Derivation of the Relation Between the Transmission Coefficient and the Ratio of the Average Width to the Average Spacing*

Nazakat Ullah†
Department of Physics, University of Toronto, Toronto 181, Ontario, Canada
(Received 20 February 1970)

The expansion developed previously for the derivation of the relation between the transmission coefficient and the ratio of the average width to the average spacing is reexamined. By rewriting the series in a more convenient form, not only is the objection raised against the expansion removed, but also an estimate of the smallness of the terms which are usually thrown out can be easily made.

An important relationship in the theory of average cross sections[1] is the connection between the transmission coefficient T_c defined as

$$T_c = 1 - |\langle S_{cc}\rangle|^2, \tag{1}$$

where $\langle S_{cc}\rangle$ is the energy average[2] of the low-energy scattering matrix S, and the ratio of the average width $\langle \Gamma_\mu \rangle$ to the average spacing D. Because of the difficulties of the unitarity constraint on S, two kinds of models have been used for this purpose: (1) the picket-fence model,[3] which contains an infinite number of resonances having the same width Γ and a constant spacing D, and (2) the finite resonances model,[4] in which the individual widths and spacings are not restricted to constant values, but the number of resonances are taken to be finite. A comment has been made[5] that the derivation of the relation

$$T_c = 1 - e^{-(2\pi \langle \Gamma_\mu \rangle / D)}, \tag{2}$$

for the case of the single channel, given in Ref. 4, is not correct, since the terms in the expansion used there may not be small. Following along the lines of Ref. 4, an alternative proof is then suggested[5] for relation (2). The purpose of the present note is to examine the validity of the objection of Ref. 5 to the derivation given in Ref. 4.

The starting point of the derivation of relation (2) is the expression for the energy average $\langle S_{cc}\rangle$, obtained using the averaging technique of Feshbach, Kerman, and Lemmer.[2] Apart from a phase factor for the single-channel case, it is given by

$$\langle S_{cc}(E_0)\rangle_{\Delta E} = \prod_\mu \frac{E_0 + iI/2 - Z_\mu^*}{E_0 + iI/2 - Z_\mu}, \tag{3}$$

with $I = 2\Delta E/\pi$, $Z_\mu = \epsilon_\mu - \frac{1}{2}\Gamma_\mu$, and the product is to be evaluated for the N resonances in ΔE.

Expression (3) is rewritten in Ref. 4 as

$$\langle S_{cc}(E_0)\rangle_{\Delta E} = \prod_\mu \frac{1 + (\pi/\Delta E)[i(\epsilon_\mu - \langle \epsilon_\mu \rangle) - \frac{1}{2}\Gamma_\mu]}{1 + (\pi/\Delta E)[i(\epsilon_\mu - \langle \epsilon_\mu \rangle) + \frac{1}{2}\Gamma_\mu]}, \tag{4}$$

where $\langle \epsilon_\mu \rangle$ is written for $\langle \epsilon_\mu \rangle = (1/N)\sum_\mu \epsilon_\mu = E_0$.

The objection raised in Ref. 5 is that if we examine the part of the second term in the above expansion which is of the form $[1/(\Delta E)^2]\sum_\mu (\epsilon_\mu - \langle \epsilon_\mu \rangle)^2$, then for a resonance ϵ_μ which happens to fall at the end of the interval ΔE this term will be of the order of unity, making the expansion invalid. But the same argument applied to the similar part of the fourth term again gives a contribution of the same order but with the opposite sign. In fact, the successive even-order terms alternate in sign. It is easy to show that we can easily get rid of these alternating terms by rearranging expression (4). To achieve this we express $\langle S_{cc}\rangle$ in the form

$$\langle S_{cc}(E_0)\rangle_{\Delta E} = \exp\left\{\sum_\mu \ln\left[\frac{1 + (\pi/\Delta E)[i(\epsilon_\mu - \langle \epsilon_\mu \rangle) - \frac{1}{2}\Gamma_\mu]}{1 + (\pi/\Delta E)[i(\epsilon_\mu - \langle \epsilon_\mu \rangle) + \frac{1}{2}\Gamma_\mu]}\right]\right\}, \tag{5}$$

which can be expanded to yield

$$\langle S_{cc}(E_0)\rangle_{\Delta E} = \exp\left(-\frac{\pi}{\Delta E}\sum_\mu \Gamma_\mu + 2i\left(\frac{\pi}{\Delta E}\right)^2 \sum_\mu \Gamma_\mu(\epsilon_\mu - \langle \epsilon_\mu \rangle) - \frac{2}{3}\left(\frac{\pi}{\Delta E}\right)^3 \left\{\sum_\mu \Gamma_\mu[\Gamma_\mu^{\,2} - 3(\epsilon_\mu - \langle \epsilon_\mu \rangle)^2]\right\} \cdots\right). \tag{6}$$

We see from expression (6) that the terms which were alternating in character have now disappeared, thus removing the objection raised in Ref. 5. So far we have not said anything about the statistical distributions of ϵ_μ, Γ_μ. Further evaluation of the expansion terms in expression (6) needs their knowledge. Let us first consider the case when the distribution of ϵ_μ, Γ_μ is such that all their statistical moments have an upper bound which is less than ΔE. If we denote by Q_μ any combination of $\epsilon_\mu^m \Gamma_\mu^n$, then the variable statistics of the quantities Q_μ implies[6] that a term like $(1/\Delta E)\sum_{\mu=1}^{N(\Delta E)} Q_\mu$ should be replaced by

$$\frac{1}{\Delta E}\int\left(\sum_{\mu=1}^{N(\Delta E)} Q_\mu\right)P(\{Q_\mu\})\pi_\mu dQ_\mu \,.$$

Since all Q_μ in ΔE are treated statistically alike, we find that if the density of μ in ΔE is taken to be constant $= N(\Delta E)/\Delta E = 1/D$, as has been done earlier,[2,4] then this term becomes $\langle Q_\mu\rangle/D$. Using this result in expression (6), together with our assumption of an upper limit on the statistical moments less than ΔE, we get the earlier result

$$\langle S_{cc}(E_0)\rangle_{\Delta E} = e^{-\pi\langle\Gamma_\mu\rangle/D} \,. \tag{7}$$

Next let us suppose that $\epsilon_\mu = \mu D$, that is, poles are uniformly spaced and all $\Gamma_\mu = \Gamma$. This corresponds to the picket-fence model (p.f.m.) which uses fixed resonance parameters. The expansion terms in expression (6) can again be evaluated by using the expressions for summations of the type $\sum_\mu^{2n}$. If these summations are carried out in expression (6), then we find that, irrespective of the value of $(\pi\Gamma/D)$, *we do not get the result given* by (7) but instead arrive at the result

$$\langle S_{cc}(E_0)\rangle_{\Delta E} = \exp\left\{-\left(\frac{\pi\Gamma}{D}\right)\left[\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1}\left(\frac{\pi}{2}\right)^{2\nu}\right]\right\} \,. \tag{8}$$

We remark here that the same result will be obtained from Eq. (2.19) of Feshbach, Kerman, and Lemmer[2] if one uses the resonance parameters of the picket-fence model. Since it is well known[6] that for small values of $\pi\langle\Gamma_\mu\rangle/D$, $\langle S_{cc}\rangle$ is given by

$$\langle S_{cc}\rangle = 1 - \pi\langle\Gamma_\mu\rangle/D \,,$$

we can ask why we arrive at the incorrect result (8) if we use p.f.m. resonance parameters in expression (6). The explanation for this is the complete coherence of resonances in the p.f.m., which does not allow us to separate the resonances in ΔE from the ones which are outside this interval.

Since our objective here is not to discuss the problem of averaging using the p.f.m. and Lorentz energy-resolution function, we conclude this note with the following remarks:

(1) The energy-averaging prescription of Feshbach, Kerman, and Lemmer[2] gives correct results provided all the statistical moments of ϵ_μ, Γ_μ have an upper bound which is less than ΔE.

(2) The expansion used in Ref. 4 can be rearranged to get rid of the terms against which the objection was raised in Ref. 5.

(3) For the coherent model, like the p.f.m., the energy-averaging prescription[2] can be used provided the effect of the resonances outside ΔE is taken into account. If this is done then one gets expression (7) instead of the incorrect expression (8). Because of the built-in coherence of this model, it cannot be used either to check the usual energy-averaging prescription[2] or the derivation given in Ref. 4. Detailed numerical calculations for the p.f.m. using various energy-resolution functions are carried out which check our point of view. These calculations are intended to be published later.

ACKNOWLEDGMENTS

I would like to thank Professor L. E. H. Trainor and Dr. T. W. Donnelly for useful discussions.

---

*Work supported by the National Research Council of Canada.

†On leave of absence from Tata Institute of Fundamental Research, Bombay, India.

[1] E. Vogt, in *Advances in Nuclear Physics,* edited by M. Baranger and E. Vogt (Plenum Press, Inc., New York, 1968), Vol. 1, p. 21.

[2] H. Feshbach, A. K. Kerman, and R. H. Lemmer, Ann. Phys. (N.Y.) 41, 230 (1967).

[3] P. A. Moldauer, Phys. Rev. 157, 907 (1967).

[4] N. Ullah, Phys. Rev. Letters 20, 1510 (1968).

[5] P. A. Moldauer, Phys. Rev. 177, 1841 (1969).

[6] C. E. Porter, *Statistical Theories of Spectra: Fluctuations* (Academic Press Inc., New York, 1965).