Investigations of the Accuracy of a Separable Approximation to Local Potentials

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We apply the unitary pole approximation (UPA) to a number of spin-dependent local potentials and use the separable potentials obtained to calculate the three-nucleon binding energy and scattering lengths. Excellent agreement is found between the binding energies obtained with the separable potentials and those calculated by Malfliet and Tjon using the local potentials. Simple analytic approximations are given for the UPA form factors, and comparisons are made between the local and the UPA two-body phase shifts.

I. INTRODUCTION

In this paper, we investigate the validity of the separable approximation for local interactions. Separable potentials were first[1] used in calculations on the nuclear three-body system because of the simplifications they brought about in the numerical treatment of the three-body equations. Later, a justification of the separable potential approach using pole-dominance arguments was given by Lovelace.[2] Early calculations used Yamaguchi[3] shapes or simple modifications of them. Such potential shapes, however, are not in keeping with the pole-dominance arguments of Lovelace, since they correspond to separable approximations to purely attractive local potentials, and it is well known that the two-nucleon interaction contains some type of short-range repulsion. In calculations of three-body binding energies, Kok, Erens, and Van Wageningen[4] showed that for purely attractive local potentials the separable approximation is not very good. Calculations[5] by Levinger's group, however, gave good results when a one-term separable approximation was made to a two-term separable potential containing repulsion. Recent investigations[6,7] on separable approximations to spin-independent local potentials containing repulsion have produced very encouraging results, giving an order of magnitude better agreement than that obtained with purely attractive local potentials.

In this paper we investigate the validity of the separable approximation for spin-dependent local potentials. A particular separable approximation called the unitary pole approximation[2,5] (UPA) is applied to the local spin-dependent potentials of Malfliet and Tjon[8] (MT). We use the separable potentials obtained to calculate the binding energy of the three-nucleon system and compare our results to those of MT, finding excellent agreement. Results for other three-body parameters are also given.

In Sec. II we present the separable approximations used, giving a convenient analytic approximation for the form factors. The two-body scattering parameters of the local and separable potentials are compared. In Sec. III the results of our three-body calculations are given and compared with the results of MT.

II. DETERMINATION OF THE SEPARABLE POTENTIALS

The approximation we use is the UPA, which has been investigated by a number of authors.[2,4-7,9] In this section we outline how our separable approximation is obtained, referring to Ref. 7 for more details.

The solutions of the homogeneous Lippmann-Schwinger equation

$$|\psi_n\rangle = \lambda_n V G_0(s)|\psi_n\rangle \tag{1}$$

may be used as a complete set of functions with which to expand the two-body potential.[10] If the two-body system possesses a bound state and s is taken as $-B$ where B is the two-body binding energy, then Eq. (1) will have a solution with $\lambda_1^B = 1$. The corresponding eigenfunction $|\psi_1^B\rangle$ will be related to the bound-state wave function $|\chi_1^B\rangle$ by

$$|\psi_1^B\rangle \propto (-B-K)|\chi_1^B\rangle\ , \tag{2}$$

where K is the kinetic energy operator. The UPA for a partial wave with a bound state is then

$$V_u^B = -|\psi_1^B\rangle\langle\psi_1^B|\ , \tag{3}$$

$$T_u^B(E) = \frac{-|\psi_1^B\rangle\langle\psi_1^B|}{1+\langle\psi_1^B|G_0(E)|\psi_1^B\rangle}\ , \tag{4}$$

where we have chosen the normalization

$$\langle\psi_1|G_0(s)|\psi_1\rangle = -1\ . \tag{5}$$

If there is no bound state but an antibound state, then choosing $s=0$ in Eq. (1), we find that there is

a λ_1^A slightly greater than 1. This may be seen from Eq. (1), which says that the potential $\lambda_n V$ gives a bound state at the energy s, and if an antibound state is present, we need a slightly stronger potential than V to bind the system. The UPA for the antibound-state case is then given by

$$V_u^A = -\frac{|\psi_1^A\rangle\langle\psi_1^A|}{\lambda_1^A}, \tag{6}$$

$$T_u^A = \frac{-|\psi_1^A\rangle\langle\psi_1^A|}{\lambda_1^A + \langle\psi_1^A|G_0(E)|\psi_1^A\rangle}, \tag{7}$$

where the normalization of Eq. (5) is again chosen.

It may easily be shown[2] that the UPA T matrix in the bound-state case reproduces exactly the bound-state pole in the actual T matrix. The T matrix of Eq. (7) does not reproduce the antibound-state pole exactly, but is expected to come close. As we shall see, the method gives excellent results. It should be noted that this reproduction of the poles of the T matrix is the major justification for the separable approximation.

To find the UPA, we must therefore solve Eq. (1) for the two-body potentials of interest. This was done using a variational method. Defining the vector $|\chi_n\rangle$ in analogy with Eq. (2) by

$$|\psi_n\rangle = (s-K)|\chi_n\rangle, \tag{8}$$

it is easy to show that $|\chi_n\rangle$ satisfies the equation

$$(s-K)|\chi_n\rangle = \lambda_n V|\chi_n\rangle, \tag{9}$$

i.e., the Schrödinger equation with the potential $\lambda_n V$. Contracting Eq. (9) with $\langle\chi_n|$, we obtain the expression

$$\frac{1}{\lambda_n} = \frac{\langle\chi_n|V|\chi_n\rangle}{\langle\chi_n|s-K|\chi_n\rangle}, \tag{10}$$

which may easily be shown to be a variational expression for $1/\lambda_n$.

As our trial functions, we used the linear combination

$$\chi(r) = \sum_n a_n \phi_n(r) \tag{11}$$

with ϕ_n given by

$$\phi_n(r) = \frac{e^{-\alpha r} - e^{-n\mu r}}{r} \tag{12}$$

with $\alpha = \sqrt{-s}$. Our variational parameters are thus the a_n and the range μ. In practice, μ^{-1} was chosen as the range of the two-body potential. When the expression (11) is inserted in the variational principle and the a_n varied, we obtain the following system of linear equations

$$\underline{G}\vec{a} = \lambda \underline{V}\vec{a} \tag{13}$$

with

$$(\underline{G})_{n,m} = \langle\phi_n|\, s-K|\,\phi_m\rangle \tag{14}$$

and

$$(\underline{V})_{n,m} = \langle\phi_n|\, V|\phi_m\rangle. \tag{15}$$

For the potentials of interest here, all integrals could be performed easily in coordinate space. The linear system was solved by an iteration method.

Once we have found the a_n for the desired λ, we obtain $\psi_1(p)$ from

$$\psi_1(p) = \langle p|\psi_1\rangle = \langle p|\,(s-K)|\chi_1\rangle. \tag{16}$$

TABLE I. Comparison of the scattering lengths and effective ranges of local potentials and the UPA to them.

Local potential[a]	Scattering length (F) Local	Scattering length (F) UPA	Effective range (F) Local	Effective range (F) UPA
Singlet I	−23.3	−23.6	2.8	2.86
Singlet II	−23.3	−23.1	2.8	2.77
Reid[b]	−17.1	−17.2	2.8	2.66
Triplet III	5.45	5.55	1.8	1.95
Triplet IV	5.45	5.63	1.8	2.03

[a] Potentials I–IV are from Ref. 8. I and III contain a soft-core repulsion, while II and IV are purely attractive.
[b] The Reid potential is from Ref. 11. We use his soft-core singlet potential.

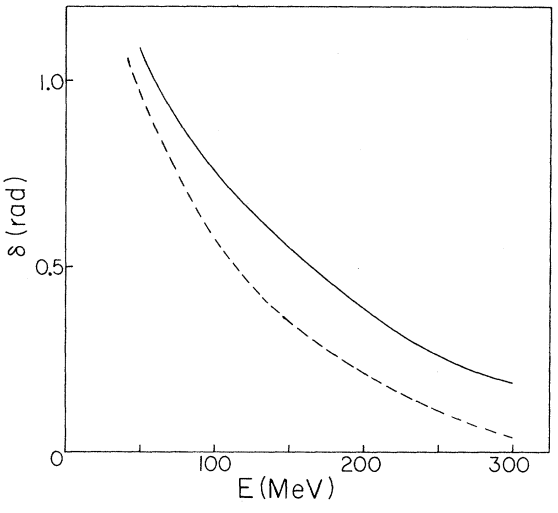

FIG. 1. Local (———) and UPA (– – –) phase shifts for the triplet potential III of MT.

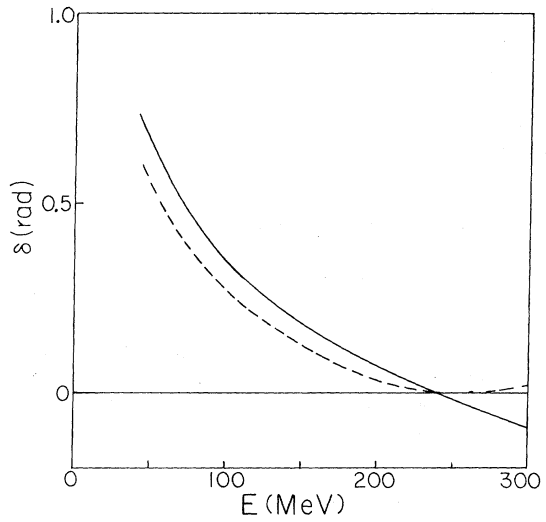

FIG. 2. Local (———) and UPA (– – –) phase shifts for the singlet potential I of MT.

We find that

$$\psi_1(p) = \sum_n c_n \frac{1}{p^2 + (n\mu)^2}, \tag{17}$$

where the c_n are simply related to the a_n.

We wish to compare separable-potential calculations with the local-potential three-body calculations of MT.[8] These authors use four different local potentials: two potentials containing soft-core repulsion and fit to singlet (potential I) and triplet (potential III) two-nucleon data, and two purely attractive Yukawa potentials fit to the low-energy singlet (potential II) and triplet (potential IV) data.

In Table I we give the low-energy properties of these potentials and compare them with the low-energy properties obtained with the UPA. The strength of the UPA to potential II was adjusted to give rough agreement with the local-potential scattering length. Also included are results for the Reid singlet soft-core two-nucleon potential.

Phase shifts obtained with the UPA's to the soft-core potentials are compared in Figs. 1 and 2 with the MT local-potential phase shifts. In making the UPA, we have neglected both attractive and repulsive parts of the local potential, and to some extent it is the partial cancellation of these neglected components which makes the UPA a good approximation.[6,7] The effects of these neglected parts of the interaction are separated in the two-body phase shift, since at intermediate energies the UPA phase shifts are less than, and at higher energies greater than, the corresponding local-potential phase shifts.

The coefficients c_n for the momentum-space UPA form factors [see Eq. (17)] obtained with potentials I and III are given in Table II, as is the singlet strength [see Eq. (6)]. Form factors are normalized according to Eq. (5). We plot the various form factors used in Figs. 3 and 4.

TABLE II. Coefficients for the UPA form factors. We give the coefficients for the UPA form factors to potentials I and III of Ref. 8. The form factors are given in terms of the C_n and μ by [Eq. (17)]

$$\psi_1(p) = \sum_{n=1}^{9} C_n \frac{1}{p^2 + (n\mu)^2}.$$

The quantity λ_s is the singlet strength [Eq. (6)]. Form factors are normalized according to Eq. (5).

	C_n	
n	Triplet	Singlet
1	2.240 754	3.937 232
2	95.920 87	75.599 92
3	−900.3338	−902.8878
4	4196.698	4484.466
5	−12 438.34	−13 426.14
6	22 896.51	24 706.38
7	−24 903.14	−26 872.25
8	14 602.12	15 774.22
9	−3553.804	−3844.966
	λ_s = 1.079 71	
	μ = 1.55 F^{-1}	

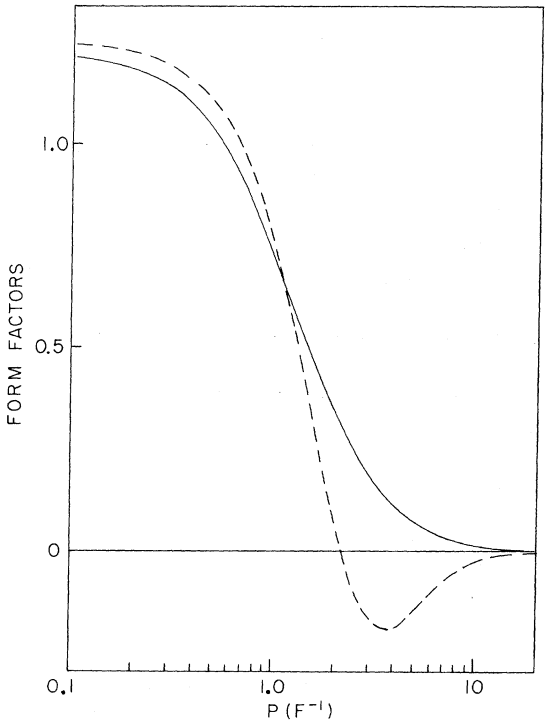

FIG. 3. UPA form factors for the triplet potentials III (– – –) and IV (———) of MT. Potential III contains a soft-core repulsion.

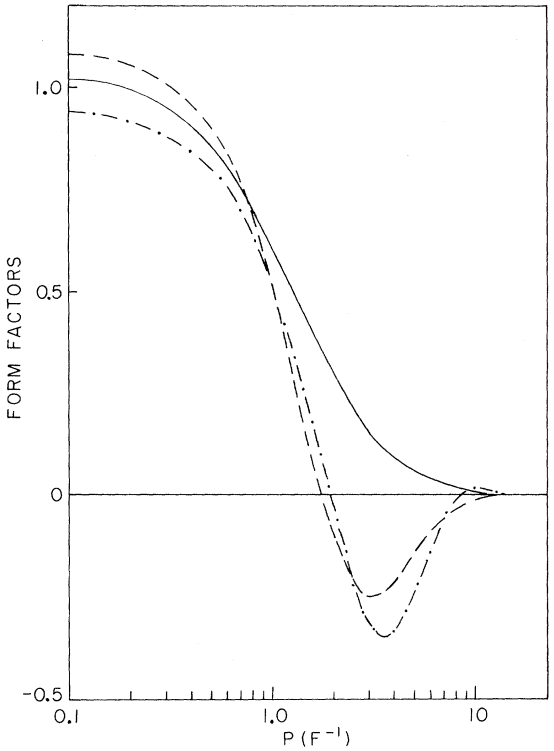

FIG. 4. UPA form factors for the singlet potentials I (– – –) and II (———) of MT and for the singlet S soft-core potential of Reid (– · – · –). Potential I and the Reid potential contain a soft-core repulsion.

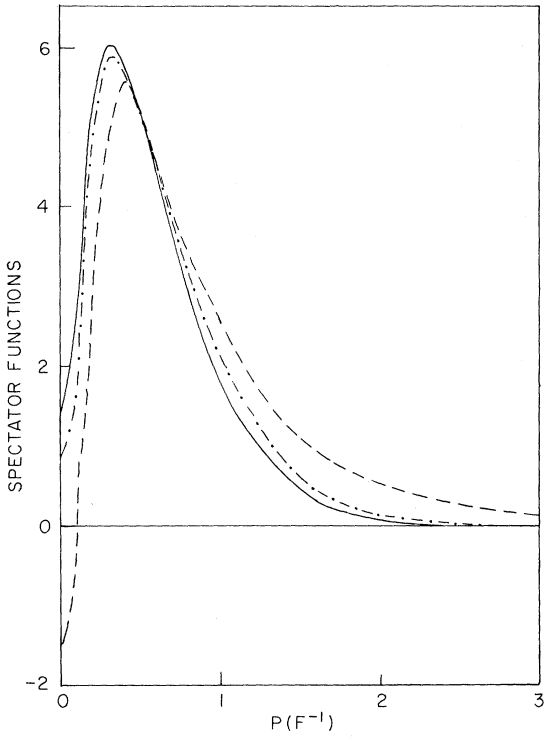

FIG. 5. The triplet part of the doublet spectator functions obtained in calculations of the doublet scattering lengths in cases A (———), C (– – –), and D (– · – · –) of Table III.

III. THREE-BODY CALCULATIONS

The three-body binding energies and scattering lengths were calculated in the UPA to the local potentials described above, and the binding energies were compared with the local-potential results of MT. The scattering and bound-state equations with separable interaction are well known.[2]

In Table III we compare the UPA separable-potential calculations with the local-potential results of MT. Excellent agreement is found in the cases where the UPA is made to a potential including repulsion (cases A and B). Case C, however, where the UPA is made to a purely attractive potential, shows a discrepancy of the order of 14%. This is in agreement with previous results, both for potentials with repulsion,[6,7] and for purely attractive potentials.[4] This discrepancy would indicate that the UPA is not as effective in the case of purely attractive potentials as it is if a certain amount of repulsion is included. This is partly because errors in the approximation tend to cancel them-

TABLE III. Comparison of separable-potential calculations with the local-potential results of Malfliet-Tjon. The local potentials I through IV are from Ref. 8. I and III contain short-range repulsion, while II and IV are purely attractive Yukawa potentials. The UPA binding energies and scattering lengths are obtained from the separable potentials discussed in Sec. II of this work. We include results obtained with a separable approximation to the Reid singlet S soft-core potential (Ref. 11).

			Binding energy (MeV)		Scattering lengths (F)	
	Singlet	Triplet	UPA	Local potential[a]	^2a	^4a
A	I	III	8.464	8.3 ± 0.1	0.92	6.47
B	I	IV	8.378	8.4 ± 0.1	0.78	6.57
C	II	IV	10.409	12.1 ± 0.1	−1.11	6.57
D	Reid[b]	III	8.741	...	0.53	6.47

[a]See Ref. 8.

[b]See Ref. 11.

selves if repulsion is included, whereas they are unable to do so in the attractive case. Also, the UPA gives a fairly good approximation to the local-potential T matrix for potentials with repulsion.[6,7]

From Table II it is seen that the UPA to the Reid singlet potential (case D) gives a greater binding energy than the UPA to the MT soft-core singlet potential (case A). The UPA to Reid is seen from Table I to give a smaller effective range than that from the UPA to potential I. A smaller effective range is usually associated with a greater three-body binding energy. Alternatively, we can try to explain this difference between the binding energies in terms of the difference in the shape of the UPA potentials [see Eq. (6)]. A comparison of the appropriate form factors in Fig. 4 shows that while the MT form factor is larger at low momentum, the absolute value of the Reid form factor is greater at high momentum than that of the MT form factor, reaching a relative maximum at about 3 F^{-1}. It would seem, then, that the higher binding energy of the Reid potential depends on the high-momentum behavior of the form factor, and therefore the far off-shell T matrix becomes important in the three-body calculation. This conclusion is strengthened when we consider the phase-space enhancement at high momentum appearing in the three-body equations.

In Fig. 5 we plot the triplet spectator functions obtained in the calculation of the doublet scattering lengths for cases A, C, and D of Table III. The doublet scattering lengths are proportional to the spectator functions evaluated at zero momentum. The fact that the spectator function varies rapidly in the low-momentum region indicates why the doublet scattering length is sensitive to the interaction. The scattering lengths obtained here are in agreement with those obtained with potentials producing similar three-body binding energies.

IV. SUMMARY

We have applied the UPA to a number of local potentials. The binding energies obtained with the separable approximation give excellent agreement with those obtained using the local potentials,[8] especially in the case of potentials containing repulsion. This is in agreement with previous results.[6,7] The advantage of the separable approximation is that it may easily be extended to calculations of other aspects of the three-body problem, including scattering and three-body breakup, as has been done in this work in the calculation of the three-body scattering lengths. The analytic approximations given here for the UPA form factors should be useful in such calculations.

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