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Low-Energy Theorem for Nucleon-Nucleon Bremsstrahlung*

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The low-energy theorem is derived for nucleon-nucleon bremsstrahlung. We assume that the two nucleons interact through a potential which can be any model, nonlocal as well as local. For those potentials which depend explicitly upon momentum and/or angular momentum operators, the gauge terms arising from these operators are included in the derivation. These gauge terms are always important in the study of the off-energy-shell effects of the two-nucleon interaction, and for the $np\gamma$ process they are essential in the derivation of the low-energy theorem. It is found that the gauge terms are canceled precisely by parts of the terms which represent the photons emitted by the internal nucleon lines.

I. INTRODUCTION

Recently, Heller[1-3] has derived the low-energy theorem for bremsstrahlung in a potential model. He has shown that when the bremsstrahlung amplitude $\vec{M}$ is expanded in powers of the photon momentum K,

$$\vec{M}=\vec{A}/K+\vec{B}+\vec{C}K+\ldots, \tag{1}$$

the coefficients $\vec{A}$ and $\vec{B}$ are independent of the off-energy-shell effects. Without introducing the concept of potential, the theorem was first derived by Low[4] using the formalism of quantum field theory.

In Heller's derivation the potential was assumed to be independent of momentum and angular momentum operators. If the potential depends explicitly upon these operators, as is the case for most realistic potentials, one in general must include the gauge terms arising from this dependence. Heller's method has general validity only for momentum-independent potentials, since it neglects the gauge terms. We have found,[3] however, that the low-energy theorem can be derived for the particular case of proton-proton bremsstrahlung ($pp\gamma$) without explicit inclusion of the gauge terms. This cannot be done for neutron-proton bremsstrahlung ($np\gamma$), because the gauge terms contribute to both $\vec{B}$ and $\vec{C}$ for $np\gamma$ while only to $\vec{C}$ for $pp\gamma$ in Eq. (1).

In most of the recent nucleon-nucleon bremsstrahlung calculations, realistic two-nucleon potentials[5] have been used, but the gauge terms arising from these potentials have been ignored completely. The off-energy-shell effects contributed by the gauge terms are therefore missing. Furthermore, current conservation is obviously violated in the $np\gamma$ calculations.

The purpose of this paper is to derive the low-energy theorem in the case of the general potential. The gauge terms arising from the momentum and/or angular momentum operators are included in the derivation.

II. DERIVATION OF THE LOW-ENERGY THEOREM

We consider the interaction of two particles where particle 1 is charged and particle 2 is uncharged. The masses for particle 1 and particle 2 are m_1 and m_2, respectively. Our method for deriving the low-energy theorem is quite different from the one used by Heller.[6] We use the following operator identity[3]:

$$t(E')G_0(E')[Q_1G_0(E)^{-1} - G_0(E')^{-1}Q_2]G_0(E)t(E) \equiv t(E')Q_2 - Q_1t(E) - [1+t(E')G_0(E')] \times (V_NQ_2 - Q_1V_N)[1+G_0(E)t(E)]. \quad (2)$$

Here Q_1 and Q_2 are arbitrary operators; t is the nucleon-nucleon transition operator; G_0 is the free-particle Green's function; V_N is the two-nucleon potential, and the quantities E and E' are, respectively, the energies of the initial and final nucleons. This method is very general; it is valid whether the particles have spin or not. Therefore, for the sake of simplicity, we shall derive the theorem by ignoring the spin. The appearance of the formulas when spin is included will be given at the end of this section.

The bremsstrahlung matrix T can be written in a standard way

$$T = T_E + T_I, \quad (3)$$

where T_E and T_I are the sum of those terms in which the photon is emitted from an external charged-particle line and an internal line, respectively. Denoting the gauge terms by T_G and the terms which represent the photons emitted from the internal nucleon lines by T_R, we further write

$$T_I = T_R + T_G. \quad (4)$$

The electromagnetic interaction Hamiltonian $V_{\text{e.m.}}$ consists of two terms:

$$V_{\text{e.m.}} = V^1_{\text{e.m.}} + V^2_{\text{e.m.}}. \quad (5)$$

Here $V^1_{\text{e.m.}}$ is the term arising from the kinetic energy operators, and $V^2_{\text{e.m.}}$ is the term arising out of the potential.[7] To the lowest order in the electromagnetic interaction, the matrices, T_E, T_R, and T_G can be written, in terms of t, G_0, $V^1_{\text{e.m.}}$, and $V^2_{\text{e.m.}}$, as

$$T_E = V^1_{\text{e.m.}}G_0(E)t(E) + t(E')G_0(E')V^1_{\text{e.m.}}, \quad (6a)$$

$$T_R = t(E')G_0(E')V^1_{\text{e.m.}}G_0(E)t(E), \quad (6b)$$

$$T_G = V^2_{\text{e.m.}} + V^2_{\text{e.m.}}G_0(E)t(E) + t(E')G_0(E')V^2_{\text{e.m.}} + t(E')G_0(E')V^2_{\text{e.m.}}G_0(E)t(E) = [1+t(E')G_0(E')]V^2_{\text{e.m.}}[1+G_0(E)t(E)]. \quad (6c)$$

Using the Coulomb gauge, we then define the matrix elements of T and T_X ($X = E, I, R, G$) by

$$\langle T\rangle \equiv \langle \vec{P}_1', \vec{P}_2', \vec{K}|T|\vec{P}_1, \vec{P}_2\rangle = N\hat{e}\cdot\vec{M}, \quad (7a)$$

$$\langle T_X\rangle \equiv \langle \vec{P}_1', \vec{P}_2', \vec{K}|T_X|\vec{P}_1, \vec{P}_2\rangle = N\hat{e}\cdot\vec{M}_X, \quad (7b)$$

where $N = -\epsilon/2\pi m_1\sqrt{K}$, ϵ is the charge, $\vec{K}$ is the photon momentum, $\hat{e}$ is the polarization vector of the photon, and $\vec{P}_i$ and $\vec{P}_i'$ represent the initial and final momenta of the ith nucleon. Corresponding to Eqs. (3) and (4), we have

$$\vec{M} = \vec{M}_E + \vec{M}_I, \quad (8a)$$

$$\vec{M}_I = \vec{M}_R + \vec{M}_G. \quad (8b)$$

To derive $\vec{M}_R$, we write the matrix elements of $V^1_{\text{e.m.}}$ as follows:

$$\langle \vec{K}|V^1_{\text{e.m.}}|0\rangle = N\vec{\mathbf{P}}_1\cdot\hat{e}\,e^{-i\vec{K}\cdot\vec{r}_1}. \quad (9)$$

Here, $\vec{\mathbf{P}}_1$ is the momentum operator of the particle 1, and we have neglected the magnetic-moment effects, since they contribute to $\vec{C}$ of Eq. (1). Inserting $Q_1 = Q_2 = Q_0 = e^{-i\vec{K}\cdot\vec{r}_1}$ into Eq. (2), we obtain

$$\langle \vec{P}_1', \vec{P}_2'|t(E')G_0(E')[Q_0G_0(E)^{-1} - G_0(E')^{-1}Q_0]G_0(E)t(E)|\vec{P}_1, \vec{P}_2\rangle \equiv \langle \vec{P}_1', \vec{P}_2'|t(E')Q_0 - Q_0t(E)|\vec{P}_1, \vec{P}_2\rangle - \langle \vec{P}_1', \vec{P}_2'|[1+t(E')G_0(E')](V_NQ_0 - Q_0V_N)[1+G_0(E)t(E)]|\vec{P}_1, \vec{P}_2\rangle. \quad (10)$$

Let us write

$$\langle \vec{P}_1', \vec{P}_2' | t(E')G_0(E')[Q_0 G_0(E)^{-1} - G_0(E')^{-1}Q_0]G_0(E)t(E)|\vec{P}_1, \vec{P}_2\rangle = \vec{K}\cdot\vec{M}_1,$$

$$\langle \vec{P}_1', \vec{P}_2' | t(E')Q_0 - Q_0 t(E) | \vec{P}_1, \vec{P}_2\rangle = \vec{K}\cdot\vec{M}_2,$$

$$\langle \vec{P}_1', \vec{P}_2' | [1 + t(E')G_0(E')](V_N Q_0 - Q_0 V_N)[1 + G_0(E)t(E)] | \vec{P}_1, \vec{P}_2\rangle = \vec{K}\cdot\vec{M}_0.$$

Here, we have used $K^2 = \vec{K}\cdot\vec{K}$, and $K = \vec{K}\cdot(\vec{K}/K)$ in defining $\vec{M}_i$ $(i = 0, 1, 2)$. Since we use the Coulomb gauge, $\hat{e}\cdot\vec{K} = 0$, the terms containing photon momentum $\vec{K}$ in expressions of the form $\hat{e}\cdot\vec{M}_i$ are zero. In terms of $\vec{M}_0$, $\vec{M}_1$, and $\vec{M}_2$, the identity (10) becomes

$$\vec{K}\cdot(\vec{M}_1 - \vec{M}_2 + \vec{M}_0) \equiv 0. \tag{11}$$

If we assume that $\vec{M}_i$ $(i = 0, 1, 2)$ are analytic at $\vec{K} = 0$, we obtain

$$\vec{M}_1 = \vec{M}_2 - \vec{M}_0 + O(K)$$

or

$$\hat{e}\cdot\vec{M}_1 = \hat{e}\cdot\vec{M}_2 - \hat{e}\cdot\vec{M}_0 + O(K). \tag{12}$$

It can be shown that

$$\hat{e}\cdot\vec{M}_1 = -(1/m_1)\hat{e}\cdot\vec{M}_R, \\ = -(1/m_1 N)\langle T_R\rangle. \tag{13}$$

If we parameterize the t matrix elements, as Heller did, in terms of the following scalar variables: the average of the initial and final kinetic energies in the c.m. system ν, the square of the momentum transfer u, and the amount that the initial (final) state is off the energy shell Δ_i (Δ_f), and then expand them in powers of K, we obtain

$$\hat{e}\cdot\vec{M}_2 = -\frac{1}{m_1}\left[\tfrac{1}{2}\hat{e}\cdot(\vec{q}_i + \vec{q}_f)\frac{\partial t}{\partial\nu} - \hat{e}\cdot\vec{P}_1'\frac{\partial t}{\partial\Delta_f} - \hat{e}\cdot\vec{P}_1\frac{\partial t}{\partial\Delta_i}\right] + O(K). \tag{14}$$

Here, $\vec{q}_i = \bar{m}[(\vec{P}_1/m_1) - (\vec{P}_2/m_2)]$, $\vec{q}_f = \bar{m}[(\vec{P}_1'/m_1) - (\vec{P}_2'/m_2)]$, $\bar{m} = m_1 m_2/(m_1 + m_2)$, and the derivatives of the t functions are evaluated at $\nu = (1/4\bar{m})(q_f^2 + q_i^2)$, $u = (\vec{P}_2' - \vec{P}_2)^2$, and $\Delta_i = \Delta_f = 0$. In the derivation of Eq. (14) we have used $\hat{e}\cdot\vec{K} = 0$. Substituting Eqs. (13) and (14) into Eq. (12), gives

$$\hat{e}\cdot\vec{M}_R = \tfrac{1}{2}\hat{e}\cdot(\vec{q}_i + \vec{q}_f)\frac{\partial t}{\partial\nu} - \hat{e}\cdot\vec{P}_1'\frac{\partial t}{\partial\Delta_f} - \hat{e}\cdot\vec{P}_1\frac{\partial t}{\partial\Delta_i} + m_1\hat{e}\cdot\vec{M}_0 + O(K). \tag{15}$$

Except for an additional term $m_1\hat{e}\cdot\vec{M}_0$ appearing in Eq. (15), this result agrees with Heller's. If the potential used does not depend upon either momentum or angular momentum, then $\vec{M}_0 = 0$. In this case Heller's result is reproduced. On the other hand, if it does, then $\vec{M}_0$ contributes to both $\vec{B}$ and $\vec{C}$ of Eq. (1). Therefore $\vec{M}_0$ cannot be neglected in the derivation of the low-energy theorem.

We next discuss the derivation of $\vec{M}_G$ and $\vec{M}_I$. The gauge terms $\vec{M}_G$ and the amplitude $\vec{M}_0$ are far too complicated to be calculated. However, they are not completely independent. As we have proved in the Appendix, they satisfy, independent of the potential model, the following relation:

$$\hat{e}\cdot\vec{M}_G + m_1\hat{e}\cdot\vec{M}_0 = 0. \tag{16}$$

Combining Eqs. (8b), (15), and (16), we obtain the internal scattering amplitude

$$\hat{e}\cdot\vec{M}_I = \tfrac{1}{2}\hat{e}\cdot(\vec{q}_i + \vec{q}_f)\frac{\partial t}{\partial\nu} - \hat{e}\cdot\vec{P}_1'\frac{\partial t}{\partial\Delta_f} - \hat{e}\cdot\vec{P}_1\frac{\partial t}{\partial\Delta_i} + O(K). \tag{17}$$

The external scattering amplitude is given by

$$\hat{e}\cdot\vec{M}_E = m_1\left[\frac{\hat{e}\cdot\vec{P}_1'}{P_{1\mu}'K_\mu}\langle\vec{P}_1' + \vec{K}, \vec{P}_2'|t(E)|\vec{P}_1, \vec{P}_2\rangle - \frac{\hat{e}\cdot\vec{P}_1}{P_{1\mu}K_\mu}\langle\vec{P}_1', \vec{P}_2'|t(E')|\vec{P}_1 - \vec{K}, \vec{P}_2\rangle\right]. \tag{18}$$

Here we have used

$$P_{1\mu}'K_\mu \approx m_1 K - \vec{K}\cdot\vec{P}_1' - \tfrac{1}{2}K^2,$$

$$P_{1\mu}K_\mu \approx m_1 K - \vec{K}\cdot\vec{P}_1 + \tfrac{1}{2}K^2.$$

If we perform a similar expansion to $\vec{M}_E$, then Eq. (18) reduces to

$$\hat{e}\cdot\vec{M}_E = m_1\left(\frac{\hat{e}\cdot\vec{P}_1'}{P_{1\mu}'K_\mu} - \frac{\hat{e}\cdot\vec{P}_1}{P_{1\mu}K_\mu}\right)t + \tfrac{1}{2}\left(\frac{\hat{e}\cdot\vec{P}_1'}{P_{1\mu}'K_\mu}\vec{K}\cdot\vec{q}_f + \frac{\hat{e}\cdot\vec{P}_1}{P_{1\mu}K_\mu}\vec{K}\cdot\vec{q}_i\right)\frac{\partial t}{\partial\nu} + \hat{e}\cdot\vec{P}_1'\frac{\partial t}{\partial\Delta_f} + \hat{e}\cdot\vec{P}_1\frac{\partial t}{\partial\Delta_i} + O(K). \tag{19}$$

Adding Eq. (19) to Eq. (17), we finally obtain

$$\hat{e}\cdot\vec{M}=\hat{e}\cdot\vec{M}_E+\hat{e}\cdot\vec{M}_I,$$

$$=m_1\left(\frac{\hat{e}\cdot\vec{P}_1'}{P_{1\mu}'K_\mu}-\frac{\hat{e}\cdot\vec{P}_1}{P_{1\mu}K_\mu}\right)t+\left[\frac{\hat{e}\cdot\vec{q}_i+\hat{e}\cdot\vec{q}_f}{2}+\frac{\hat{e}\cdot\vec{P}_1'}{P_{1\mu}'K_\mu}\frac{\vec{K}\cdot\vec{q}_f}{2}+\frac{\hat{e}\cdot\vec{P}_1}{P_{1\mu}K_\mu}\frac{\vec{K}\cdot\vec{q}_i}{2}\right]\frac{\partial t}{\partial\nu}+O(K). \tag{20}$$

We see that the total bremsstrahlung amplitude $\vec{M}$ does not contain the terms of the off-energy-shell derivatives of the t matrices. The coefficients $\vec{A}$ and $\vec{B}$ of Eq. (1) are completely determined by the elastic scattering t matrix and its derivatives with respect to energy. This is the low-energy theorem for bremsstrahlung produced in potential scattering, where the potential is allowed to contain momentum and/or angular momentum operators.

As we have already mentioned, Eq. (16) is valid whether the interacting particles have spin or not. Therefore, the derivation given above can be easily extended for two spin-$\frac{1}{2}$ particles. Many of the formulas when spin is included have already been published in Ref. 1. We refer to this paper for the details. However, since the method used in Ref. 1 is valid only for those potentials which are independent of the momentum operators, some important differences appear when potentials used in the derivation contain momentum operators. So, without going into the detailed derivation of the theorem for two spin-$\frac{1}{2}$ particles, we shall just discuss these differences.

When spin is included, the initial and final states can be written as $|\vec{P}_1, \vec{P}_2;\ Sm_z\rangle$ and $|\vec{P}_1', \vec{P}_2'; S'm_z'\rangle$, respectively. Here S is the total spin, and m_z is the sum of the spin components of the two particles along the z axis. The spin generalization of Eq. (15) is given by

$$\hat{e}\cdot\vec{M}_R=\sum_n\Big\{\langle S'm_z'|A_n|Sm_z\rangle\left[\frac{\hat{e}\cdot(\vec{q}_i+\vec{q}_f)}{2}\frac{\partial t_n}{\partial\nu}-\hat{e}\cdot\vec{P}_1'\frac{\partial t_n}{\partial\Delta_f}-\hat{e}\cdot\vec{P}_1\frac{\partial t_n}{\partial\Delta_i}\right]$$
$$+\bar{m}\langle S'm_z'|\hat{e}\cdot[\vec{Q}\times\vec{\nabla}_N A_n+Q^2\vec{\nabla}_p A_n-\vec{Q}(\vec{Q}\cdot\vec{\nabla}_p A_n)]|Sm_z\rangle t_n\Big\}+m_1\hat{e}\cdot\vec{M}_0+O(K). \tag{21}$$

In the derivation of Eq. (21) we have used $\hat{e}\cdot\vec{K}=0$. The operators A_n belong to the set of operators, $\{1, \vec{\sigma}_a\cdot\vec{L}_i, (\vec{\sigma}_1\cdot\vec{L}_i)(\vec{\sigma}_2\cdot\vec{L}_j)\}$, and A_n and their derivatives are evaluated at

$$\vec{L}_1=\vec{Q}=\vec{q}_f-\vec{q}_i,$$
$$\vec{L}_2=\vec{N}=\vec{q}_i\times\vec{q}_f,$$
$$\vec{L}_3=\vec{P}=\vec{Q}\times\vec{N}.$$

The functions t_n and their derivatives are evaluated at $\nu=(1/4\bar{m})(q_i^2+q_f^2)$, $u=Q^2$, $\Delta_i=0$, and $\Delta_f=0$. Again, the result given by Eq. (21) is different from Heller's by an additional term, namely $m_1\hat{e}\cdot\vec{M}_0$, which contributes to both $\vec{B}$ and $\vec{C}$ of Eq. (1) for those potentials depending upon momentum and/or angular momentum operators. Therefore, using $\hat{e}\cdot\vec{M}_R$ alone, the low-energy theorem cannot be derived. The internal scattering amplitude is given by

$$\hat{e}\cdot\vec{M}_I=\hat{e}\cdot\vec{M}_R+\hat{e}\cdot\vec{M}_G. \tag{22}$$

From Eq. (A13) of the Appendix, we have

$$\hat{e}\cdot\vec{M}_G+m_1\hat{e}\cdot\vec{M}_0=0. \tag{23}$$

Then, Eq. (22) becomes

$$\hat{e}\cdot\vec{M}_I=\sum_n\Big\{\langle S'm_z'|A_n|Sm_z\rangle\left[\frac{\hat{e}\cdot(\vec{q}_i+\vec{q}_f)}{2}\frac{\partial t_n}{\partial\nu}-\hat{e}\cdot\vec{P}_1'\frac{\partial t_n}{\partial\nu_f}-\hat{e}\cdot\vec{P}_1\frac{\partial t_n}{\partial\Delta_i}\right]$$
$$+\bar{m}\langle S'm_z'|\hat{e}\cdot[\vec{Q}\times\vec{\nabla}_N A_n+Q^2\vec{\nabla}_p A_n-\vec{Q}(\vec{Q}\cdot\vec{\nabla}_p A_n)]|Sm_z\rangle t_n\Big\}+O(K). \tag{24}$$

If we perform the similar expansion to $\hat{e}\cdot\vec{M}_E$, and add it to Eq. (24), then we obtain Eq. (33) of Ref. 1. In that equation the terms of the off-energy-shell derivatives of the t matrices have disappeared completely. The coefficients $\vec{A}$ and $\vec{B}$ of Eq. (1) are determined by the amplitude of the corresponding nonradiative process.

III. DISCUSSION AND CONCLUSION

The internal scattering amplitude derived in the last section is quite general; it can be applied to any bremsstrahlung process in the potential scattering. For example, the amplitude given by Eq. (24) can be used to calculate the $np\gamma$ cross section. For the $pp\gamma$ case, the internal scattering amplitude is given by

$$\hat{e}\cdot\vec{M}_I=\frac{\hat{e}\cdot(\vec{P}_1+\vec{P}_2)}{(P_1+P_2)_\mu K_\mu}[\langle\vec{P}_1',\vec{P}_2';S'm_z'|t(E')|\vec{P}_1-\vec{K},\vec{P}_2;Sm_z\rangle+\langle\vec{P}_1',\vec{P}_2';S'm_z'|t(E')|\vec{P}_1,\vec{P}_2-\vec{K};Sm_z\rangle$$
$$-\langle\vec{P}_1'+\vec{K},\vec{P}_2';S'm_z'|t(E)|\vec{P}_1,\vec{P}_2;Sm_z\rangle-\langle\vec{P}_1',\vec{P}_2'+\vec{K};S'm_z'|t(E)|\vec{P}_1,\vec{P}_2;Sm_z\rangle]+\hat{e}\cdot\vec{R}(K)\,, \qquad (25)$$

where $\vec{R}(K)$ contributes to $\vec{C}$ of Eq. (1). Since we use the gauge in which $e_0=0$ and $\hat{e}\cdot\vec{K}=0$, the internal scattering amplitude given by Eq. (17) or Eq. (25) can be extended to a four-vector form $e_\mu M_{I\mu}$. It can be shown that this extended amplitude, $M_{I\mu}$, does satisfy the gauge condition or an approximate gauge condition

$$(M_I+M_E)_\mu K_\mu=0 \quad\text{or}\quad O(K^2)\,. \qquad (26)$$

If we neglect the mass difference of the nucleons, $m_1=m_2=m$, then Eq. (17) can be rearranged in the form

$$\vec{M}_I=-\tfrac{1}{2}(\vec{P}_1+\vec{P}_2)\left(\frac{\partial t}{\partial\Delta_i}+\frac{\partial t}{\partial\Delta_f}\right)-\left[\vec{q}_i\left(\frac{\partial t}{\partial\Delta_i}-\frac{1}{2}\frac{\partial t}{\partial\nu}\right)+\vec{q}_f\left(\frac{\partial t}{\partial\Delta_f}-\frac{1}{2}\frac{\partial t}{\partial\nu}\right)\right]+O(K)\,. \qquad (27)$$

This equation can be used to derive Heller's conditions for S-wave scattering. As we know, for a a pure S wave $\vec{M}_I$ would contribute nothing in the c.m. system in which the total momentum vanishes, $\vec{P}_1+\vec{P}_2=0$. Since $\vec{q}_i$ and $\vec{q}_f$ are linearly independent, Eq. (27) gives us Heller's conditions for S-wave scattering

$$\frac{\partial t}{\partial\Delta_i}-\frac{1}{2}\frac{\partial t}{\partial\nu}=0\,,\qquad \frac{\partial t}{\partial\Delta_f}-\frac{1}{2}\frac{\partial t}{\partial\nu}=0\,. \qquad (28)$$

We have derived the low-energy theorem for bremsstrahlung with particular attention given to the nucleon-nucleon bremsstrahlung process. In our derivation, the potential is allowed to have possible momentum dependence or to have explicit nonlocality, and the gauge terms are taken into account. The theorem shows that all of the interesting information about the off-energy-shell effects is contained in the coefficients $\vec{C}$, $\vec{D}$, ... of Eq. (1). The gauge terms are found to be essential in the derivation of the low-energy theorem for $np\gamma$, but they are not essential for $pp\gamma$. However, since they are important in the study of the off-energy-shell effects of the two-nucleon interaction, they must be calculated exactly in either the $pp\gamma$ or the $np\gamma$ calculation. Both the gauge terms, and T_R (which represent the photons emitted by the internal nucleon lines) are very difficult to calculate.[8] Fortunately, they are not independent; the gauge terms are canceled precisely by parts of T_R.

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APPENDIX. CANCELLATION OF THE GAUGE TERMS BY PARTS OF T_R

We consider the scattering of two particles where particle 1 is charged with charge ϵ and particle 2 is uncharged. These two particles are assumed to interact through a potential V_N. The electromagnetic interaction arising out of the potential V_N will be denoted by $V^2_{\text{e.m.}}$. Let (m_i, P_i, S_i, m_{zi}) and $(m_i, P_i', S_i', m_{zi}')$ be the mass, momentum, spin, and the z component of the ith particle in the initial and final states, respectively. We denote the momentum of the emitted photon by $\vec{K}$, and its polarization vector by $\hat{e}$.

I. We first prove that if we define

$$[V_N, Q_0]\equiv V_N Q_0-Q_0V_N\,,\quad =\vec{K}\cdot\vec{F}_0\,,$$

and

$$\langle\vec{K}|V^2_{\text{e.m.}}|0\rangle=m_1 N\hat{e}\cdot\vec{G}\,,$$

where

$$Q_0=e^{-i\vec{K}\cdot\vec{r}_1}\,,\quad N=-\epsilon/2\pi m_1\sqrt{K}\,,$$

then, we have

$$\vec{G}+\vec{F}_0=0 \qquad (A1)$$

for any potential model.

Proof: This result is true for a potential with an arbitrary number of factors of the momentum operator. Let us consider a potential of the form

$$V_N(\vec{P}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2) = [\vec{F}_{n1}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)\cdot\vec{P}q_{n1}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)][\vec{F}_{n2}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)\cdot\vec{P}q_{n2}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)]\cdots$$
$$\times[\vec{F}_{nn}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)\cdot\vec{P}q_{nn}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)] = \prod_{i=1}^{n}[\vec{F}_{ni}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)\cdot\vec{P}q_{ni}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)]. \quad \text{(A2)}$$

Here $\vec{r}$ is the relative position vector, $\vec{P}=\frac{1}{2}(\vec{P}_1-\vec{P}_2)$ is the relative momentum operator, and $\vec{F}_{ni}$ and q_{ni} are functions of $\vec{r}$ and the Pauli spin operators $\vec{\sigma}_i$ of the nucleons. $V^2_{\text{e.m.}}$ is obtained by changing $\vec{P}_1$ everywhere in the potential given by Eq. (A2) to $\vec{P}_1-\epsilon\vec{A}(\vec{r}_1, t)$ and keeping only terms to first order in ϵ. We obtain

$$\langle\vec{K}|V^2_{\text{e.m.}}|0\rangle = m_1 N\hat{e}\cdot\vec{G}, \quad \text{(A3)}$$

where

$$\vec{G}=\frac{1}{2}\left\{(\vec{F}_{n1}Q_0q_{n1})\prod_{j=2}^{n}(\vec{F}_{nj}\cdot\vec{P}q_{nj})+\sum_{S=2}^{n-1}\left[\prod_{i=1}^{S-1}(\vec{F}_{ni}\cdot\vec{P}q_{ni})(\vec{F}_{ns}Q_0q_{ns})\prod_{j=S+1}^{n}(\vec{F}_{nj}\cdot\vec{P}q_{nj})\right]+\prod_{i=1}^{n-1}(\vec{F}_{ni}\cdot\vec{P}q_{ni})(\vec{F}_{nn}Q_0q_{nn})\right\}. \quad \text{(A4)}$$

Now, using the method of mathematical induction, we can show that if $[A_i, C]\equiv A_iC-CA_i$, then

$$\left[\prod_{i=1}^{n}A_i, C\right.=[A_1, C]\prod_{i=2}^{n}A_i+\sum_{S=2}^{n-1}\left\{\prod_{i=1}^{S-1}A_i[A_s, C]\prod_{j=S+1}^{n}A_j\right\}+\prod_{i=1}^{n-1}A_i[A_n, C]. \quad \text{(A5)}$$

Choosing A_i to be $(\vec{F}_{ni}\cdot\vec{P}q_{ni})$ and C to be Q_0, Eq. (A5) gives

$$\left[\prod_{i=1}^{n}(\vec{F}_{ni}\cdot\vec{P}q_{ni}), Q_0\right]=[(\vec{F}_{n1}\cdot\vec{P}\,q_{n1}), Q_0]\prod_{j=2}^{n}(\vec{F}_{nj}\cdot\vec{P}q_{nj})$$
$$+\sum_{S=2}^{n-1}\left\{\prod_{i=1}^{S-1}(\vec{F}_{ni}\cdot\vec{P}q_{ni})[(\vec{F}_{ns}\cdot\vec{P}q_{ns}), Q_0]\prod_{j=S+1}^{n}(\vec{F}_{nj}\cdot\vec{P}q_{nj})\right\}$$
$$+\sum_{i=1}^{n-1}(\vec{F}_{ni}\cdot\vec{P}q_{ni})[(\vec{F}_{nn}\cdot\vec{P}q_{nn}), Q_0]. \quad \text{(A6)}$$

Since

$$[(\vec{F}_{nm}\cdot\vec{P}q_{nm}), Q_0]=(\vec{F}_{nm}\cdot\vec{P})[q_{nm}, Q_0]+[(\vec{F}_{nm}\cdot\vec{P}), Q_0]q_{qnm},$$
$$=[\vec{F}_{nm}\cdot\vec{P}, Q_0]_{nm},$$
$$=-\frac{1}{2}(\vec{F}_{nm}\cdot\vec{K}Q_0)q_{nm},$$

we obtain

$$[V_n, Q_0]=\left[\prod_{i=1}^{n}(\vec{F}_{ni}\cdot\vec{P}q_{ni}), Q_0\right]=\vec{K}\cdot\vec{F}_0, \quad \text{(A7)}$$

where

$$\vec{F}_0=-\frac{1}{2}\left\{(\vec{F}_{n1}Q_0q_{n1})\prod_{j=2}^{n}(\vec{F}_{nj}\cdot\vec{P}q_{nj})+\sum_{S=2}^{n-1}\left[\prod_{i=1}^{S-1}(\vec{F}_{ni}\cdot\vec{P}q_{ni})(\vec{F}_{ns}Q_0q_{ns})\prod_{j=S+1}^{n}(\vec{F}_{nj}\cdot\vec{P}q_{nj})\right]+\prod_{i=1}^{n-1}(\vec{F}_{ni}\cdot\vec{P}q_{ni})(\vec{F}_{nn}Q_0q_{nn})\right\}. \quad \text{(A8)}$$

Combining Eq. (A4) and Eq. (A8) gives Eq. (A1).

If we define $\vec{F}_{nm}$ to be $\vec{F}'_{nm}\times\vec{r}$, then $\vec{F}_{nm}\cdot\vec{P}=\vec{F}'_{nm}\cdot\vec{L}$ since $\vec{L}=\vec{r}\times\vec{P}$. The potential given by Eq. (A2) becomes

$$V'_n(\vec{L}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)=\prod_{i=1}^{n}[\vec{F}'_{ni}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)\cdot\vec{L}q_{ni}(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)]. \quad \text{(A9)}$$

There are many other forms which can be constructed. For example, we can have a potential of the form

$$V''_n(\vec{P}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2)=\prod_{i=1}^{n}(d_{ni}\vec{P}\cdot f_{ni}\vec{P}h_{ni}), \quad \text{(A10)}$$

or

$$V_n'''(\vec{L}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2) = \prod_{i=1}^{n} \{d'_{ni}\vec{L}\cdot f'_{ni}\vec{L}h'_{ni}\}. \tag{A11}$$

Here the scalar functions d_{ni}, f_{ni}, h_{ni}, d'_{ni}, f'_{ni}, and h'_{ni} are functions of $\vec{r}$, $\vec{\sigma}_1$, and $\vec{\sigma}_2$. Following a similar procedure, Eq. (A1) can also be proved for the potentials given by Eqs. (A10) or (A11). Therefore, Eq. (A1) is true for a general potential of the form

$$V_N = V_0(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2) + \sum_{n=1}^{\infty} V_n(\vec{P}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2) + \sum_{n=1}^{\infty} V'_n(\vec{L}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2) + \sum_{n=1}^{\infty} V''_n(\vec{P}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2) + \sum_{n=1}^{\infty} V'''_n(\vec{L}, \vec{r}, \vec{\sigma}_1, \vec{\sigma}_2) + \text{similar combinations.} \tag{A12}$$

As special examples, we give the explicit expressions of $\vec{G}$ and $\vec{F}_0$ for the following particular potentials:
(i) The momentum-independent potential:

$$\vec{G} = \vec{F}_0 = 0.$$

(ii) The local potentials: We consider the potential of the form

$$V_N = V_c(r) + V_T(r)S_{12} + V_{LS}(r)\vec{S}\cdot\vec{L} + V_{LL}(r)L_{12}.$$

Here V_c, V_T, V_{LS}, and V_{LL} are central, tensor, spin-orbit and quadratic spin-orbit potentials, respectively, and L_{12} is defined by

$$L_{12} = [\delta_{LJ} + (\vec{\sigma}_1\cdot\vec{\sigma}_2)]L^2 - (\vec{L}\cdot\vec{S})^2.$$

The expressions are

$$\vec{G} = -\vec{F}_0 = \tfrac{1}{2}V_{LS}(r)(\vec{S}\times\vec{r})Q_0 + \tfrac{1}{2}V_{LL}(r)\{(\delta_{LJ} + \vec{\sigma}_1\cdot\vec{\sigma}_2)[(\vec{L}\times\vec{r})Q_0 - Q_0(\vec{r}\times\vec{L})] + (\vec{L}\cdot\vec{S})(\vec{r}\times\vec{S})Q_0 + (\vec{r}\times\vec{S})Q_0(\vec{L}\cdot\vec{S})\}.$$

(iii) The velocity-dependent potentials: We consider the potential of the form

$$V_N = V_1(r) + P^2V_2(r) + V_3(r)P^2.$$

The expressions are

$$\vec{G} = -\vec{F}_0 = \tfrac{1}{2}(\vec{P}Q_0 + Q_0\vec{P})V_2(r) + \tfrac{1}{2}V_3(r)(\vec{P}Q_0 + Q_0\vec{P}).$$

(iv) The nonlocal potentials: We consider the potential of the form

$$V_N = \sum_{n=0}^{\infty} V_n(r)(\vec{r}\cdot\vec{P})^n.$$

The expressions are

$$\vec{G} = -\vec{F}_0 = \tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{s=1}^{n} V_n(r)(\vec{r}\cdot\vec{P})^{n-s}(\vec{r}Q_0)(\vec{r}\cdot\vec{P})^{s-1}$$

II. Secondly we prove the following relation:

$$\vec{M}_G + m_1\vec{M}_0 = 0, \tag{A13}$$

where

$$\vec{M}_G = m_1\langle\vec{P}'_1, \vec{P}'_2; S'm'_z|[1 + t(E')G_0(E')]\vec{G}[1 + G_0(E)t(E)]|\vec{P}_1, \vec{P}_2; Sm_z\rangle,$$

$$\vec{M}_0 = \langle\vec{P}'_1, \vec{P}'_2; S'm'_z|[1 + t(E')G_0(E')]\vec{F}_0[1 + G_0(E)t(E)]|\vec{P}_1, \vec{P}_2; Sm_z\rangle. \tag{A14}$$

Here S (S') is the total spin of the initial (final) state, $m_z = m_{z1} + m_{z2}$, and $m'_z = m'_{z1} + m'_{z2}$. *Proof*: Combining Eq. (A1) and Eq. (A14), we obtain Eq. (A13).

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[6]We have modified Heller's method to obtain our result, and demonstrated that the extended Heller's method is equivalent to ours; M. K. Liou, to be published.

[7]Let us denote the momentum-dependent potential by

$$V_N(\vec{P}_1, \ldots).$$

By changing $\vec{P}_1$ everywhere in the potential to $\vec{P}_1 - \epsilon\vec{A}(\vec{r}_1, t)$, we define $V^2_{\text{e.m.}}$ by

$$V_N(\vec{P}_1 - \epsilon\vec{A}, \ldots) = V_N(\vec{P}_1, \ldots) + V^2_{\text{e.m.}} + O(\epsilon^2).$$

The terms represented by $O(\epsilon^2)$ are terms of second order in the electromagnetic coupling constant.

[8]T_R has been calculated numerically by V. R. Brown [Phys. Rev. 177, 1498 (1969)], M. I. Sobel [Phys. Rev. 152, 1385 (1966)], and W. A. Pearce *et al.* [Nucl. Phys. B3, 241 (1967)].