VI. CONCLUSIONS

The magnetic dipole moment of ^{12}B has been recalculated using new measurements of nuclear spin-lattice relaxation times of ^{12}B implanted in Au, Pd, and Pt. The large number of relaxation mechanisms and the short half-life of ^{12}B make it difficult to measure relaxation times greater than a few hundred msec. Nevertheless, such measurements can be used to make meaningful calculations of the magnetic effects of the stopping foils.

Clearly, the stopping foil must be made of some fcc metal in order to make accurate resonant-depolarization measurements of the effective moment of ^{12}B. The relaxation-time measurements show that Pt is probably the most desirable metal to use because the Knight shift for ^{12}B in Pt is relatively small.

The electric quadrupole moment of ^{12}B has been measured directly for the first time using the asymmetry of the resonance line of ^{12}B in Be foil under magnetic field reversal, and an estimate of the field gradient at the ^{9}Be lattice position.

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Pion-Deuteron Elastic Scattering at Intermediate Energies*

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Pion-deuteron elastic scattering is calculated via the impulse approximation (including double scattering) at intermediate energies. Good agreement is found with experiment at 87-, 142-, and 180-MeV/c^2 incident pion energy. Thus the impulse approximation seems to give reliable results even in the region of a large resonance.

I. INTRODUCTION

The elastic scattering of pions by deuterons is for several reasons a good test of our ability to understand composite systems of strongly interacting particles, using a bound-state model and a generalized impulse approximation. First, the deuteron is well described as a bound state of a neutron and a proton with a wave function whose properties are known. Second, the weak binding of the deuteron and the large distance between its components encourages the belief that the scattering amplitude can be well approximated by a simple expression involving amplitudes for pions scat-

tering on free nucleons. Such an expression will be some version of the impulse approximation,[1] and fortunately the requisite accurate information on the pion-nucleon scattering amplitude is available. Third, a theoretical calculation can immediately be compared with nature, as there are a number of differential cross-section measurements, including a recent experiment[2] with an incident pion lab energy of 180 MeV/c^2.

If the comparison shows that pion-deuteron scattering can be calculated well, then our knowledge of the deuteron wave function is corroborated and we can be more confident of other calculations of scattering from deuterons. In particular, we can be more sanguine in situations where information about scattering on neutrons is obtained by calculation from experiments on scattering by deuterons.

The calculation presented here is based on the impulse approximation. In the simple version of the impulse approximation, the pion-deuteron scattering amplitude is expressed as a superposition of scattering amplitudes for pions on a set of free neutrons and protons which have the same momentum distribution that they would have inside the deuteron. This leads to formulas similar to

$$A_d = \int d^3p\, A_p(\vec{p})\phi_F^*(\vec{p})\phi_I(\vec{p}) + \text{term for neutron},$$

where A_d, A_p, and A_n are scattering amplitudes for pions on deuterons, protons, and neutrons, respectively, and ϕ_F and ϕ_I are deuteron wave functions. The proton and neutron scattering amplitudes are customarily brought out of the integral on the grounds that they are slowly varying, and evaluated at some average value of p. However, in a bound system, in addition to the possibility that the pion will undergo just one scattering (which leads to the simple impulse approximation), there may be significant effects due to multiple scattering and the binding potential. These corrections are considered here, following a generalization of the impulse approximation by Chew and Goldberger.[3]

The preceding considerations are described in detail in Sec. II, and in Sec. III the calculation is compared with the experimental data.

II. IMPULSE APPROXIMATION AND FIRST-ORDER CORRECTIONS

Our starting point is the generalized impulse approximation of Chew and Goldberger.[3] We consider a scatterer which is made up of several constituents bound by a potential U, and an incident particle which interacts with the kth particle of the scatterer through a potential V_k. The total Hamiltonian is

$$H = K + U + V, \tag{1}$$

where K is the kinetic energy operator and

$$V = \sum_k V_k . \tag{2}$$

For convenience, we define operators $\omega_k^{(\pm)}$ such that $\omega_k^{(+)}$, operating to the right on an eigenstate of K with energy E, is given by

$$\omega_k^{(+)} = 1 + (E - K - V_k + i\epsilon)^{-1} V_k , \tag{3a}$$

and $\omega_k^{(-)}$ is a similar operator acting to the left,

$$\omega_k^{(-)} = 1 + V_k (E - K - V_k + i\epsilon)^{-1} . \tag{3b}$$

The scattering operators for two-body scattering are given by

$$t_k^{(+)} = V_k \omega_k^{(+)} , \tag{4a}$$

$$t_k^{(-)} = \omega_k^{(-)} V_k , \tag{4b}$$

and the scattering amplitudes are given by matrix elements of the $t_k^{(\pm)}$ between initial and final states consisting of free particles. (We might note that if the initial and final states have the same energies, then there is no difference between $t_k^{(+)}$ and $t_k^{(-)}$.)

The impulse approximation relates the scattering operator for scattering on the entire bound system to the two-body operators. The total T matrix is

$$T^{(+)} = \sum_k t_k^{(+)} + \sum_{k'} \sum_{k' \neq k} t_{k'}^{(-)} (\omega_k^{(+)} - 1) + \sum_{k',k} \sum (\omega_{k'}^{(-)} - 1)[U, \omega_k^{(+)}] , \tag{5}$$

where this operator is to be evaluated between eigenstates of the unperturbed Hamiltonian for the entire composite system, which is

$$H_0 = K + U . \tag{6}$$

The first term here is just single scattering and gives the simple impulse approximation; the second term gives the multiple scattering corrections; and the last term shows the effects of the binding potential on the individual two-body scatterings.

The necessary matrix elements of the two-body scattering operators $t_k^{(\pm)}$ are known from available analyses of pion-nucleon scattering data, and as a practical matter we note that

$$\omega_k^{(+)} - 1 = (E - K + i\epsilon)^{-1} t_k^{(+)} \tag{7}$$

which can be evaluated easily. Thus the single- and double-scattering terms in the expression for $T^{(+)}$ are written in terms of known quantities. We will argue that the binding corrections are small.

The results of analyses of pion-nucleon scattering are a set of phase shifts[4] which give the scattering amplitude in the c.m. system. The necessary relations between these phase shifts and the matrix elements of t_k are given here for the definite example of the reaction $\pi^+ + p \to \pi^+ + p$. Let us

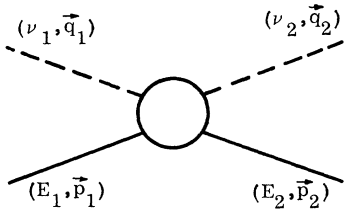


FIG. 1. Pion-nucleon scattering kinematics.

say we are scattering from an initial pion-proton state χ_a to a final state χ_b, with the spin projection of the initial proton i, and of the final proton j. The kinematics are shown in Fig. 1. A matrix element of t_k contains a δ function of three-momentum which will be taken out so that it does not appear explicitly in subsequent formulas.

$$\langle\chi_b|t_k|\chi_a\rangle=(2\pi)^3\delta^3(\vec{p}_1+\vec{q}_1-\vec{p}_2-\vec{q}_2)F_{ji}(p_2,q_2;p_1q_1)\,. \tag{8}$$

In the c.m. system, there is a conventionally defined scattering amplitude matrix, f, related to F by

$$(\nu_1\nu_2E_1E_2)^{1/2}\,F_{ji}=-2\pi W f_{ji}\,, \tag{9}$$

where W is the total c.m. energy, and the quantity on the left is a Lorentz scalar. If $|i\rangle$ and $|j\rangle$ represent Pauli spinors for the proton spin states, we have[5]

$$f_{ji}=\langle j|f|i\rangle\,, \tag{10}$$

with

$$f=f_1+\vec{\sigma}\cdot\hat{q}_1\vec{\sigma}\cdot\hat{q}_2f_2\,, \tag{11a}$$

$$f_1=\sum_{l=0}^{\infty}a_{l_+}(W)P'_{l+1}(\cos\theta)+\sum_{l=2}^{\infty}a_{l_-}(W)P'_{l-1}(\cos\theta)\,, \tag{11b}$$

$$f_2=\sum_{l=1}^{a}[a_{l_-}(W)-a_{l_+}(W)]P'_l(\cos\theta)\,, \tag{11c}$$

and finally

$$a_{l\pm}=(1/q)e^{i\delta_{l\pm}}\sin\delta_{l\pm}\,. \tag{12}$$

$\vec{q}_1$ and $\vec{q}_2$ are the pion momenta in the c.m., q their magnitude, and θ the angle between them.

We must also establish the notation to be used for the deuteron wave function. The deuteron has total angular momentum 1, spin 1, and the orbital state is a combination of S wave and D wave. Its wave function in coordinate space is

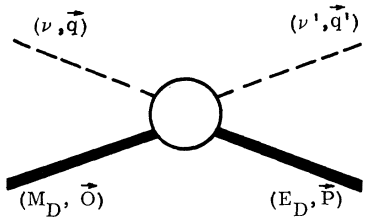


FIG. 3. Pion-deuteron kinematics in lab system.

$$\phi^{(m)}(\vec{x})=\frac{N}{r}[u(r)Y_{00}(\theta,\phi)\chi^m+w(r)\times\sum_n Y_{2,m-n}(\theta,\phi)\chi^nC_{m-n,n}]\,, \tag{13}$$

where χ^m is a combination of Pauli spinors in a total spin-1 state, with a magnetic quantum number m; $C_{m-n,n}$ is a Clebsch-Gordan coefficient,

$$C_{m-n,n}=(2,m-n;\ 1,n|\,2,\ 1;\ 1,m) \tag{14}$$

in Edmonds's[6] notation; the normalization constant N is determined by the condition

$$\int dr[u^2(r)+w^2(r)]=1\,. \tag{15}$$

The forms of the S- and D-state radial wave functions $u(r)$ and $w(r)$ are taken from the work[7] of Moravcsik, and of Lomon and Feshbach.

A. Single-Scattering Terms

Keeping only the single-scattering terms in the formula for $T^{(+)}$ gives what is often called the "simple impulse approximation." We have

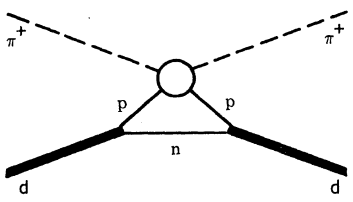
$$T^{(+)}=\sum t_k^{(+)} \tag{16}$$

and this is to be evaluated between initial and final pion-deuteron states. The pion may scatter from either the proton or neutron inside the deuteron, and these processes are diagrammed in Fig. 2.

The kinematics for scattering from a deuteron in the lab system are shown in Fig. 3, and the initial and final deuterons have spin projections m and m', respectively. The matrix element of $T^{(+)}$ will contain a δ function of three-momentum, which will again be written explicitly so that it will not appear in subsequent formulas,

$$\langle\phi_b|T^{(+)}|\phi_a\rangle=(2\pi)^3\delta^{(3)}(\vec{P}+\vec{q}'-\vec{q})A_{m'm}(\vec{q},\vec{q}')\,. \tag{17}$$

Now it can be seen that



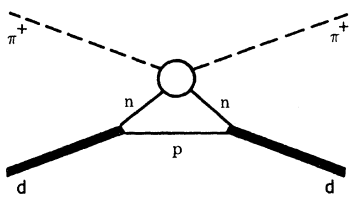


FIG. 2. Pion-deuteron single-scattering processes.

$$A_{m'm}(\vec{q},\vec{q}') = \int \frac{d^3p}{(2\pi)^3}\, \phi^{m'}_{i'j'}{}^*(\tfrac{1}{2}\vec{P}+\vec{p})\phi^{m}_{ij}(\vec{p}) \times[\delta_{j'j}F_{i'i}(\vec{P}+\vec{p},\vec{q}';\vec{p},\vec{q}) + \delta_{i'i}G_{j'j}(\vec{P}+\vec{p},\vec{q}';\vec{p},\vec{q})] \,. \quad (18)$$

G is defined in the same way as F but for the process $\pi^+ + n \to \pi^+ + n$.

If F and G are slowly varying as functions of energy, we can remove them from the integral, evaluating them at some average value of $\vec{p}=\vec{p}_0$. We can find some optimal value of $\vec{p}_0$ (following Pendelton[8]) by expanding F and G in a Taylor series,

$$F(\vec{P}+\vec{p},\vec{q}';\vec{p},\vec{q}) = F(\vec{P}+\vec{p}_0,\vec{q}';\vec{p}_0,\vec{q}) + \vec{\nabla}_p F(\vec{P}+\vec{p}_0,\vec{q}';\vec{p}_0,\vec{q})(\vec{p}-\vec{p}_0)+\cdots . \quad (19)$$

If we are not right at a resonance energy, we can assume the term linear in $\vec{p}$ will not have a zero coefficient, so that we can choose $\vec{p}_0$ by requiring that the linear term not contribute to the integral. Thus,

$$\vec{p}_0 = -\tfrac{1}{4}\vec{P} \,, \quad (20)$$

and

$$A_{m'm}(\vec{q}',\vec{q}) = [\delta_{j'j}F_{i'i}(\tfrac{3}{4}\vec{P},\vec{q}';-\tfrac{1}{4}\vec{P},\vec{q}) + \delta_{i'i}G_{j'j}(\tfrac{3}{4}\vec{P},\vec{q}';-\tfrac{1}{4}\vec{P},\vec{q})] \times \int \frac{d^3p}{(2\pi)^3}\, \phi^{m'}_{i'j'}{}^*(\tfrac{1}{2}\vec{P}+\vec{p})\phi^{m}_{ij}(\vec{p}) \,. \quad (21)$$

The integral that remains is a "form factor" that measures the probability that the deuteron will stay together when given a momentum transfer $\vec{P}$. Writing the wave function as a combination of S and D waves leads to four integrals. The first integral is what the form factor would be if there were no D-state part (and can be integrated analytically for a Hulthén-type wave function),

$$E = \frac{1}{4\pi}\int \frac{d^3p}{(2\pi)^3}\, u^*(\tfrac{1}{2}\vec{P}+\vec{p})u(\vec{p}) \,,$$
$$= \frac{1}{4\pi}\int d^3x\, e^{i\frac{1}{2}\vec{P}\cdot\vec{x}}\left|\frac{u(r)}{r}\right|^2 ,$$
$$= \int dr\, j_0(\tfrac{1}{2}Pr)u^2(r) \,. \quad (22a)$$

The other integrals are

$$D = \int dr\, j_2(\tfrac{1}{2}Pr)u(r)w(r) \,, \quad (22b)$$

$$C = \int dr\, j_0(\tfrac{1}{2}Pr)w^2(r) \,, \quad (22c)$$

$$B = \int dr\, j_2(\tfrac{1}{2}Pr)w^2(r) \,. \quad (22d)$$

Finally, let us write down the differential cross section in the lab for a calculation where only single scattering is included.

$$\frac{d\sigma}{d\Omega} = R \times \tfrac{1}{3}\sum_{m,m'} |A_{m'm}|^2 = R(|F_{++}+G_{++}|^2\mathcal{F}_1{}^2 + \tfrac{2}{3}|F_{-+}+G_{-+}|^2\mathcal{F}_2{}^2) , \quad (23)$$

where

$$\mathcal{F}_1 = \left[(E+C)^2 + 4\left(D-\frac{1}{2\sqrt{2}}B\right)^2\right]^{1/2} , \quad (24a)$$

$$\mathcal{F}_2 = E + \frac{1}{\sqrt{2}}D - \tfrac{1}{2}C + \tfrac{1}{2}B \,, \quad (24b)$$

and

$$R = (2\pi)^{-2}\nu\nu'\frac{E'}{W}\frac{q'}{q}\left(1-\frac{\nu' q}{Wq'}\cos\theta\right)^{-1} . \quad (25)$$

B. Double-Scattering Terms

The double-scattering contributions to the T matrix are given by

$$\sum_{k'\neq k}\sum_{k} t_{k'}^{(-)}(E-K+i\epsilon)^{-1}t_k^{(+)} \,. \quad (26)$$

Three processes may contribute, and these are diagrammed in Fig. 4. For simplicity in this paper, formulas will be written down only for the reaction in Fig. 4(a), which consists of a π^+ scattering

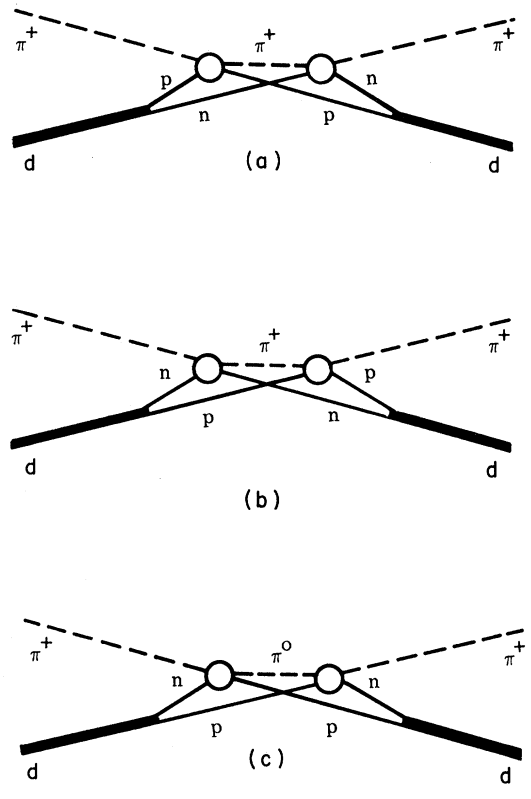


FIG. 4. Pion-deuteron double-scattering processes.

elastically first off the proton and then off the neutron.

Some approximations will be made. Terms in the double scattering of order m_π/m_n will be neglected, and only the δ-function part of the propagator will be kept, so that

$$(E-K+i\epsilon)^{-1} \to -i\pi\delta(E-K). \tag{27}$$

The deuteron will be taken to be in an S state only, and again the energy dependence of the pion-nucleon scattering amplitude will be neglected. This time, however, we must integrate over an angular variable for the intermediate pion, so that it is necessary to make the expansion

$$F_{\alpha i}(\tfrac{1}{2}\vec{P}-\vec{p}',\vec{q}'';-\vec{p},\vec{q}) = \sum f_{\alpha i}^{l_1 m_1} Y_{l_1 m_1}(\Omega_1), \tag{28a}$$

$$G_{\beta j}(\tfrac{1}{2}\vec{P}+\vec{p}',\vec{q}';\vec{p},\vec{q}'') = \sum g_{\beta j}^{l_2 m_2} Y_{l_2 m_2}(\Omega_2), \tag{28b}$$

where Ω_1 and Ω_2 are the angles between $(\vec{q}'',\vec{q})$ and $(\vec{q}',\vec{q}'')$, respectively.

With these approximations, the contribution to πd elastic scattering can be calculated straightforwardly. Let the energy and momentum of the intermediate pion be ν'' and q''. We have

$$A^{(2)}_{m'm} = -i\pi\int \frac{d^3q''}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\chi^{m'}_{\alpha\beta} F_{\alpha i}(\tfrac{1}{2}\vec{P}-\vec{p}',\vec{q}'';-\vec{p},\vec{q}) \times\delta(\nu-\nu'')G_{\beta j}(\tfrac{1}{2}\vec{P}+\vec{p}',\vec{q}';\vec{p},\vec{q}'')\chi^{m}_{ij}\phi(p')\phi(p); \tag{29}$$

this is reduced to a longer but actually more manageable form in the Appendix.

Some comment on binding corrections is also in order. Using the techniques of Bander,[9] one can show that the binding corrections are of order m_π/m_n compared with double scattering corrections, so they may be consistently neglected.

III. RESULTS AND COMMENTS

Experimental data on the differential cross section for pion-deuteron elastic scattering are available for five values within the range that could be called the "intermediate"-energy region. The kinetic energies of the incident pions in these five experiments are 61,[10] 87,[11] 142,[12] 180,[2] and 300[13] MeV/c^2. At low energies it is difficult to determine if the deuteron has remained intact after scattering the pion, so that data on purely elastic scattering are not available below 61 MeV/c^2. At high energies, one cannot treat the nucleus nonrelativistically, so that the calculation presented here is not applicable above 300 MeV/c^2. This is why we have restricted our attention to the experimental energies listed above.

Perhaps the most interesting data are those taken at 180 MeV/c^2. This is only 10 or 15 MeV/c^2 lower than the energy required to excite the 33 resonance at its peak. Since the individual scattering amplitudes are large near a resonance, one expects that the contribution of the double-scattering terms will be largest here. Also, it has been suggested that the approximations that are made in the impulse approximation are, for various reasons, not valid near a resonance region.

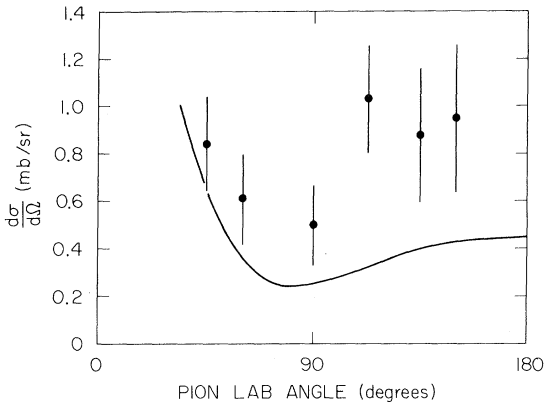


FIG. 5. π-d elastic scattering at 61 MeV/c^2. The solid line shows our calculation; the experimental data are from Ref. 10.

The results of this calculation and the experimental data are shown graphically in Figs. 5–9. To within distances that can be judged in a graph, there is no difference between using the Moravcsik or Lomon-Feshbach wave function. The agreement with the experimental data is not good at 61 and 300 MeV/c^2; on the other hand, the curves for 87 and 142 MeV/c^2 are quite acceptable. The fit at 180 MeV/c^2 is too high in the backward direction.

The agreement at 180 MeV/c^2 is, however, good in the forward direction, indicating that the impulse approximation is valid in this energy region. Tables I, II, and III show the effect of the various corrections at some angles at the energies 87, 142, and 180 MeV/c^2, with the calculations done using

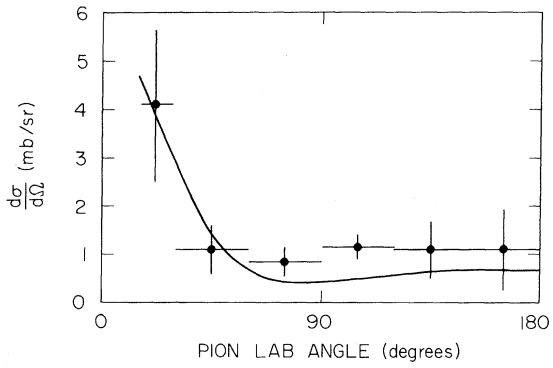


FIG. 6. π-d elastic scattering at 87 MeV/c^2. The solid line shows our calculations; the experimental data are from Ref. 11.

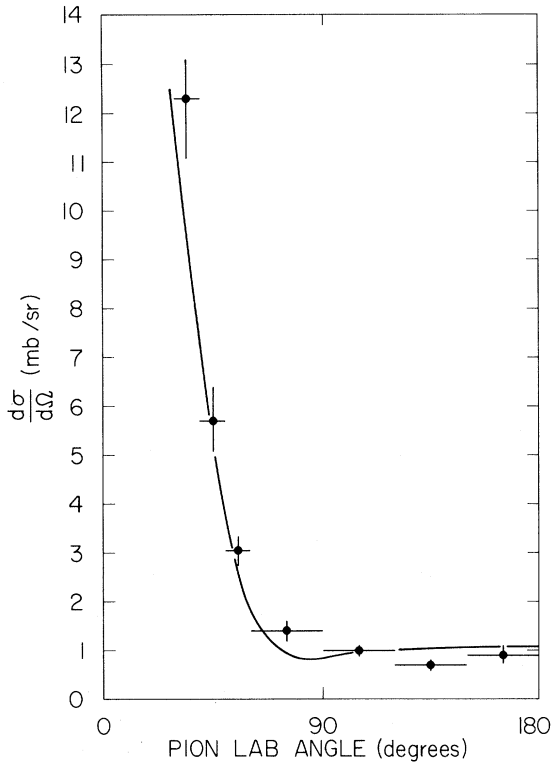


FIG. 7. π-d elastic scattering at 142 MeV/c^2. The solid line shows our calculation; the experimental data are from Ref. 12.

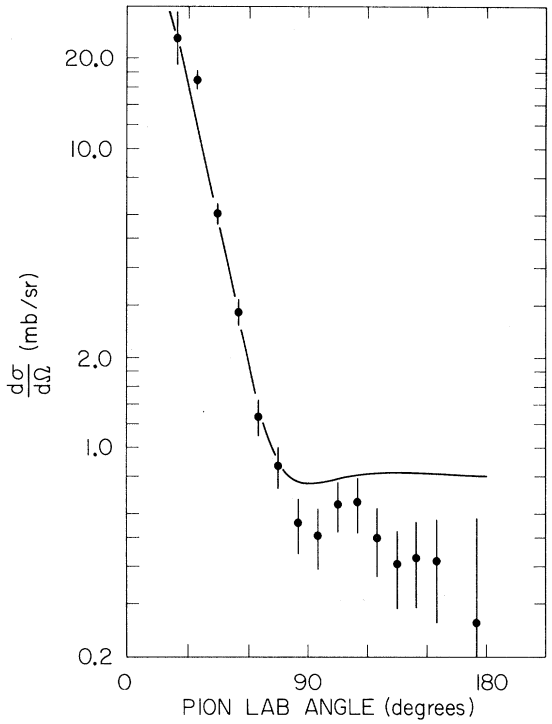


FIG. 8. π-d elastic scattering at 180 MeV/c^2. The solid line shows our calculation; the experimental data are from Ref. 2.

the Lomon-Feshbach wave function. Listed in the tables are the results of calculating pion-deuteron elastic scattering first with a pure S-state deuteron and no double scattering; then including the D state but not double scattering; then including double scattering but not the D state; and finally both the D state of the deuteron wave function and double scattering are included. Some experimental results are listed for comparison.

It is seen that the effect of the D-state part of the wave function is to increase the predicted cross section in the backward direction. This occurs because of the D-state radial wave function; the pure S-state form factor falls off rapidly with momentum transfer, while the form factors involving the D state fall less rapidly.

The double-scattering terms interfere with the single-scattering terms and also have their greatest effect at large angles. This is because of the rapid decrease of the form factors with momentum transfer. The form factors measure the probability that the deuteron will stay together when one of its constituents is given a certain momentum transfer. Attempting to deflect a pion through a large angle with just one scattering will probably knock the deuteron apart, and it becomes relatively easier

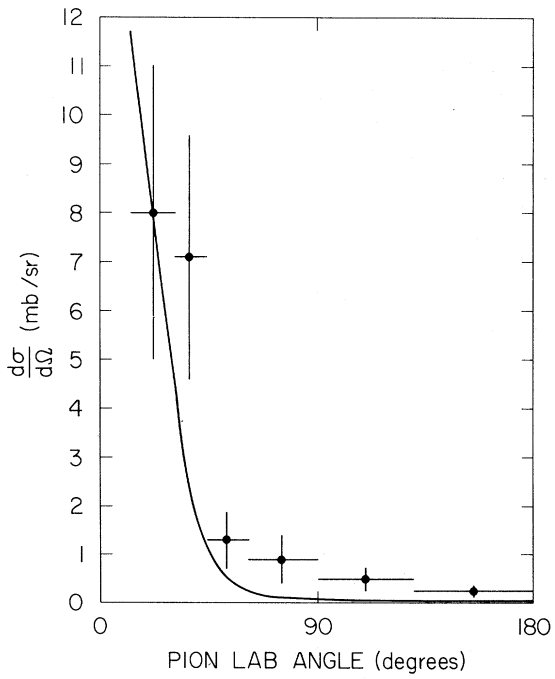


FIG. 9. π-d elastic scattering at 300 MeV/c^2. The solid line shows calculation; the experimental data are from Ref. 13.

TABLE I. Pion-deuteron elastic scattering cross sections at 87 MeV/c^2 (mb/sr).

	45°	75°	105°	135°	175°
Pure S state, single scattering	1.63	0.52	0.50	0.58	0.61
Including D state, single scattering	1.56	0.47	0.50	0.62	0.66
Pure S state with double scattering	1.58	0.53	0.53	0.61	0.63
Including D state with double scattering	1.51	0.47	0.53	0.64	0.69
Experiment (Ref. 11)	1.1 ± 0.5	0.85 ± 0.30	1.15 ± 0.25	1.1 ± 0.6	1.1 ± 0.85

to deflect the pion by scattering it twice through smaller angles.

The fact that there is not a large difference between the results with the Moravcsik wave function and the Lomon-Feshbach wave function should not be surprising, in spite of the fact that the former has 7.0% D state and the latter has 5.2% D state. The D state gives significant contributions in the backward directions, but it also happens that the S-wave form factor from the Moravcsik wave functions falls more quickly with momentum transfer, and the decreased S-wave contribution compensates for the increased D-wave contribution. The reason for this is that both wave functions were calculated from potentials, which were in turn fitted to the nucleon-nucleon scattering data. If the scattering data are the same, it could be expected that the two wave functions predict the same observable consequences, even if their internal details differ.

The results here suggest that triple scattering, temporary binding of pion and nucleon to produce an N^*, removing the pion-nucleon amplitude from the integral, and various other things which have been suggested as reasons for the invalidity of the impulse approximation at certain energies do not in fact radically affect the result.[14] This is reasonable. The width of the 33 resonance is about 120 MeV, while the width of the pion wave function, which determines the "width" of the integrals, is about 60 MeV. Thus it seems all right to remove the amplitude from the integrals. The impulse approximation assumes that the interaction takes place in a short time compared with other time scales within the deuteron, so that if the pion and nucleon bind together in an N^* for an appreciable period the approximation is wrong. However, with the average size of the deuteron being 4.3 F and with the pions at our energies, any N^* that is produced can travel only a small fraction of the distance between the nucleus before it decays. Finally, the triple scattering is probably small.[15]

Earlier calculations[16] of pion-deuteron elastic scattering near 140 MeV/c^2 gave a result that was much too large in the backward direction. The main difference between the present calculation and earlier calculations is not in the formal input, but in the wave function used. We have used here Moravcsik's[7] best analytic approximation to the Gartenhaus[17] wave function and also the Lomon-Feshbach wave function with 5.2% D state. These wave functions were calculated by solving the deuteron bound-state problem with a potential which was inferred from an analysis of nucleon-nucleon scattering data. The Moravcsik, Lomon-Feshbach, and other wave functions obtained by the same methods[18] fall off more quickly in momentum space than commonly available Hulthén or Hulthén with hard-core wave functions. Thus the form factors or "overlap functions" decrease more quickly

TABLE II. Pion-deuteron elastic scattering cross sections at 142 MeV/c^2 (mb/sr).

	45°	75°	105°	135°	175°
Pure S state, single scattering	5.43	0.95	0.80	0.86	0.85
Including D state, single scattering	5.34	0.90	0.86	0.99	1.02
Pure S state with double scattering	5.35	1.09	0.94	0.96	0.94
Including D state with double scattering	5.26	1.04	1.00	1.10	1.10
Experiment (Ref. 12)	5.7 ± 0.6	1.4 ± 0.2	1.0 ± 0.1	0.7 ± 0.1	0.9 ± 0.15

TABLE III. Pion-deuteron elastic scattering cross sections at 180 MeV/c^2 (mb/sr).

	45°	75°	105°	135°	175°
Pure S state, single scattering	6.31	0.76	0.58	0.59	0.56
Including D state, single scattering	6.33	0.75	0.66	0.73	0.73
Pure S state with double scattering	6.28	0.99	0.76	0.70	0.64
Including D state with double scattering	6.28	0.98	0.84	0.85	0.82
Experiment (Ref. 2)	6.07 ± 0.48	0.87 ± 0.14	0.65 ± 0.13	0.41 ± 0.12	0.25 ± 0.36

with increasing momentum transfer than if they were calculated with a Hulthén wave function, and thus the calculated cross section is more strongly suppressed in the backward direction.

Our knowledge of the deuteron wave function, though improved, is still a hindrance to calculating scattering at high momentum transfers. The momentum transfer involved in backward scattering at 140 or 180 MeV/c^2 are already high enough to put us in an area of marginal certainty for the wave function. Unfortunately, the pion-deuteron scattering experiment probably cannot be used as a probe of the wave function, because at higher momentum transfers multiple scattering becomes increasingly important, for reasons already mentioned, and the calculation becomes muddled.

APPENDIX

Double-Scattering Terms

Making the approximations stated in the text, the amplitude is

$$A^{(2)}_{m'm} = -i\pi\int d^3x\,\phi^2(x)\int\frac{d^3q''}{(2\pi)^3}\,e^{i(\vec{q}''-\vec{Q})\cdot\vec{x}}$$
$$\times\delta(\nu-\nu'')\chi^{m'\dagger}_{\alpha\beta}F_{\alpha i}G_{\beta_d}\chi^{m}_{ij}$$
$$= -i\pi(2\pi)^{-3}\nu q\sum_{l_1m_1l_2m_2}\chi^{m'\dagger}_{\alpha\beta}f^{l_1m_1}_{\alpha i}g^{l_2m_2}_{\beta j}\chi^{m}_{ij}$$
$$\times\int d^3x\,\phi^2(x)e^{-i\vec{Q}\cdot\vec{x}}\int d\Omega''e^{i\vec{q}''\cdot\vec{x}}$$
$$\times Y_{l_1m_1}(\Omega_1)Y_{l_2m_2}(\Omega_2), \tag{A1}$$

where

$$\vec{Q}=\tfrac{1}{2}(\vec{q}+\vec{q}'), \tag{A2}$$

Ω_1 represents the angle between q and q'', and Ω_2 represents the angles between q'' and q. Ω_3 represents the angle between $\vec{x}$ and $\vec{q}''$; then the exponential factor in the last integral may be expanded,

$$e^{i\vec{q}''\cdot\vec{x}}=\sum_l i^l[4\pi(2l+1)]^{1/2}j_l(qx)Y_{l0}(\Omega_3). \tag{A3}$$

This leaves integrals of the form

$$I=\int d\Omega''Y_{l_1m_1}(\Omega_1)Y_{l_2m_2}(\Omega_2)Y_{l0}(\Omega_3), \tag{A4}$$

which cannot be integrated immediately, because the arguments of the spherical harmonics measure the direction of $\vec{q}''$ in three different reference systems. One may, however, rotate each of these reference systems to some standard coordinate system. If the Euler angles for rotating the standard system to a system with its z axis along $\vec{q}$ are ω_1, then

$$Y_{l_1m_1}(\Omega_1)=\sum_{m'}D^{l_1}_{m'_1m_1}(\omega_1)Y_{l_1m_1}(\Omega_0), \quad \text{etc.} \tag{A5}$$

Now with the help of some formulas from Edmonds,[6] the integral can be done:

$$I=\sum_{m'm'_1m'_2}D^{l}_{m'0}(\omega_3)D^{l_1}_{m'_1m_1}(\omega_1)D^{l_2}_{m'_2m_2}(\omega_2)$$
$$\times\left(\frac{(2l+1)(2l_1+1)(2l_2+1)}{4\pi}\right)^{1/2}\begin{pmatrix}l & l_1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l & l_1 & l_2\\ m' & m'_1 & m'_2\end{pmatrix}, \tag{A6}$$

Edmonds's convention is used for the 3-j symbols.

Choosing the z axis of the standard system parallel to $\vec{Q}$ allows the angular part of the x integral to be done easily. One finds

$$A^{(2)}_{m'm} = -i(2\pi)^{-1}\nu q\sum_{\substack{l_1m_1l_2m_2\\ lm'_1m'_2}}(2l+1)(2l_1+1)^{1/2}$$
$$\times(2l_2+1)^{1/2}\begin{pmatrix}l & l_1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l & l_1 & l_2\\ 0 & m'_1 & m'_2\end{pmatrix}$$
$$\times\chi^{\dagger m'}_{\alpha\beta}f^{l_1m_1}_{\alpha i}g^{l_2m_2}_{\beta j}\chi^{m}_{ij}d^{l_1}_{m'_1m_1}(\theta_1)d^{l_2}_{m'_2m_2}(-\theta_2)\Phi_l, \tag{A7}$$

where θ_1 and θ_2 are the angles between $(\vec{q},\vec{Q})$ and $(\vec{q}',\vec{Q})$, respectively, and Φ_l is the radial integral,

$$\Phi_l=\int_0^\infty dx\,x^2j_l(qx)j_l(Qx)\phi^2(x). \tag{A8}$$

If the wave function is of Hulthén type,

$$\phi(x) = N\sum C_i \frac{e^{-\alpha_i x}}{x}\,, \tag{A9}$$

then these radial integrals can be done analytically. One uses

$$j_l(qx) = \tfrac{1}{2}(-i)^l \int_{-1}^{1} d\zeta\, P_l(\zeta) e^{iqx\zeta} \tag{A10}$$

to show that

$$\Phi_l = \tfrac{1}{4}(-1)^l N^2 \sum_i \sum_j C_i C_j \int d\zeta d\eta\, P_l(\zeta) \times \frac{1}{\alpha_i + \alpha_j - iq\zeta - iQ\eta} P_l(\eta)\,. \tag{A11}$$

The last integrals are quite manageable.

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Extended Shell-Model Description of Nucleon-Transfer Form Factors*

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A flexible microscopic model is introduced in an attempt to illuminate problems associated with the calculation of single-nucleon-transfer form factors. The model employs a truncated shell-model basis augmented by a set of single-nucleon channels. It can be used even when the residual nucleus is unbound, but ignores contributions from many-particle breakup channels. The model wave function is fully antisymmetric, and it is shown that all quantities which appear in the model are calculable. An approximate single-channel calculation of form factors for the reactions ^{40}Ca$(d,p)^{41}$Ca(2.01 MeV) and ^{40}Ca$(^3$He$,d)^{41}$Sc(2.10 MeV) is used to illustrate some of the features of the model.

I. INTRODUCTION

It is well known that the nucleon-transfer form factor appears as an essential ingredient in distorted-wave analyses of single-nucleon transfer reactions. Because of the current reliance upon the distorted-wave method for the extraction of spectroscopic information from reaction data and because the extracted results are found to depend on the details of the form factor,[1-5] the form factor itself has become the subject of considerable interest.[6-20]