive lore concerning tensor forces in this reaction.

[33]H. Kottler, M.S. thesis, Case Institute of Technology, 1965 (unpublished).

[34]J. Krauss, Ph.D. thesis, Case Western Reserve University, 1970 (unpublished).

[35]We follow the conventions of Ref. 10 in writing these terms as operators in (ordinary) spin space.

[36]By this we mean that the ease of solution of (2.8), given (2.9), is independent of the form of the interaction. However, it is obvious that the computation of the input (2.9) will be more difficult starting from local potentials than for separables, for example. This additional difficulty does not appear to be computationally prohibative. These comments do not necessarily apply if one desires an improved approximation for $\bar{U}$.

[37]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954); Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. 95, 1635 (1954).

[38]See Ref. 31 and also D. Bollé and K. L. Kowalski, Nuovo Cimento 67A, 523 (1970).

[39]H. P. Noyes and H. Fiedeldey, Ref. 5, p. 195.

[40]H. P. Noyes, Ref. 6, p. 2.

[41]A. C. Phillips, Nucl. Phys. A107, 209 (1968).

[42]T. Brady, M. Fuda, E. Harms, J. S. Levinger, and R. Stagat, Phys. Rev. 186, 1069 (1969).

[43]K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. 126, 881 (1962).

[44]We employ the notation and phase conventions for the Clebsch-Gordan coefficients and spherical harmonics contained in, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Appendix A.

[45]The description of this circumstance in Ref. 9 is rather misleading.

[46]D. C. Kocher and T. B. Clegg, Nucl. Phys. A132, 455 (1969).

[47]S. Kikuchi, J. Sanda, S. Sowa, I. Hayashi, K. Nisimura, and K. Fukinaga, J. Phys. Soc. Japan 15, 9 (1960).

[48]J. D. Seagrave, Phys. Rev. 97, 757 (1955).

[49]D. O. Caldwell and J. R. Richardson, Phys. Rev. 98, 28 (1955).

[50]S. W. Bunker, J. M. Cameron, R. F. Carlson, J. R. Richardson, P. Tomãs, W. T. H. Van Oers, and J. Verba, Nucl. Phys. A113, 461 (1968).

[51]J. H. Williams and M. K. Brussel, Phys. Rev. 110, 136 (1958).

[52]H. E. Conzett, H. S. Goldberg, E. Shield, R. J. Slobodrian, and S. Yambe, Phys. Rev. Letters 11, 68 (1964).

[53]The spline fits of the N-d cross sections by J. D. Seagrave (Ref. 5) are very useful in estimating the heights of the forward and backward peaks in the cross sections. The statements made in the text concerning the accuracy of our predicted cross sections are made with this information in mind. Reference should also be made to Seagrave's article for those data which are not plotted here.

[54]See Refs. 1–6 and especially the discussions in Ref. 4. The model of Ref. 21 apparently suffers less from this defect than any other. Cf. R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. 140, B1291 (1965); and also A. C. Phillips, Phys. Rev. 142, 984 (1966).

[55]It is amusing to note that, in contrast to the forward-

peak behavior, the model of Ref. 21 suffers more from an over-suppression of the backward peak than any other. The exact calculation of Phillips (Ref. 54) also has too small of a backward peak; this has been attributed to the inclusion of too few (N-d) partial waves.

[56]The use of somewhat more realistic two-particle transition operators (Ref. 24) and higher N-d partial waves, possibly in the Born approximation, are two very evident improvements.

[57]The data for these are rather sparse. See Haeberli, Ref. 6.