# Migdal's Quasiparticle Model and Radiative Muon Capture in $Ca^{40}$

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In this paper we evaluate the radiative photon spectrum and the branching ratio $R$ of radiative-to-ordinary muon capture rate in $Ca^{40}$, taking into account the nucleon-nucleon residual interaction by means of Migdal's quasiparticle model.

The most important effect of the residual interaction is to reduce both the ordinary and radiative rate about 40% lower than that obtained in the closure-harmonic-oscillator model. This result differs from that obtained by Fearing using the giant-dipole-resonance model, while it confirms the prediction of Rood and Tolhoek (RT) that the branching ratio $R$ would be nearly independent of the nuclear model. We also evaluate the circular polarization of the photons, $P_\gamma$. We find that our values of $P_\gamma$ are greater than the RT values by about 15% in the high-energy range. In the low-energy range, our values of $P_\gamma$ are in close agreement with those of RT. For the maximum energy of the photon spectrum, we obtain the average value $k_{max} = 90.5$ MeV, which is very close to the RT value.

Comparing our results with the experimental data of Conversi, Diebold, and Di Lella, we find that our calculation requires $g_P = (12.4 \pm 2.8)g_A$ to fit the experimental values, where $g_P$ is the induced pseudoscalar coupling constant and $g_A$ the axial-vector coupling constant.

Finally, we think that the disagreement between our result and the more currently accepted value of $g_P$ is not so great as to call into question the theoretical mechanism of the radiative capture, if one takes into account the large experimental uncertainties.

## I. INTRODUCTION

Since Conversi, Diebold, and Di Lella (CDD)[1] measured the photon spectrum for the radiative muon capture process in $Ca^{40}$, some authors[2-5] have tried to explain the experimental results using several approaches. These calculations are of particular interest in that the radiative spectrum is quite sensitive to $g_P$, the induced pseudoscalar coupling constant of the weak interaction.

In the experimental work of CDD only photons with an energy above a threshold of about 55 MeV could be used in the analysis because the background due to electrons from the free decay of muons prevents the observation of the relatively few photons from the radiative process below 55 MeV. CDD used the theoretical formulas of Rood and Tolhoek (RT)[2] in the analysis of the experimental results, varying the values of $k_{max}$ (the maximum energy of the photon spectrum for an average excitation of the nucleus) and $g_P$. They obtained good agreement between experiment and theory with $k_{max} = (88 \pm 4)$ MeV and $g_P = (13.3 \pm 2.8)g_A$, where $g_A$ is the axial-vector coupling constant.

The value of $R$ (the branching ratio of radiative-to-ordinary muon capture rate) was obtained by extrapolating the measured high-energy tail of the photon spectrum to low photon energies. CDD obtained

$$R = (3.1 \pm 0.6)10^{-4}.$$

RT[3] derived their theoretical formulas starting from an "effective" Hamiltonian[6] and assuming the "closure approximation"[7] for the evaluation of the nuclear matrix elements. Assuming $g_P = 8g_A$ as suggested theoretically and $k_{max} = 91$ MeV, they found for the branching ratio in $Ca^{40}$ $R = 2.14 \times 10^{-4}$ using a simple "model-independent" approximation, and $R = 2.39 \times 10^{-4}$ using a shell model with harmonic-oscillator wave functions.

In view of the discrepancy between the theoretical value $g_P \simeq 8g_A$ of Goldberger and Trieman and the experimental result, Borchi and Gatto[4] explored the dependence of the photon spectrum on a possible induced tensor current and determined the tensor coupling constant $g_T$ needed to fit the experimental result. Using $g_P = 8g_A$ and $k_{max} = 85$ MeV, they obtained $g_T \simeq 20g_V$ where $g_V$ is the vector coupling constant.

A different approach was tentatively explored by Fearing[5] who used the giant-dipole-resonance (GDR) model developed by Foldy and Walecka (FW)[8] to improve the calculation of nuclear matrix elements. In this model FW attributed the failure of the independent-particle models to the neglect of the effect of internucleonic interaction. To take this effect into some account, they established a relationship between the important dipole matrix elements in muon capture and the ones which appear in electric dipole photoabsorption in the parent nucleus. Then these dipole matrix elements were replaced by the empirical ones deduced directly from photoabsorption data. However, when calculating the total rate, the monopole and quadrupole contributions were taken unchanged from their shell-model values.

On the other hand, the assumption that only the nonrelativistic dipole terms are affected by dynamical correlation does not seem justified in the FW model. It is hardly understandable why dynamical correlation should affect only the dipole part. Calculations show that if one takes adequately into account the residual interaction, the other multipoles are reduced about 40% from the shell-model values. For this reason the results obtained in the GDR model are often overestimated.

Fearing,[5] comparing his results with the data of CDD[1] on the branching ratio $R$, found that the GDR calculation using the Goldberger-Treiman value for $g_P$ requires an induced tensor coupling constant $g_T \gtrsim 35 g_V$ to fit the experiment. Alternatively, by taking $g_T = 0$, he obtained agreement between the theoretical and experimental data with $g_P = (16.5 \pm 3.1) g_A$. In view of the compatibility of other experiments with a value $g_P/g_A$ between 7 and 11 and $g_T = 0$ (see Sec. V) this result may appear very surprising, as it is apparently based on phenomenological grounds. It is still more surprising in view of its discrepancy with the RT result, because RT[3] found that the branching ratio $R$ is nearly independent (less than 10%) of the specific nuclear model. However, we think that in Fearing's calculation the dipole term is not important enough to justify the application of the GDR model, as it contributes only 34% to ordinary capture and only 50% to radiative capture.

As the other multipoles are evaluated in the closure-harmonic-oscillator approximation, the method appears to be poorly self-consistent in this case. Moreover, as the contribution of the dipole term is somewhat lower for ordinary capture than for radiative capture, ordinary capture appears to be overestimated more than the radiative capture. As a result, the branching ratio $R$ is underestimated.

In order to obtain a greater quantitative insight into this problem we evaluate in this paper radiative capture in $Ca^{40}$ taking into account the residual interaction between nucleons on the basis of Migdal's quasiparticle model.[9] This model has been applied to a great number of calculations with some remarkable successes.[10-13] Even though this model may be subjected to some criticisms[14,15] it, nevertheless, has been seen to eliminate the difficulty encountered with nuclear structure. Moreover, it is our opinion that it appears to be more self-consistent than the GDR model.

In Sec. II we give an interaction Hamiltonian for the $\mu$ radiative capture. In Sec. III we give a brief account of Migdal's quasiparticle model, and in this model we derive the photon spectrum for radiative muon capture by nuclei. In Secs. IV, V, and VI numerical results are given for ordinary and radiative muon capture in $Ca^{40}$ and a comparison with other calculations is presented. Section VII summarizes our conclusions.

## II. INTERACTION HAMILTONIAN

We outline here the derivation of the general formulas for radiative muon capture in a nucleus, following, for the most part, the work by Borchi,[16] to which the reader is referred for details and for references to earlier calculations (the reader is also referred to the work of RT[3] for similar calculations).

The most general effective Hamiltonian for the process $\mu^- + p \to n + \nu$ in a theory with $V$ and $A$ coupling and with first-class interactions only is[17]

$$H_{\text{eff}} = \frac{1}{\sqrt{2}} \bar{\psi}_n \left( g_V \gamma_\lambda - i g_m \frac{1}{m_\mu} \sigma_{\lambda\rho} q_\rho + g_A \gamma_5 \gamma_\lambda + g_P \frac{1}{m_\mu} \gamma_5 q_\lambda \right) \psi_p \bar{\psi}_\nu (1-\gamma_5) \gamma_\lambda \psi_\mu \qquad (2.1)$$

in which $q_\lambda = (p_p - p_n)_\lambda$ is the momentum of the proton minus the momentum of the neutron. The form factors $g_V$, $g_A$, $g_m$, $g_P$ are known, respectively, as vector, axial-vector, weak-magnetism, and induced-pseudoscalar form factors. The dependence on $q^2$ in all the couplings, except $g_P$, is small enough to be neglected. The dependence of $g_P$ on momentum is written as

$$g_P(q^2) = g_P (m_\pi{}^2 + m_\mu{}^2)/(m_\pi{}^2 - q^2) .$$

The constants $g_V$, $g_A$, and $g_m$ are connected with the Fermi constant $G$, which we take to be $1.417 \times 10^{-49}$ erg cm$^3$, by the relations $g_V = 0.97G$, $g_A = -1.18G$, $g_m = (m_\mu/2M)(\mu_p - \mu_n) 0.97G$; and a plausible value for the induced pseudoscalar constant, as given by Goldberger and Treiman, is $g_P \simeq 8 g_A$.

To obtain the interaction for radiative muon capture one adds a photon line in all possible ways to the diagram corresponding to Eq. (2.1). The corrective terms introduced by Adler and Dothan,[18] which take into account the structure effects, are not considered.

In deriving the interaction Hamiltonian for radiative $\mu^-$ capture on nuclei, the problem is greatly simplified if we do not consider the momentum of the nucleons, and if we consider the muon at rest. Indeed, the effect of the nucleon velocity terms on the ratio of radiative capture to normal capture is negligible, although they contribute about +10% in both of these processes separately, as shown by RT.[3] It is convenient to consider separately the emission of right- and left-circularly-polarized photons. The polarization vectors are $\vec{\epsilon}_L = (1/\sqrt{2}) \times (\hat{i} + i\hat{j})$ and $\vec{\epsilon}_R = \vec{\epsilon}_L^{\,*}$, with $\hat{i}$, $\hat{j}$, and $\vec{k}$ (photon momentum) forming a right-handed frame. We shall therefore write the effective Hamiltonian for a nucleus with $A$ nucleons in the form[19]

$$H_{\text{eff}}^{(L)}=\frac{1}{m_\mu\sqrt{2}}(1-\vec{\sigma}^l\cdot\hat{p}_\nu)\Big\{\sum_{i=1}^{A}\tau_i^{(-)}\Big[A\vec{\sigma}_i\cdot\vec{\epsilon}_L+B\Big(\vec{\sigma}_i\cdot\hat{k}\vec{\epsilon}_L\cdot\hat{p}_\nu+\frac{p_\nu}{k}\vec{\sigma}_i\cdot\hat{p}_\nu\vec{\epsilon}_L\cdot\hat{p}_\nu\Big)+C\vec{\sigma}_i\cdot\hat{p}_\nu\vec{\sigma}_i\cdot\vec{\epsilon}_L$$
$$+\vec{\sigma}^l\cdot\vec{\epsilon}_L(D\vec{\sigma}_i\cdot\hat{p}_\nu+E\vec{\sigma}_i\cdot\hat{k}+F)+G\vec{\sigma}^l\cdot\vec{\sigma}_i\vec{\sigma}_i\cdot\vec{\epsilon}_L+H\vec{\sigma}^l\cdot\hat{k}\vec{\sigma}_i\cdot\vec{\epsilon}_L\Big]\Big\} \tag{2.2a}$$

with

$$A=-\frac{m_\mu}{2M}\Big[g_A+(1+\mu_p-\mu_n)g_V+g_P^{(N)}+\frac{k}{2M}(1+\mu_p-\mu_n)g_P^{(N)}\Big],$$
$$B=\frac{m_\mu}{M}\frac{p_\nu k}{m_\pi^2+(\vec{p}_\nu+\vec{k})^2}g_P^{(N)},\quad C=-\frac{m_\mu p_\nu}{4M^2}(1+\mu_p-\mu_n)g_P^{(N)},\quad D=\frac{p_\nu}{2M}g_P^{(I)},\quad E=\frac{k}{2M}g_P^{(I)}, \tag{2.3a}$$
$$F=-\frac{m_\mu}{2M}(\mu_p-\mu_n)g_A,\qquad G=-\frac{m_\mu}{2M}(g_V+g_A),\qquad H=-g_m;$$

where

$$g_P^{(N)}=g_P(m_\mu^2+m_\pi^2)/(p_\nu^2-k^2+m_\pi^2),$$
$$g_P^{(I)}=g_P(m_\mu^2+m_\pi^2)/[(\vec{p}_\nu+\vec{k})^2+m_\pi^2],$$

and

$$H_{\text{eff}}^{(R)}=\frac{1}{m_\mu\sqrt{2}}(1-\vec{\sigma}^l\cdot\hat{p}_\nu)\Big\{\sum_{i=1}^{A}\tau_i^{(-)}\Big[a\vec{\sigma}^l\cdot\hat{k}\vec{\sigma}_i\cdot\vec{\epsilon}_R+bi\vec{\sigma}^l\cdot\hat{p}_\nu\times\vec{\epsilon}_R+c\vec{\sigma}_i\cdot\vec{\epsilon}_R+d\hat{p}_\nu\cdot\vec{\epsilon}_R+e\vec{\sigma}_i\cdot\hat{p}_\nu\vec{\sigma}_i\cdot\vec{\epsilon}_R$$
$$+Bp_\nu\cdot\vec{\epsilon}_R\Big(\vec{\sigma}_i\cdot\hat{k}+\frac{p_\nu}{k}\vec{\sigma}_i\cdot\hat{p}_\nu\Big)+f\vec{\sigma}^l\cdot\vec{\epsilon}_R+g\vec{\sigma}_i\cdot\hat{p}_\nu+h\vec{\sigma}_i\cdot\hat{k}+l\vec{\sigma}^l\cdot\vec{\sigma}_i\hat{p}_\nu\cdot\vec{\epsilon}_R+m\vec{\sigma}_i\cdot\vec{\epsilon}_R\Big]\Big\} \tag{2.2b}$$

with

$$a=-\frac{k}{2M}g_V-\frac{k}{m_\mu}g_m,\quad b=\frac{p_\nu}{2M}g_V,$$
$$c=\frac{m_\mu}{2M}\Big[(1+\mu_p-\mu_n)g_V-g_A+\frac{p_\nu-k}{m_\mu}g_V-g_P^{(N)}-\frac{k}{2M}(1+\mu_p-\mu_n)g_P^{(N)}\Big]+\frac{p_\nu-k}{m_\mu}g_m-g_A,$$
$$d=\Big[\frac{p_\nu}{2M}(\mu_p-\mu_n)-\frac{p_\nu}{m_\mu}\Big]g_m,\quad e=\frac{p_\nu}{m_\mu}g_m+\frac{p_\nu}{2M}\Big[g_V+\frac{m_\mu}{2M}(1+\mu_p-\mu_n)g_P^{(N)}\Big], \tag{2.3b}$$
$$f=-g_V-g_A+g_m+\frac{m_\mu}{2M}(\mu_p-\mu_n)g_A-\frac{k}{2M}g_V,\quad g=\frac{p_\nu}{2M}g_A,\quad h=\frac{k}{2M}g_A,\quad l=\frac{p_\nu}{m_\mu}g_m+\frac{p_\nu}{2M}g_V,$$
$$m=g_A-g_m-\frac{m_\mu}{2M}(g_V-g_A).$$

## III. NUCLEAR MODEL AND RADIATIVE CAPTURE OF THE MUON

In this section we first outline the method of finite Fermi systems, developed by Migdal[9] for the calculation of the nuclear matrix elements, and then we consider the $\mu^-$ radiative capture in detail.

In Migdal's theory the excited states of the nucleus are described in terms of the interacting quasiparticles. The central point of the theory is that the application of an external field $V^0$ on the system causes the quasiparticles to be acted upon by a certain effective field $V^{\text{eff}}$ which, in systems without pair correlations, is determined from the equation

$$V^{\text{eff}}=e_q V^0+FGGV^{\text{eff}}, \tag{3.1}$$

where $F$ is the quasiparticle interaction amplitude near the Fermi surface, $GG$ are two Green's functions, one of a particle and the other of a hole, and $e_q$ is the effective charge of the quasiparticle denoting the difference of the external field acting on quasiparticles from that applied to particles. The amplitude $F$ is an operator in spin and isospin space and has, accurate to $(N-Z)/A$, the isotopically invariant form

$$F=\Big(\frac{dn}{d\epsilon_0}\Big)^{-1}[f+g\vec{\sigma}\cdot\vec{\sigma}'+(f'+g'\vec{\sigma}\cdot\vec{\sigma}')\vec{\tau}\cdot\vec{\tau}'], \tag{3.2}$$

where $\vec{\sigma}$, $\vec{\sigma}'$, $\vec{\tau}$, $\vec{\tau}'$ are the spin and isospin operators of the interacting quasiparticles, $dn/d\epsilon_0$ is the derivative of the nucleon density with respect to the limiting Fermi energy $\epsilon_0$, and $f$, $g$, $f'$, $g'$ are dimensionless quantities of the order of unity, which must be found by comparing theory with experiment.

As we think this brief summary to be sufficient to give some account (for further details the reader is referred to Migdal[9] and Rho[20]) of the ideas which support Migdal's theory, we will now proceed to extend the treatment given by Bunatyan[12] for ordinary muon capture to radiative muon capture. We regard radiative muon capture as a transition of the nucleus from an initial state with energy $E_0$ into a state $S$ with energy $E_S$ and express the probability of this transition, in accordance with the Lehman expansion, in terms of the residue of the polarization operator $P(\omega)$ calculated for $\omega_S = (E_S - E_0)/\hbar$:

$$W_S(x, \hat{k}, \hat{p}_\nu, \lambda)dx d\Omega_{\vec{k}} d\Omega_{\vec{\nu}}$$

$$= \frac{e^2 m_\mu{}^2}{2(2\pi)^4\hbar^5} x(1-x)^2 dx d\Omega_{\vec{k}} d\Omega_{\vec{\nu}} \left(1 - \frac{\hbar\omega_S}{m_\mu c^2}\right)^4 \mathrm{Res} P_\lambda(\omega_S), \tag{3.3}$$

where the circular polarization of the final photon is specified by $\lambda$ and $x = kc/(m_\mu c^2 - \hbar\omega_S)$ while the other symbols are self-explaining. The shape of the photon spectrum is obviously

$$N(x)dx = \sum_\lambda \sum_{\omega_S} \int d\Omega_{\vec{k}} \int d\Omega_{\vec{\nu}} W_S(x, \hat{k}, \hat{p}_\nu, \lambda) dx. \tag{3.4}$$

The polarization operator in systems without pair correlations is defined by the relation

$$P = e_q V^0 G^p G^n V^{\text{eff}}, \tag{3.5}$$

where $G^p$ and $G^n$ are the Green's functions for the proton and the neutron.

Using the representation of the nucleon wave functions $\varphi_\lambda(\vec{r})$ determined from the expression

$$H_{\text{eff}} \varphi_\lambda = \epsilon_\lambda \varphi_\lambda, \tag{3.6}$$

where $H_{\text{eff}}$ is the effective self-consistent potential $[\lambda = (n, l, j, M_j)]$, one obtains for $P(\omega)$ in this representation

$$P(\omega) = e_q \sum_{\lambda_1, \lambda_2} V^0_{\lambda_1\lambda_2}{}^*(\omega) A^{np}_{\lambda_1\lambda_2}(\omega) V^{\text{eff}}_{\lambda_2\lambda_1}(\omega), \tag{3.7}$$

$A^{np}_{\lambda_1\lambda_2}$ being the integral, with respect to the energy, of the product of the pole parts of the Green's functions near the Fermi surface

$$P(\omega) = e_q \sum_{\lambda_1 < \lambda_2} V^0_{\lambda_1\lambda_2}{}^*(\omega)(-1)\left[\frac{n^n_{\lambda_2} - n^p_{\lambda_1}}{\hbar\omega - (\epsilon^n_{\lambda_2} - \epsilon^p_{\lambda_1})} + \frac{n^n_{\lambda_1} - n^p_{\lambda_2}}{\hbar\omega - (\epsilon^n_{\lambda_1} - \epsilon^p_{\lambda_2})}\right] V^{\text{eff}}_{\lambda_2\lambda_1}(\omega). \tag{3.8}$$

Here $n_\lambda$ are the occupation numbers of the neutrons and protons and the sum in (3.8) is taken under the condition that the state $\varphi_{\lambda_1}$ is lower in energy than the state $\varphi_{\lambda_2}$. Substituting in (3.8) the expression (2.2) for $V^0$ and expanding the exponential $e^{i\vec{s}\cdot\vec{r}}$, where $\vec{s} = (\vec{p}_\nu + \vec{k})/\hbar$, of the neutrino plus photon field in spherical functions, we obtain for the photon spectrum in radiative capture[21]

$$N(x)dx = \frac{e^2}{\hbar c \pi} \frac{Z^4_{\text{eff}}}{Z} U \int \frac{d\Omega_{\vec{\nu}}}{4\pi} \int \frac{d\Omega_{\vec{k}}}{4\pi} x(1-x)^2 dx \sum_\lambda \sum_{\omega_S > 0} \left(1 - \frac{\hbar\omega_S + \mathcal{E}_\mu}{m_\mu c^2}\right)^4 \Big\{ G^{\text{I}}_r e^{\text{I}}_q \sum_L (2L+1) \sum_\alpha \mathrm{Res} U^{\text{I}}_\alpha(L, \omega_S)$$
$$\times \Theta^{\text{I}}_\alpha(L, \omega_S)[j_L(sr)]_\alpha + G^{\text{II}}_r e^{\text{II}}_q \sum_L (2L+1) \sum_\alpha \mathrm{Res} U^{\text{II}}_\alpha(L, \omega_S) \Theta^{\text{II}}_\alpha(L, \omega_S)[j_L(sr)]_\alpha \Big\}. \tag{3.9}$$

Here

$$U = \frac{m_\mu{}^5 c^4}{2\pi^2\hbar^7}\left(\frac{e^2}{\hbar c}\right)^3, \quad \alpha \equiv (n_1 l_1 n_2 l_2),$$

$$e^{\text{I}}_q \simeq 1, \quad e^{\text{II}}_q \simeq 0.9,$$

$\mathcal{E}_\mu$ is the binding energy of the $\mu^-$ meson in the $K$ orbit, the symbol $\Theta^{\text{I}}_\alpha(L, \omega_S)$ $[\Theta^{\text{II}}_\alpha(L, \omega_S)]$ is defined by

$$\Theta_\alpha(L, \omega) = \frac{2(2l_1+1)(2l_2+1)}{|\Delta\epsilon_{n_1 l_1 n_2 l_2}|^2 - \hbar^2\omega^2} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix}^2 2|\Delta\epsilon_{n_1 l_1 n_2 l_2}| \tag{3.10}$$

with

$$\Delta\epsilon_{n_1 l_1 n_2 l_2} = \epsilon_{n_1 l_1} - \epsilon_{n_2 l_2},$$

where the superscript I [II] indicates the fact that $\omega_S$ is the pole of $U^{\text{I}}_\alpha(L, \omega)$ $[U^{\text{II}}_\alpha(L, \omega)]$, and $G^{\text{I}}_r$, $G^{\text{II}}_r$ are rather complicated expressions which we give in the Appendix. $U^{\text{I}}_\alpha(L, \omega)$ and $U^{\text{II}}_\alpha(L, \omega)$ are determined, as in Bunatyan,[12] from the equations

$$e^{\text{I},\text{II}}_q [j_L(sr)]_\alpha = \sum_\beta [S^{\text{I},\text{II}}(L, \omega)]^\beta_\alpha U^{\text{I},\text{II}}_\beta(L, \omega), \tag{3.11}$$

where

$$\beta \equiv (n l n' l'),$$

$$[S^{\text{I}}(L, \omega)]^\beta_\alpha = \left[-\left(\frac{dn}{d\epsilon_0}\right)^{-1} \frac{f^{\text{I}}}{4\pi} C^\beta_\alpha \Theta_\beta(L, \omega) + \delta_{\alpha\beta}\right]. \tag{3.12}$$

$C_\alpha^\beta$ is defined in terms of the radial part of the wave function $\varphi_\lambda(\vec{r})$:

$$C_\alpha^\beta = \int R_{nl} R_{n'l'} R_{n_1 l_1} R_{n_2 l_2} r^2 dr \,, \tag{3.13}$$

$S^{\mathrm{II}}(L,\omega)$ differs from $S^{\mathrm{I}}(L,\omega)$ in that $f^{\mathrm{I}}$ is replaced by $f^{\mathrm{II}}$, where $f^{\mathrm{I}}$ is usually taken as $\simeq 0.8$ and $f^{\mathrm{II}} \simeq 1$.

$$j_L(sr) = (\pi/2sr)^{1/2} J_{L+1/2}(sr) \,, \tag{3.14}$$

where $J_{L+1/2}$ is a Bessel function of the first kind, and

$$[j_L(sr)]_\alpha = \int R_{n_1 l_1} j_L(sr) R_{n_2 l_2} r^2 dr \,. \tag{3.15}$$

From (3.11) one obtains

$$U_\alpha^{\mathrm{I,II}}(L,\omega) = D_\alpha^{\mathrm{I,II}}(L,\omega)/D^{\mathrm{I,II}}(L,\omega) \,, \tag{3.16}$$

where $D^{\mathrm{I,II}}(L,\omega)$ is the determinant of the matrix $S^{\mathrm{I,II}}(L,\omega)$ and $D_\alpha^{\mathrm{I,II}}$ is the determinant of the matrix obtained from $S^{\mathrm{I,II}}$ by replacing the corresponding column with the left side of (3.11). The poles $\omega_S$ of $U^{\mathrm{I,II}}(L,\omega)$ are then determined from the equation

$$D^{\mathrm{I,II}}(L,\omega) = 0 \,, \tag{3.17}$$

and the residues of $U^{\mathrm{I,II}}(L,\omega)$ are equal to

$$\mathrm{Res}\, U_\alpha^{\mathrm{I,II}}(L,\omega)\big|_{\omega=\omega_S} = D_\alpha^{\mathrm{I,II}}(L,\omega_S) \bigg/ \frac{\partial D^{\mathrm{I,II}}(L,\omega)}{\hbar\,\partial\omega}\bigg|_{\omega=\omega_S} \tag{3.18}$$

## IV. ORDINARY MUON CAPTURE IN $Ca^{40}$

In this section we report for completeness the results obtained for ordinary $\mu$ capture in $Ca^{40}$. This calculation is closely related to that of Bunatyan.[12] The nucleon radial wave functions have been taken in the harmonic-oscillator model with the oscillator length parameter $b = (\hbar/M\omega)^{1/2} = 2.03$ F.

For the nucleus $Ca^{40}$, it is shown in earlier works,[12,22] that one can confine oneself with good accuracy (about 95%) to the few main transitions: $(1p\text{-}2p)$, $(1d\text{-}2d)$, and $(2s\text{-}3s)$ for monopole terms $(L=0)$; $(1d\text{-}1f)$, $(2s\text{-}2p)$, and $(1d\text{-}2p)$ for dipole terms $(L=1)$; and $(1d\text{-}1g)$, $(2s\text{-}2d)$, and $(1p\text{-}1f)$ for quadrupole terms $(L=2)$. The values for $\Delta\epsilon_\alpha$ and $C_\alpha^\beta$ are given in Table I.

Assuming also that $(dn/d\epsilon_0)^{-1}(1/4\pi) = 35$ $\mathrm{F}^3$ MeV we obtain, using the formulas of Bunatyan,[12] the results listed in Table II. We obtain for the $\mu$ capture probability the value $\Lambda^{\mu c}(Ca^{40}) = 22.07 \times 10^5$ $\mathrm{sec}^{-1}$. Taking into account that the relativistic and multipole terms with $L > 2$, which we have neglected, would contribute from 10 to 15%, this result is in good agreement with the experimental data

$$\Lambda^{\mu c}_{\mathrm{expt}} = 24.44 \pm 0.23 \times 10^5 \ \mathrm{sec}^{-1},^{23}$$
$$= 25.5 \ \pm 0.5 \ \ \times 10^5 \ \mathrm{sec}^{-1}.^{24}$$

Bunatyan[12] obtained $\Lambda^{\mu c}(Ca^{40}) = 22.7 \times 10^5$ $\mathrm{sec}^{-1}$. This slight difference is probably due to a slightly different choice of the parameters.

TABLE I. $\Delta\epsilon_\alpha$ and $C_\alpha^\beta$ for $L=0$, 1, and 2.

| $\Delta\epsilon_\alpha = 20.1$ MeV for $L=0$, 2; $\Delta\epsilon_\alpha = 10.07$ MeV for $L=1$; $C_\alpha^\beta$ $(\mathrm{F}^{-3})$ | | |
|---|---|---|
| $L=0$ | $L=1$ | $L=2$ |
| $C_{1p2p}^{1p2p} = 0.023$ | $C_{1d1f}^{1d1f} = 0.020$ | $C_{1d1g}^{1d1g} = 0.015$ |
| $C_{1d2d}^{1d2d} = 0.014$ | $C_{2s2p}^{2s2p} = 0.028$ | $C_{2s2d}^{2s2d} = 0.011$ |
| $C_{1p2p}^{1d2d} = 0.014$ | $C_{1d1f}^{2s2p} = 0.009$ | $C_{1d1g}^{2s2d} = 0.002$ |
| $C_{1d2d}^{2s3s} = 0.006$ | $C_{2s2p}^{1d2p} = 0.001$ | $C_{1p1f}^{1p1f} = 0.019$ |
| $C_{1p2p}^{2s3s} = 0.011$ | $C_{1d1f}^{1d2p} = -0.003$ | $C_{1d1g}^{1p1f} = 0.015$ |
| $C_{2s3s}^{2s3s} = 0.047$ | $C_{1d2p}^{1d2p} = 0.014$ | $C_{2s2d}^{1p1f} = 0.001$ |

## V. PHOTON SPECTRUM FOR RADIATIVE CAPTURE: NUMERICAL RESULTS AND DISCUSSION

In this section we give the numerical results for the photon spectrum and for the branching ratio $R$ in radiative $\mu$ capture in $Ca^{40}$. We will not report in detail the numerical calculations involved in the radiative photon spectrum, as this doubtless would be rather tedious for the reader. However, to give a brief account, we mention the essential points of the calculation. In evaluating the photon spectrum by Eq. (3.9), one has to take into account the fact that $G_r$, Res $U(L,\omega_S)$, and $[j_L(sr)]_\alpha$ are functions of $k$, $\omega_S$, and $\theta$ (the angle between

TABLE II. Contributions to the ordinary muon capture rate $\Lambda^{\mu c}$ from the various multipole terms $(0 \leq L \leq 2)$ and excitation energies $\hbar\omega_S$. I and II refer to Fermi and Gamow-Teller transitions, respectively. The values of $\hbar\omega_S$ are given in MeV, and $\Lambda_{LS}$ in $10^5$ $\mathrm{sec}^{-1}$.

| $L$ | $\hbar\omega_S^{\mathrm{I}}$ | $\Lambda_{LS}^{\mathrm{I}}$ | $\hbar\omega_S^{\mathrm{II}}$ | $\Lambda_{LS}^{\mathrm{II}}$ |
|---|---|---|---|---|
| 0 | 20.887 | 0.01 | 21.101 | 0.04 |
| | 22.286 | 0.02 | 22.805 | 0.06 |
| | 26.403 | 0.28 | 27.744 | 0.99 |
| 1 | 11.117 | 0.03 | 11.366 | 0.13 |
| | 11.577 | 0.04 | 11.922 | 0.13 |
| | 13.316 | 3.80 | 14.010 | 12.90 |
| 2 | 20.299 | 0.02 | 20.348 | 0.08 |
| | 20.699 | 0.08 | 20.847 | 0.29 |
| | 23.639 | 0.73 | 24.443 | 2.44 |

the photon and neutrino momentum).

The matrix elements $[j_L(sr)]_\alpha$ defined in Eq. (3.15) have been evaluated by means of the general formula

$$\int_0^\infty e^{-x^2} x^{2n+\mu+1} J_\mu(2x\sqrt{z})dx = \frac{n!}{2} e^{-z} z^{\mu/2} L_n^\mu(z), \quad (5.1)$$

where $L_n^\mu(z)$ are the Laguerre polynomials.

Using the calculated $[j_L(sr)]_\alpha$ one obtains, for every excitation frequency, the residues of $U(L,\omega_S)$ from Eq. (3.18). In order to derive the photon spectrum as a function of $k$ only, one has to remove the dependence on $\theta$, integrating over the photon solid angle expressions of the form $e^{-\zeta_S \cos\theta} \times P(\cos\theta)$, where $P(\cos\theta)$ is a polynomial in $\cos\theta$.[25] This integration is easily accomplished by means of the incomplete $\gamma$ functions $B_K(\zeta_S) = \int_{-1}^{+1} x^K e^{-\zeta_S x} dx$. We have evaluated the radiative photon spectrum and the radiative capture rate in $Ca^{40}$ with various values of the coupling constant $g_P$ ranging from $g_P = 7g_A$ to $g_P = 18g_A$. The dependence of the above-mentioned quantities on the variations of the other coupling constants $g_A$, $g_V$, and $g_m$ has been seen to be quite negligible by several authors,[1,3] and for this reason we have not taken it into account.

A central role in determining the branching ratio $R$ is played by the average value of the maximum photon energy $k_{\max}$, as a slight lowering of $k_{\max}$ would cause a noticeable decrease in the value of $R$. We obtain for $k_{\max}$ the value $k_{\max} = 90.5$ MeV. This value should be compared with the value $k_{\max} = (88 \pm 4)$ MeV of CDD[1] and with the value $k_{\max} = 91$ MeV determined by RT.[3]

The function $R(x) = N(x)/\Lambda^{\mu c}$ is plotted in Fig. 1 as a function of $x = k/k_{\max}$ for some values of $g_P$. Figure 1 shows that, as pointed out in previous calculations,[3,5] the shape of the spectrum is not very sensitive to $g_P$, and the main effect of varying $g_P$ is to increase or decrease the magnitude of the spectrum and of the branching ratio $R$.

In Table III we have listed, for several values of $g_P/g_A$, the results which we obtain for the ordinary muon capture rate $\Lambda^{\mu c}$ and for the branching ratio $R$. We also find that the dipole part amounts to 89% of total radiative rate, while it contributes 77% to the ordinary capture rate. As one may see from Table III, we obtain for the total radiative capture rate with $g_P/g_A = 7$ the value 493 sec$^{-1}$, which is very close to the value 519 sec$^{-1}$ obtained by Fearing[5] in the GDR model, and lower by about 40% than the value 841 sec$^{-1}$ which Fearing obtained in the closure-harmonic-oscillator calculation.[5] On the other hand, with the same coupling constants we obtain for the total ordinary capture rate the value $\Lambda^{\mu c} = 22.40 \times 10^5$ sec$^{-1}$, compared with the values $\Lambda^{\mu c}(\text{GDR}) = 30.4 \times 10^5$ sec$^{-1}$ and $\Lambda^{\mu c}(\text{closure}) = 38.9 \times 10^5$ sec$^{-1}$ obtained by Fearing.

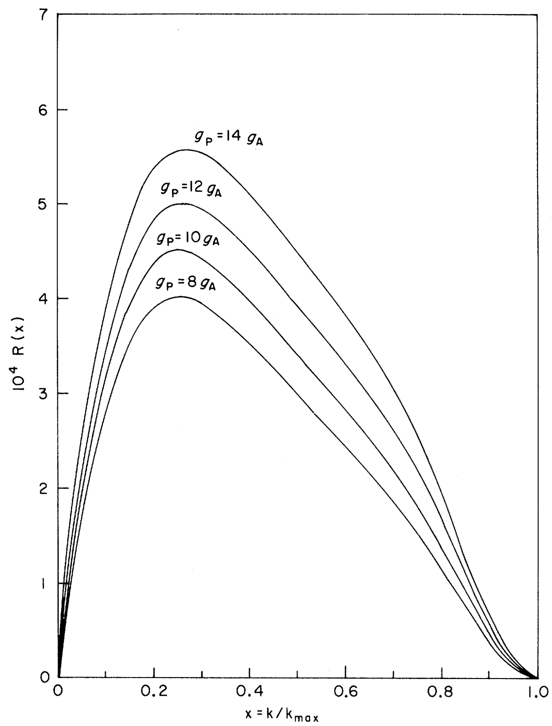


FIG. 1. Results for the photon spectra $R(x)$ for radiative muon capture, in comparison with the ordinary capture rate for $Ca^{40}$; $x = k/k_{\max}$; units are such that $\int_0^1 R(x)dx = \Lambda^{\mu c}_{\text{rad}}/\Lambda^{\mu c}$; $k_{\max} = 90.5$ MeV.

As we have obtained a reduction of about 40% for both total radiative and ordinary capture in comparison with the closure-harmonic-oscillator calculation, our value for the branching ratio $R$ is very close to that obtained in the closure-harmonic-oscillator calculation. This result confirms the prediction of RT[3] who indicated that the branch-

TABLE III. Theoretical values of the ordinary muon capture rate $\Lambda^{\mu c}$ and of the branching ratio $R$ of radiative-to-ordinary muon capture in $Ca^{40}$ as functions of $g_P/g_A$ values.

| $g_P/g_A$ | $\Lambda^{\mu c}$ (in $10^5$ sec$^{-1}$) | $10^4 \times R$ |
|---|---|---|
| 7 | 22.40 | 2.20 |
| 8 | 22.07 | 2.34 |
| 9 | 21.75 | 2.49 |
| 10 | 21.45 | 2.65 |
| 11 | 21.18 | 2.83 |
| 12 | 20.93 | 3.02 |
| 13 | 20.70 | 3.22 |
| 14 | 20.49 | 3.43 |
| 15 | 20.31 | 3.65 |
| 16 | 20.15 | 3.89 |
| Expt. | $24.44 \pm 0.23$[a] | $3.1 \pm 0.6$[c] |
| | $25.5 \pm 0.5$[b] | |

[a]See Ref. 23. [b]See Ref. 24. [c]See Ref. 1.

ing ratio $R$ is nearly independent of the nuclear model.

At this point we would like to compare our theoretical spectrum with the measured points by CDD. Unfortunately this is not possible, as the measured quantity is not the actual photon spectrum but a spectrum somewhat distorted by effects of the finite resolution of the counters used. However, we think that the comparison between our value of the branching ratio $R$ and the one given by CDD should give at least a general understanding of the agreement between the present theoretical calculation and the experimental data. Using the "standard" values for the other coupling constants we get good agreement between our results and the experimental data of CDD with

$$g_P = (12.4 \pm 2.8) g_A ,$$

where the error indicated is of experimental origin only and does not contain contributions from possible uncertainties of theoretical origin.

On the other hand, in a recent experimental work on muon capture in gaseous hydrogen Alberigi Quaranta *et al.*[26] obtained for the induced pseudoscalar coupling constant the value $g_P = (-7.3 \pm 3.7) g_V$. Moreover combining, by a least-squares-fit procedure, their value of $g_P$ together with other results obtained from $\mu$ capture in liquid hydrogen, they derived for $g_P$:

$$\bar{g}_P = (8.9 \pm 1.9) g_A ,$$

which is not incompatible with our range of values for $g_P$.

In conclusion, taking into account the experimental uncertainties, we find that our value of $g_P$ does not conflict with the more accepted values as much as the previous calculations do.

## VI. CIRCULAR POLARIZATION OF THE PHOTONS

In Fig. 2 the circular polarization of the photons

$$P_\gamma(x) = \frac{N^+(x) - N^-(x)}{N^+(x) + N^-(x)}$$

is plotted for some values of $g_P$ as a function of $x$. Here

$$N^\pm(x) = \int d\Omega_{\hat{k}} \int d\Omega_{\hat{\nu}} \, W(x, \hat{k}, \hat{p}_\nu, \lambda = \pm 1).$$

We find, with $g_P = 8 g_A$, that our values of $P_\gamma(x)$

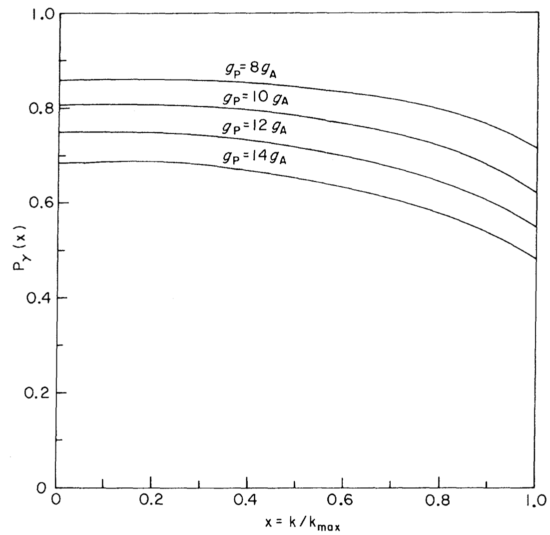


FIG. 2. Results for the degree of circular polarization $P_\gamma(x)$ of the photons emitted in radiative muon capture for $Ca^{40}$.

are greater than the RT[3] values about 15% in the high-energy range. In the low-energy range our values are in close agreement with those of RT. As RT pointed out, a measurement of $P_\gamma(x)$ would be very interesting in view of its dependence on $g_P$, and would provide further support to the correctness of the theory.

## VII. CONCLUSIONS

In this paper we have studied ordinary and radiative muon capture by $Ca^{40}$ taking into account the residual interaction between the nucleons using the finite Fermi-system theory. We obtain for both total radiative and ordinary capture a reduction of about 40% in comparison with the closure-harmonic-oscillator calculation. For this reason our value for the branching ratio $R$ is in good agreement with the RT value, while it is somewhat different from Fearing's value obtained taking into account the residual interaction in the GDR model.

As we get good agreement with the experimental data of CDD for $g_P = (12.4 \pm 2.8) g_A$, we think that the disagreement between our result and the more currently accepted values of $g_P$ is not so large to call into question the theoretical mechanism of radiative capture if the large experimental uncertainties are taken into account.

## APPENDIX

Here we give the following expression for $G_r^{\mathrm{I}}$ and $G_r^{\mathrm{II}}$ in terms of the quantities defined in Eqs. (2.3):

$$G_r^{\mathrm{I}}(\lambda = -1) = (\tfrac{1}{2} C^2 + F^2 - CF) - (\hat{p}_\nu \cdot \hat{k}) F^2 + (\hat{p}_\nu \cdot \hat{k})^2 (CF - \tfrac{1}{2} C^2) ,$$

$$G_r^{\mathrm{I}}(\lambda=+1)=(f+m)^2+\tfrac{1}{2}(d+e)^2+b^2-b(f+m)-(f+m)(d+e)+(\hat{p}_\nu\cdot\hat{k})(f+m-b)^2$$
$$+(\hat{p}_\nu\cdot\hat{k})^2\left[(f+m)(d+e)-\tfrac{1}{2}(d+e)^2-b(f+m)\right],$$

$$G_r^{\mathrm{II}}(\lambda=-1)=A\left(A+\frac{p_\nu B}{k}-D\right)+B\left(\frac{B}{2}-C-E+\frac{p_\nu{}^2B}{2k^2}\right)+C\left(\frac{C}{2}+E\right)+D\left(D+G-\frac{p_\nu B}{k}\right)+E^2+G^2+(\hat{p}_\nu\cdot\hat{k})$$
$$\times\left[2A(C-G)+B\left(\frac{p_\nu B}{k}-D-\frac{p_\nu E}{k}-\frac{p_\nu G}{k}\right)-(D-E)^2\right]+(\hat{p}_\nu\cdot\hat{k})^2\left[B\left(-\frac{p_\nu A}{k}-\frac{B}{2}+C+E+\frac{p_\nu D}{k}-\frac{p_\nu{}^2B}{2k^2}\right)\right.$$
$$\left.+C\left(\frac{C}{2}-E-2G\right)+D(A-2E-G)\right]+(\hat{p}_\nu\cdot\hat{k})^3B\left(D+\frac{p_\nu E}{k}+\frac{p_\nu G}{k}-\frac{p_\nu{}^2B}{k^2}\right),$$

$$G_r^{\mathrm{II}}(\lambda=+1)=c\left(c+\frac{p_\nu B}{k}-g-l\right)+e\left(\frac{e}{2}-m+B-h\right)+g\left(g+m-a+l-\frac{p_\nu B}{k}\right)+h(h+2m-l-B)+a(a-2m+l)$$
$$+l\left(\frac{3l}{2}-2m-\frac{p_\nu B}{k}\right)+2m^2+B\left(\frac{B}{2}+\frac{p_\nu{}^2B}{2k^2}-m\right)+(\hat{p}_\nu\cdot\hat{k})\left[2c(m-a-e)+g(g+2h+2m-B)\right.$$
$$\left.+h(h+2m)+B\left(-\frac{p_\nu h}{k}-\frac{p_\nu a}{k}-l+\frac{p_\nu B}{k}\right)+m^2\right]+(\hat{p}_\nu\cdot\hat{k})^2\left[a(2e+g-l)+c\left(g+l+\frac{p_\nu B}{k}\right)\right.$$
$$\left.+e\left(\frac{e}{2}+h-m-B\right)+g\left(2h-l+m+\frac{p_\nu B}{k}\right)+h(l+B)+l\left(-\frac{3l}{2}+2m\right)+B\left(\frac{p_\nu l}{k}+m-\frac{B}{2}-\frac{p_\nu{}^2B}{2k^2}\right)\right]$$
$$+(\hat{p}_\nu\cdot\hat{k})^3B\left(g+l+\frac{p_\nu a}{k}+\frac{p_\nu h}{k}-\frac{p_\nu B}{k}\right).$$

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$$|\varphi_\mu|^2=|\varphi_\mu^0|^2\,Z_{\mathrm{eff}}^4/Z^4,$$

where

$$|\varphi_\mu^0|^2=(1/\pi)(m_\mu e^2 Z/\hbar^2)^3\ .$$

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