Spurious solutions in few-body equations

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After Faddeev and Yakubovskii showed how to write connected few-body equations which are free from discrete spurious solutions various authors have proposed different connected few-body scattering equations. Federbush first pointed out that Weinberg's formulation admits the existence of discrete spurious solutions. In this paper we investigate the possibility and consequence of the existence of spurious solutions in some of the few-body formulations. Contrary to a proof by Hahn, Kouri, and Levin and by Bencze and Tandy the channel coupling array scheme of Kouri, Levin, and Tobocman which is also the starting point of a formulation by Hahn is shown to admit spurious solutions. We can show that the set of six coupled four-body equations proposed independently by Mitra, Gillespie, Sugar, and Panchapakesan, by Rosenberg, by Alessandrini, and by Takahashi and Mishima and the seven coupled four-body equations proposed by Sloan are related by matrix multipliers to basic sets which correspond uniquely to the Schrödinger equation. These multipliers are likely to give spurious solutions to these equations. In all these cases spuriosities are shown to have no hazardous consequence if one is interested in studying the scattering problem.

[NUCLEAR REACTIONS Scattering theory, spurious solutions in three- and four-body equations.]

I. INTRODUCTION

After Faddeev[1] and Yakubovskii[2] proposed n-body equations with a two-body connected kernel, which becomes n-body connected after a minimal number of iterations, various authors[3-8] suggested different types of equations. The kernels of some of the other formulations are two-body connected[8] and become fully connected after a certain number of iterations, which may exceed the minimal number. Also there are formulations[3-6] with kernels of higher connectivity. (We shall call in this paper a kernel that becomes connected after a certain number of iterations a CAI kernel.) But Faddeev's and Yakubovskii's equations have rightfully become very popular because of the elegance and sophistication of their formulations. Another important feature of their formulations is that they are completely equivalent to the underlying Schrödinger equation.

The existence of a discrete set of square-integrable solutions at real or complex energies in any formulation over and above the ones predicted by the Schrödinger theory is called spuriosity. We shall not have to be concerned with spuriosities due to continuum states. This is because throughout the paper we shall always encounter the following situation. The scattering problem is given by an equation or set of equations of the type

$$(1-K)\psi=\phi, \tag{1.1}$$

which is in unique correspondence to the Schrödinger equation. The kernel is disconnected and noncompact. Suppose we find a multiplier $(1+M)$ such that $(1+M)\,(1-K)=(1-K_c)$, where K_c is a CAI kernel and hence compact. Both M and K and hence K_c depend on the complex energy variable z. Then multiplying Eq. (1.1) by $(1+M)$ from the left we get

$$(1-K_c)\psi=(1+M)\phi\,. \tag{1.2}$$

In general the multiplier $(1+M)$ alone will annihilate states which belong to the continuous and discrete spectrum of M. But since K_c is assumed to be compact, the continuum states will not show up as spuriosities in Eq. (1.2). So throughout the rest of the paper we shall be restricted to the consideration of discrete spurious solutions defined by

$$[1+M(z)]\xi(z)=0. \tag{1.3}$$

Weinberg[3] proposed a connected kernel method for the n-body problem by decomposing the full resolvent operator into terms of increasing connectivity. This then leads to a single equation for the full wave function, in contrast to a set of coupled equations for the Faddeev-Yakubovskii components. Federbush[9] was the first to point out that the homogeneous part of that equation allows discrete spurious solutions. This problem was later studied by Newton,[10] who showed that these spurious discrete solutions arise due to a factorization of the Weinberg kernel. So it may be hazardous to use this equation in the calculation of bound states. Since then, to the best of our knowledge, consequences of these spurious solutions on scattering states have not been studied. Also the Weinberg decomposition has not attracted too much attention, though essentially the same formulation has been recovered more recently[11] in a different guise under the name "cluster decomposition" which is the basic tool in the derivation of the Bencze-Redish (BR) equation[6]—a generalization of the four-body Sloan equation[5] to the n-body system.

In this paper we study the general question of the applicability of the few-body equations with discrete spurious solutions to the analysis of scattering. For simplicity we restrict ourselves to the study of three- and four-body equations. But our conclusions can be generalized to more than four particles. To get some insight into the problem we first study the three-body equations related to the Weinberg[3] kernel. The homogeneous equation with the Weinberg kernel has discrete spurious solutions on the first sheet of the complex energy plane. If one is interested in scattering, one should look into the consequences of these discrete spurious states on transition amplitudes. It is shown that the homogeneous coupled set of equations for the transition operators does not have any spuriosity at the energies where the homogeneous equation for the wave function has discrete spurious solutions. We find that the wave function spuriosity has disappeared in the coupled set of equations for the transition amplitude which surprisingly has developed a new spuriosity. These three-body connected equations for the transition operators are easily recognized to be the once iterated Alt-Grassberger-Sandhas (AGS) equations.[12] Hence these equations for the transition operators will have exactly the same solution as the AGS equations because the two sets of equations differ by a multiplier $(1+M)=(1+K)$ which can be canceled. Here K is the kernel of the AGS equations. Though the homogeneous part of the coupled equations for the transition amplitudes have spuriosities where $(1+K)$ has its discrete spectrum, it will not show up in the physical scattering problem.

Next we study a different multichannel formulation of the n-body problem due to Kouri, Levin, and Tobocman (KLT).[8] This has been shown[13] to be equivalent to the fundamental equation of an independent formulation of Hahn.[7] Contrary to a "proof" by Hahn, Kouri, and Levin[14] and confirmation by Bencze and Tandy[15] we show that the channel coupling array scheme of KLT leads to discrete spurious solutions in the first sheet of the complex energy plane. Again for simplicity we restrict ourselves to the three-channel three-body problem. The equations satisfied by the KLT transition amplitudes are related to the AGS equations[12] by a multiplier which is responsible for spurious solutions. Nevertheless the spurious multiplier cancels from both sides of the KLT equations and the solutions for the KLT transition operators are exactly equal to the solutions of the AGS equations. Hence the equations can be used in the study of the scattering problems.

Then we consider the six coupled four-body equations for the t matrix proposed independently by Mitra, Gillespie, Sugar, and Panchapakesan, by Rosenberg, by Alessandrini, and by Takahaski and Mishima.[4] These equations have a CAI kernel and were needed because the simple Faddeev equations for the four-body t matrices have a disconnected or noncompact kernel. These equations of Ref. 4 are shown to be related to the simple Faddeev equations by a spurious matrix multiplier which is likely to give rise to spurious discrete solutions. However, this will not show up in the solution of the scattering problem.

Next we apply the same considerations to the seven coupled four-body equations for the transition amplitudes proposed by Sloan.[5] We again find out that this is also likely to have discrete spurious solutions which will not affect the scattering problem. This is contrary to a proof by Bencze and Tandy[15] who essentially followed and generalized the proof by Hahn, Kouri, and Levin[14] to a wider class of equations. This proof by Bencze and Tandy has the same defect as the one in Ref. 14. Also, their proof does not apply to Sloan equations, since Eq. (9) with choices (10) and (12) of Ref. 15 is not the Sloan or the Bencze-Redish set of equations as claimed in Ref. 15. In the case of four particles, for instance, it still contains transition operators of the form $T^{d\alpha}\cdots$ where d refers to a three-body fragmentation channel and α refers to a two-body fragmentation channel. It is only after the insertion of the explicit expressions for these transition operators that the disconnected pieces in the kernel can be shown to cancel. This type of insertion is not an identical rewriting of the equations and as we shall see in the text it may give rise to additional spuriosities.

Due to the cancellations of the spurious multipliers all these few-body equations with spurious solutions can be safely used to study the scattering problem. This was already known in some circles[16] in a different guise. Of course special numerical precautions might be needed if this spurious solution occurs at or very near the physical scattering energy.

In Sec. II we discuss the three-body problem in general. In Sec. II A1 we consider the problem of spuriosities in the scattering equations with the Weinberg kernel.[3] In Sec. II A2 the spuriosities in the coupled set of equations with the Weinberg kernel for the transition operator are considered. In Sec. II B we show that the channel coupling array scheme proposed in KLT (Ref. 8) allows for the existence of spurious solutions. In Sec. III we generalize our results to the case of four particles and show that the scattering equations of Refs. 4 and 5 are also likely to have spurious solutions.

Finally in Sec. IV we give a summary and some concluding remarks.

II. THREE-BODY PROBLEM

The three-body problem within the framework of the Schrödinger equation has been given a unique formulation by Faddeev. The set of three coupled integral equations for the Faddeev amplitudes is at the same time an immediate consequence of the set of three basic Lippmann-Schwinger equations[17] which define the Schrödinger state vector for scattering uniquely. However, the basic set of equations allows us to write another set of three coupled integral equations which is in unique correspondence to the Schrödinger equation. This set was first proposed[17,18] by one of the present authors (WG) and was later recovered[19] in the framework of the channel-coupling array scheme as the so-called Faddeev or the Faddeev-Lovelace choice.

There are other formulations for solving the three-body problem where the kernel or an iterate of it is connected. We shall investigate spuriosities in one of the other interesting formulations by Weinberg[3] not only for the state vectors but also for the transition operators. The set of three coupled equations for transition operators especially is of interest. We show that the spuriosities which appear in the single equation for the state vector disappear for the coupled set of equations for the transition operators and new types of spuriosities occur. Since the algebra for three particles is simple and transparent we shall exhibit these mathematical structures in detail. This knowledge will help us to understand the more complex situation in the four-body case. We shall also consider the channel-coupling array scheme of Ref. 8 and contrary to a proof in Refs. 14 and 15 we show that this scheme is not in unique correspondence to the Schrödinger equation.

A. Weinberg-type equations

1. Scattering states:

Let us begin our discussion with the equations which have three-body connected Weinberg-type kernels. Instead of using a cluster decomposition[11] of the full resolvent to derive these equations we make use of various Lippmann-Schwinger equations corresponding to different subsystem resolvent operators. This is an equivalent and intimately related procedure and applicable in the same manner for an arbitrary number of particles.

Our notation is the usual one. The indices μ, ν, α, β, λ will be used to denote a pair. G_0, G_μ, and G are the resolvent operators to the free Hamiltonian H_0, the channel Hamiltonian $H_\mu = H_0 + V_\mu$, $\mu = 1, 2, 3$, and the full Hamiltonian $H = H_0 + V$, respectively, and satisfy $G_\mu = G_0 + G_0 V_\mu G_\mu$ and $G = G_0 + G_0 V G$, where V is the total interaction and V_μ is a pair interaction. We shall use also the channel interaction $\overline{V}_\mu \equiv V - V_\mu$. In this paper the energy dependence of the energy dependent operators will not in general be explicitly shown.

The channel eigenstates are products of a two-body bound state and a momentum eigenstate of relative motion between the spectator particle and the pair α and are denoted by ϕ_α. The scattering state $\Psi_\alpha^{(+)}$ which refer to the initial channel α is defined by

$$\Psi_\alpha^{(+)} = i\epsilon G \phi_\alpha \ , \tag{2.1}$$

in the limit when $\epsilon \to 0$. Here $G = (z - H)^{-1}$ where $z = E + i\epsilon$. The scattering states obey the Lippmann Schwinger equations

$$\Psi_\alpha^{(+)} - G_0 \sum_\mu V_\mu \Psi_\alpha^{(+)} = i\epsilon G_0 \phi_\alpha \tag{2.2}$$

and

$$\Psi_\alpha^{(+)} - G_\mu \sum_{\nu \neq \mu} V_\nu \Psi_\alpha^{(+)} = i\epsilon G_\mu \phi_\alpha \ , \tag{2.3}$$

where $G_0 = (z - H_0)^{-1}$ and $G_\mu = (z - H_\mu)^{-1}$ with $z = E + i\epsilon$.

Inserting (2.3) into (2.2) yields

$$\Psi_\alpha^{(+)} - G_0 \sum_\mu V_\mu G_\mu \sum_{\nu \neq \mu} V_\nu \Psi_\alpha^{(+)} = \left(1 + \sum_\mu G_\mu V_\mu\right) i\epsilon G_0 \phi_\alpha \tag{2.4}$$

which contains the three-body connected Weinberg kernel. This derivation of Eq. (2.4) immediately reveals the spurious multiplier. If we compare the right-hand sides of Eqs. (2.2) and (2.4) we can read off the factorization property

$$\left(1 - G_0 \sum_\mu V_\mu G_\mu \sum_{\nu \neq \mu} V_\nu\right) = \left(1 + \sum_\mu G_\mu V_\mu\right) \times \left(1 - G_0 \sum_\nu V_\nu\right). \tag{2.5}$$

This was discovered by Newton[10] who showed that the first factor on the right in Eq. (2.5) was responsible for the discrete spurious solutions of the homogeneous Weinberg equation found by Federbush.[9] We shall consider the spurious eigenvalue problem in detail. The spurious states which are not eigenstates of the Schrödinger equation are given by

$$\left(1 + \sum_\mu G_\mu V_\mu\right) \chi = 0. \tag{2.6}$$

This eigenvalue problem has a disconnected or noncompact kernel. The simplest way to have a calculational scheme for Eq. (2.6) is to decompose it into Faddeev-type equations with a CAI kernel. This is easily done by introducing the components χ_μ defined by

$$\chi = -\sum_\mu G_0 t_\mu \chi = \sum_\mu \chi_\mu \ , \qquad (2.7)$$

or

$$\chi_\mu = -G_0 t_\mu \sum_\nu \chi_\nu \ , \qquad (2.8)$$

or

$$(1+G_0 t_\mu)\chi_\mu = -G_0 t_\mu \sum_{\nu\neq\mu} \chi_\nu \ ,$$

or

$$G_\mu G_0^{-1}\chi_\mu = -G_\mu V_\mu \sum_{\nu\neq\mu} \chi_\nu ,$$

or

$$\chi_\mu = -G_0 V_\mu \sum_{\nu\neq\mu} \chi_\nu \ . \qquad (2.9)$$

These are the "spurious" Faddeev equations which are to be contrasted with the physical ones[1]

$$\psi_\mu = G_0 t_\mu \sum_{\nu\neq\mu} \psi_\nu \ . \qquad (2.10)$$

In the special case when $V_3 = 0$ Eq. (2.9) reduces to

$$\begin{aligned} &\chi_3 = 0, \\ &\chi_1 = -G_0 V_1 \chi_2, \qquad (2.11) \\ &\chi_2 = -G_0 V_2 \chi_1. \end{aligned}$$

Now putting the mass $m_3 = \infty$ we arrive at the "physical" situation for which Federbush[9] has proved the existence of discrete spurious energy eigenvalues.

We would like to add the remark that we see no reason to exclude the possibility that the spurious eigenvalues may also occur by accident for real energies above the lowest scattering threshold. The spurious eigenvalue problem does not appear to be related to a flux conserving theory such as the Schrödinger equation. Boundary conditions such as absence of incoming waves rule out discrete solutions at real energies above the lowest scattering threshold in the Schrödinger theory. These considerations do not possibly apply in the spurious problem. Anyway they will show up only at discrete real energies above the lowest scattering threshold and can and should always be avoided in calculating cross sections. In the rest of the paper we shall exclude the accidental occurrence of discrete spurious solutions at real energies in the scattering region.

After this remark let us come back to Eq. (2.4). Clearly the spurious pole of the inverse operator to Eq. (2.5) will cancel in Eq. (2.4) due to the presence of the same spurious multiplier in the inhomogeneous term. In the limit $\epsilon \to 0$ the right-hand side of Eq. (2.4) obviously simplifies to ϕ_α and we have

$$\Psi_\alpha^{(+)} - G_0 \sum_\mu V_\mu G_\mu \sum_{\nu\neq\mu} V_\nu \Psi_\alpha^{(+)} = \phi_\alpha . \qquad (2.12)$$

Now let us consider the off-shell continuation of Eq. (2.12) for arbitrary complex energy z in the first sheet of the complex energy plane defined by

$$\Psi_\alpha(z) - G_0(z) \sum_\mu V_\mu G_\mu(z) \sum_{\nu\neq\mu} V_\nu \Psi_\alpha(z) = \phi_\alpha . \ (2.13)$$

Ψ_α will have unphysical spurious poles in the first sheet of the complex energy plane in contrast to the solution $\Psi_\alpha^{(+)}$ of Eq. (2.4). On the real energy axis above the lowest scattering threshold, however, $\Psi_\alpha = \Psi_\alpha^{(+)}$. We shall demonstrate these facts for a slightly modified equation which results if we insert Lippmann-Schwinger equations of type (2.3) into each other. The new equation we get can be written as

$$\Psi_\alpha^{(+)} - G_\mu \sum_{\nu\neq\mu} V_\nu G_\nu \sum_{\lambda\neq\nu} V_\lambda \Psi_\alpha^{(+)} = \Big(1 + \sum_{\nu\neq\mu} G_\mu V_\nu G_\nu G_\mu^{-1}\Big) i\epsilon G_\mu \phi_\alpha . \qquad (2.14)$$

By comparing Eq. (2.14) with Eq. (2.3) we can again read off the following factorization property:

$$\Big(1 - G_\mu \sum_{\nu\neq\mu} V_\nu G_\nu \sum_{\lambda\neq\nu} V_\lambda\Big) = \Big(1 + \sum_{\nu\neq\mu} G_\mu V_\nu G_\nu G_\mu^{-1}\Big) \times \Big(1 - G_\mu \sum_{\lambda\neq\mu} V_\lambda\Big). \qquad (2.15)$$

Taking the limit $\epsilon \to 0$ in the right-hand side of Eq. (2.14) we get

$$\Psi_\alpha^{(+)} - G_\mu \sum_{\nu\neq\mu} V_\nu G_\nu \sum_{\lambda\neq\nu} V_\lambda \Psi_\alpha^{(+)} = (\delta_{\mu\alpha} + \bar\delta_{\mu\alpha} G_\mu G_0^{-1})\phi_\alpha , \qquad (2.16)$$

where $\bar\delta_{\mu\alpha} = (1-\delta_{\mu\alpha})$. Now let us consider the off-shell continuation of Eq. (2.16) for arbitrary complex energy z in the first sheet of the complex energy plane defined by

$$\Psi_\alpha(z) - G_\mu(z) \sum_{\nu\neq\mu} V_\nu G_\nu(z) \sum_{\lambda\neq\nu} V_\lambda \Psi_\alpha(z) = \Big(\delta_{\mu\alpha} + \bar\delta_{\mu\alpha} G_\mu(z) G_0^{-1}(z)\Big)\phi_\alpha \ . \qquad (2.17)$$

The solution Ψ_α to Eq. (2.17) will have nonphysical spurious states in the first sheet of the complex energy plane defined by the eigenvalue problem

$$L_\mu \chi^\mu \equiv \Big(1 + \sum_{\nu\neq\mu} G_\mu V_\nu G_\nu G_\mu^{-1}\Big)\chi^\mu = 0. \qquad (2.18)$$

We use Eq. (2.15) to write the formal solution of Eq. (2.17) as

$$\Psi_\alpha(z)=G\left(1+\sum_{\nu\neq\mu} V_\nu G_\nu\right)^{-1} G_\mu^{-1} \times(\delta_{\mu\alpha}+\bar{\delta}_{\mu\alpha}G_\mu G_0^{-1})\phi_\alpha\ , \quad (2.19)$$

or

$$\Psi_\alpha(z)=G\left(1+\sum_{\nu\neq\mu} V_\nu G_\nu\right)^{-1}[\delta_{\mu\alpha}(z-E)+\bar{\delta}_{\mu\alpha}G_0^{-1}]\phi_\alpha . \quad (2.20)$$

We now distinguish between $\mu=\alpha$ and $\mu\neq\alpha$. For $\mu=\alpha$ we get

$$\Psi_\alpha(z)=(z-E)G\left(1+\sum_{\nu\neq\alpha} V_\nu G_\nu\right)^{-1} \times\left(1+\sum_{\lambda\neq\alpha}V_\lambda G_\lambda-\sum_{\lambda\neq\alpha}V_\lambda G_\lambda\right)\phi_\alpha\ , \quad (2.21)$$

or

$$\Psi_\alpha(z)=(z-E)G\phi_\alpha-(z-E)G\left(1+\sum_{\nu\neq\alpha} V_\nu G_\nu\right)^{-1} \times\sum_{\lambda\neq\alpha}V_\lambda G_\lambda\phi_\alpha \quad (2.22)$$

The second term contains the square integrable function $\sum_{\lambda\neq\alpha}V_\lambda G_\lambda\phi_\alpha$ and allows for the occurrence of the spurious pole due to the inverse operator. For real energies above the lowest scattering threshold, however, $z\to E+i\epsilon$ and the spurious component of $\Psi_\alpha(z)$ vanishes and the physical solution given by Eq. (2.1) survives. In the case $\mu\neq\alpha$, Eq. (2.19) becomes

$$\Psi_\alpha(z)=G\left(1+\sum_{\nu\neq\mu} V_\nu G_\nu\right)^{-1} G_0^{-1}\phi_\alpha\ , \quad (2.23)$$

or

$$\Psi_\alpha(z)=G(G_0+G_0V_\alpha G_\alpha+G_0V_\lambda G_\lambda)^{-1}\phi_\alpha, \quad \mu\neq\lambda\neq\alpha. \quad (2.24)$$

Using the resolvent identity

$$G_\alpha=G_0+G_0V_\alpha G_\alpha\ , \quad (2.25)$$

Eq. (2.24) becomes

$$\Psi_\alpha(z)=G(G_\alpha+G_0V_\lambda G_\lambda)^{-1}\phi_\alpha\ , \quad (2.26)$$

which may be rewritten as

$$\Psi_\alpha(z)=G(1+G_\alpha^{-1}G_\lambda V_\lambda G_0)^{-1}G_\alpha^{-1}\phi_\alpha\ . \quad (2.27)$$

It is easy to see that Eq. (2.27) can be rewritten as

$$\Psi_\alpha(z)=(z-E)G(1+G_\alpha^{-1}G_\lambda V_\lambda G_0)^{-1} \times[1+G_\alpha^{-1}G_\lambda V_\lambda G_0-G_\alpha^{-1}G_\lambda V_\lambda G_0]\phi_\alpha, \quad (2.28)$$

or

$$\Psi_\alpha(z)=(z-E)G\phi_\alpha-(z-E)G(1+G_\alpha^{-1}G_\lambda V_\lambda G_0)^{-1} \times G_\alpha^{-1}G_\lambda V_\lambda G_0\phi_\alpha, \quad (2.29)$$

where $\lambda\neq\alpha$ and $\lambda\neq\mu$. Again the inverse operator in the parentheses acts on a square integrable function at complex energy z allowing for the occurrence of spurious poles and again on the real axis the physical solution survives.

In a case where the spurious pole at the energy E_{sp} occurs very near to the three-body center-of-mass energy the unwanted spurious admixture for Ψ_α which has the factor $(z-E)/(z-E_{sp})$ may require special caution in certain numerical treatments.

2. Transition operators:

Let us now discuss coupled sets of equations for transition operators. We use Eq. (2.16) to derive the coupled set of equations for the transition operators. We may therefore expect that the spuriosities introduced by the three multiplier L_μ of Eq. (2.18) show up again. This is, however, not the case. The spuriosities defined by Eq. (2.18) will not appear in the equation for transition operators but a different spuriosity will show up.

We define the three-body t matrices by

$$T_\mu\phi_\alpha=V_\mu\Psi_\alpha^{(+)}\ . \quad (2.30)$$

We operate by V_μ on Eq. (2.16) from the left and derive the coupled set of equations for the t matrices given by

$$T_\mu\phi_\alpha-V_\mu G_\mu\sum_{\nu\neq\mu}V_\nu G_\nu\sum_{\lambda\neq\nu}T_\lambda\phi_\alpha =V_\mu(\delta_{\mu\alpha}+\bar{\delta}_{\mu\alpha}G_\mu G_0^{-1})\phi_\alpha \quad (2.31)$$

It is easy to see that the "spurious" transition operators

$$\tau_\mu^\beta=V_\mu\chi^\beta,\quad \mu=1,2,3 \quad (2.32)$$

corresponding to the three types of spurious states of Eq. (2.18) do not fulfill the homogeneous set of equations corresponding to Eq. (2.31). The operators τ_μ^β will obviously fulfill the row μ but not the rows $\nu\neq\mu$. Now if we compare Eq. (2.31) with the Faddeev equations[1]

$$T_\mu\phi_\alpha-V_\mu G_\mu\sum_{\nu\neq\mu}T_\nu\phi_\alpha=\delta_{\mu\alpha}V_\mu\phi_\alpha\ , \quad (2.33)$$

it is obvious that Eq. (2.31) is identical with the first iterate of Eq. (2.33). Thus in matrix notation Eq. (2.33) and (2.31) can be written as

$$(1-K_F)T=T^0 \tag{2.34}$$

and

$$(1-K_F{}^2)T=(1+K_F)T^0, \tag{2.35}$$

where K_F is the Faddeev kernel, T represents the Faddeev components defined by Eq. (2.30), and T^0 is the inhomogeneous term on the right-hand side of Eq. (2.33). Equation (2.35) allows us to read off the spurious multiplier as $(1+K_F)$ and the spurious eigenvalue problem is defined by

$$(1+K_F)T_{sp}=0\ . \tag{2.36}$$

Thus the spurious solution T_{sp} is just the eigenvector of the Faddeev kernel with eigenvalue -1. These (repulsive) eigenvalues may occur only at negative energies below the lowest threshold. Hence the homogeneous system corresponding to Eq. (2.31) will have spurious solutions on the real axis below the lowest scattering threshold. Again it is obvious from Eq. (2.35) that due to cancellation no effect will show up in the solution of the full equation (2.31).

The situation for the physical transition operators U defined by[21]

$$U^{\lambda\alpha}\phi_\alpha\equiv\sum_{\mu\neq\lambda}V_\mu\psi_\alpha^{(+)}\equiv\sum_{\mu\neq\lambda}T_\mu\phi_\alpha \tag{2.37}$$

is very similar. From Eq. (2.31) we get

$$U^{\lambda\alpha}-\sum_{\mu\neq\lambda}V_\mu G_\mu\sum_{\nu\neq\mu}V_\nu G_\nu U^{\nu\alpha}=\sum_{\mu\neq\lambda}V_\mu(\delta_{\mu\alpha}+\bar{\delta}_{\mu\alpha}G_\mu G_0^{-1}). \tag{2.38}$$

Equation (2.38) should have the same solution as the AGS equations[12]

$$U^{\lambda\alpha}-\sum_{\mu\neq\lambda}V_\mu G_\mu U^{\mu\alpha}=\bar{\delta}_{\lambda\alpha}G_0{}^{-1}. \tag{2.39}$$

Again a glance at Eq. (2.38) shows that is equal to the once iterated AGS equation. In matrix notation Eq. (2.38) is

$$(1+K_{AGS})(1-K_{AGS})U=(1+K_{AGS})U^0, \tag{2.40}$$

where U is defined by Eq. (2.37), K_{AGS} is the AGS kernel, and U^0 has components defined by $U_{\mu\alpha}^{0AGS}=\bar{\delta}_{\mu\alpha}G_0^{-1}$. Thus again spuriosity arises due to an eigenvalue -1 of K_{AGS}, which occurs at the same energies as those for the Faddeev kernel K_F.

B. Channel coupling array scheme

In this subsection we would like to discuss another method to attack the n-body problem—the channel coupling array scheme.[8] It has been claimed that this scheme is free of spuriosities.[14,15] We shall show that this statement is wrong. Nevertheless as in the examples discussed above effects of the spuriosities cancel in the solutions for scattering problems.

The scattering states are defined uniquely by the basic set of Lippmann-Schwinger equations

$$\Psi_\alpha^{(+)}=i\epsilon G_\mu\phi_\alpha+G_\mu\sum_{\nu\neq\mu}V_\nu\Psi_\alpha^{(+)},\quad \mu=1,2,3 \tag{2.41}$$

where in the limit $\epsilon\to 0$ two of the three equations are homogeneous and are necessary to specify the boundary conditions. Still, Eq. (2.41) has a noncompact kernel. One may replace this uncoupled set of three equations by coupled sets of equations with a CAI kernel. One choice appears to be very natural from the following practical point of view. The nuclear two-body interactions are dominant in specific low angular momentum states. Hence we may assume that V_ν acts only in angular momentum states $l_\nu\leq l_\nu^{max}$. Similarly for a given total angular momentum L the orbital angular momentum l_ν' of the third particle with respect to the pair ν also assumes only a finite number of values. Then the projection operator onto a subset of angular momentum states P_ν^L is defined by

$$P_\nu^L\equiv\sum_{\substack{l_\nu\leq l_\nu^{max}\\ l_\nu'}}|(l_\nu l_\nu')L\rangle\langle(l_\nu l_\nu')L|. \tag{2.42}$$

The projection of V_ν and $\Psi_\alpha^{(+)}$ onto these states is defined by $\hat{V}_\nu=P_\nu^L V_\nu P_\nu^L$ and $\Psi_{\nu\alpha}^{(+)}=P_\nu^L\Psi_\alpha^{(+)}$, respectively. Then we are naturally led from the basic set (2.41) to the following coupled set of equations (we take $\alpha=1$ as an example):

$$\begin{aligned}\psi_{11}&=P_1^L\phi_1+P_1^L G_1(\hat{V}_2\psi_{21}+\hat{V}_3\psi_{31}),\\ \psi_{21}&=P_2^L G_2(\hat{V}_3\psi_{31}+\hat{V}_1\psi_{11}),\\ \psi_{31}&=P_3^L G_3(\hat{V}_1\psi_{11}+\hat{V}_2\psi_{21}).\end{aligned} \tag{2.43}$$

Equation (2.43) has been proposed[17] and studied[18] by one of the present authors (WG). Now if we let $l_\nu^{max}\to\infty$ and sum Eqs. (2.42) and (2.43) over all L we obtain $\hat{V}_\mu=V_\mu$. Retaining the redundant $\Psi_{\nu 1}$ notation we have from Eq. (2.43)

$$\begin{pmatrix}\psi_{11}\\ \psi_{21}\\ \psi_{31}\end{pmatrix}=\begin{pmatrix}\phi_1\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0 & G_1V_2 & G_1V_3\\ G_2V_1 & 0 & G_2V_3\\ G_3V_1 & G_3V_2 & 0\end{pmatrix}\begin{pmatrix}\psi_{11}\\ \psi_{21}\\ \psi_{31}\end{pmatrix}. \tag{2.44}$$

Equation (2.44) has been recovered later under the context of the channel array coupling scheme.[20] Equation (2.44) has the property that the only solution to the set is $\psi_{11}=\psi_{21}=\psi_{31}=\Psi_1^{(+)}$ as has been nicely demonstrated by Sandhas.[19]

There are, however, other possibilities for rewriting the basic set of equations (2.41) into a coupled set of equations with a CAI kernel. KLT (Ref. 8) proposed a general coupling scheme for transition amplitudes, which can be rewritten in our language for a particular choice of coupling as

$$\begin{pmatrix}\psi_{11}\\ \psi_{21}\\ \psi_{31}\end{pmatrix}=\begin{pmatrix}\phi_1\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0 & G_1\overline{V}_1 & 0\\ 0 & 0 & G_2\overline{V}_2\\ G_3\overline{V}_3 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_{11}\\ \psi_{21}\\ \psi_{31}\end{pmatrix}. \tag{2.45}$$

Clearly $\psi_{11}=\psi_{21}=\psi_{31}=\Psi_1^{(+)}$ is a solution to Eq. (2.45). However, one may ask whether the homogeneous set of equations has discrete spurious solutions in addition to the states predicted by the Schrödinger equation. This question can be answered by noting that the matrix operators of Eqs. (2.44) and (2.45) are related by

$$\begin{pmatrix}1 & -G_1\overline{V}_1 & 0\\ 0 & 1 & -G_2\overline{V}_2\\ -G_3\overline{V}_3 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -G_1V_3G_0G_2^{-1} & G_1V_3G_0G_3^{-1}\\ G_2V_1G_0G_1^{-1} & 1 & -G_2V_1G_0G_3^{-1}\\ -G_3V_2G_0G_1^{-1} & G_3V_2G_0G_2^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & -G_1V_2 & -G_1V_3\\ -G_2V_1 & 1 & -G_2V_3\\ -G_3V_1 & -G_3V_2 & 1\end{pmatrix}. \tag{2.46}$$

Thus if

$$\begin{pmatrix}1 & -G_1V_3G_0G_2^{-1} & G_1V_3G_0G_3^{-1}\\ G_2V_1G_0G_1^{-1} & 1 & -G_2V_1G_0G_3^{-1}\\ -G_3V_2G_0G_1^{-1} & G_3V_2G_0G_2^{-1} & 1\end{pmatrix}\begin{pmatrix}\psi_{11}\\ \psi_{21}\\ \psi_{31}\end{pmatrix}=0 \tag{2.47}$$

has nontrivial solutions, Eq. (2.45) will have spurious solutions. It is easy to see that Eq. (2.45) is produced by multiplying Eq. (2.44) from the left by the spurious multiplier of Eq. (2.47). Therefore, as before, the multiplier cancels in the solution and in the physical scattering region the correct solution of Eq. (2.44) survives.

Before we discuss the existence of nontrivial spurious solutions of Eq. (2.47) let us relate Eq. (2.45) to the corresponding equations for transition operators and the channel components of the wave function introduced in Ref. 13. Introducing the matrices

$$\mathcal{G}_0=\begin{pmatrix}G_1 & 0 & 0\\ 0 & G_2 & 0\\ 0 & 0 & G_3\end{pmatrix}, \tag{2.48}$$

$$V=\begin{pmatrix}\overline{V}_1 & 0 & 0\\ 0 & \overline{V}_2 & 0\\ 0 & 0 & \overline{V}_3\end{pmatrix}, \tag{2.49}$$

$$W=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}, \tag{2.50}$$

and the column vectors ψ with components ψ_{11}, ψ_{21}, and ψ_{31} and ϕ with components ϕ, 0, 0, Eq. (2.45) can be written as

$$\psi=\phi+\mathcal{G}_0VW\psi. \tag{2.51}$$

Here we have made a particular choice of W for simplicity. Other choices have been proposed in the literature.[8] Our conclusions in this section are independent of this particular choice.

Introducing the column vector U with components U^{11}, U^{21}, U^{31} the matrix equation for the transition operator U becomes

$$U=VW+VW\mathcal{G}_0U. \tag{2.52}$$

This is exactly Eq. (9) in Ref. 15 for the specific KLT choice (2.50).

The corresponding nonspurious set of AGS (Ref. 12) equations for the transition operators are

$$\begin{pmatrix} U^{11} \\ U^{21} \\ U^{31} \end{pmatrix} = \begin{pmatrix} 0 \\ G_0^{-1} \\ G_0^{-1} \end{pmatrix} + \begin{pmatrix} 0 & V_2G_2 & V_3G_3 \\ V_1G_1 & 0 & V_3G_3 \\ V_1G_1 & V_2G_2 & 0 \end{pmatrix} \begin{pmatrix} U^{11} \\ U^{21} \\ U^{31} \end{pmatrix} . \tag{2.53}$$

It is easy to see that Eq. (2.52) results from Eq. (2.53) by the application of a matrix multiplier from the left and the following factorization property:

$$\begin{pmatrix} 1 & -\overline{V}_1G_2 & 0 \\ 0 & 1 & -\overline{V}_2G_3 \\ -\overline{V}_3G_1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -V_3G_0 & V_3G_0 \\ V_1G_0 & 1 & -V_1G_0 \\ -V_2G_0 & V_2G_0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -V_2G_2 & -V_3G_3 \\ -V_1G_1 & 1 & -V_3G_3 \\ -V_1G_1 & -V_2G_2 & 1 \end{pmatrix} . \tag{2.54}$$

Thus again

$$\begin{pmatrix} 1 & -V_3G_0 & V_3G_0 \\ V_1G_0 & 1 & -V_1G_0 \\ -V_2G_0 & V_2G_0 & 1 \end{pmatrix} \begin{pmatrix} U^{11} \\ U^{21} \\ U^{31} \end{pmatrix} = 0 \tag{2.55}$$

defines the spurious solutions to Eq. (2.52).

Finally the eigenvalue problem to the homogeneous part of Eq. (2.52) reads

$$VW\mathcal{G}_0U = U. \tag{2.56}$$

The kernel of Eq. (2.56) has the same eigenvalues as its transpose. This leads to the following related eigenvalue problem:

$$\mathcal{G}_0W^TV^T\chi = \chi , \tag{2.57}$$

where χ is a column vector with components χ_1, χ_2, and χ_3 and is called channel components[13] of the wave functions in analogy with the Faddeev components of the wave function.[1] Equation (2.57) is the homogeneous part of Eq. (27) of Ref. 15 which is claimed to be equivalent to the Schrödinger equation.

For the sake of completeness we copy the "proof" that each solution to Eq. (2.57) is related to the Schrödinger equation. In explicit notation we have

$$G_1\overline{V}_3\chi_3 = \chi_1, \quad G_2\overline{V}_1\chi_1 = \chi_2, \tag{2.58}$$

and

$$G_3\overline{V}_2\chi_2 = \chi_3.$$

Multiplying these equations by G_1^{-1}, G_2^{-1}, and G_3^{-1}, respectively, and summing the three equations we get

$$\sum_\mu (E - H_\mu)\chi_\mu - \sum_\mu \overline{V}_\mu\chi_\mu = 0 \tag{2.59}$$

or

$$(E - H)(\textstyle\sum\chi_\mu) = 0. \tag{2.60}$$

Thus $\sum_\mu\chi_\mu$ is a solution to the Schrödinger equation. From this the authors in Refs. 14 and 15 concluded that Eq. (2.58) has no spurious solutions. But Eq. (2.60) can also be fulfilled by components $\chi_\mu \neq 0$ and $\sum_\mu\chi_\mu = 0$ which correspond to discrete eigenvalues.

The reader may wonder if this is possibly also true for the Faddeev components defined by Eq. (2.10). In explicit notation Eq. (2.10) is

$$(E - H_0)\psi_1 = V_1(\psi_1 + \psi_2 + \psi_3),$$
$$(E - H_0)\psi_2 = V_2(\psi_1 + \psi_2 + \psi_3), \tag{2.61}$$

and

$$(E - H_0)\psi_3 = V_3(\psi_1 + \psi_2 + \psi_3).$$

It is easy to see that the $\sum_\mu\psi_\mu$ satisfies the Schrödinger equation (2.60) and Eq. (2.61) can not be fulfilled by components $\psi_\mu \neq 0$ and $\sum_\mu\psi_\mu = 0$ which correspond to discrete eigenvalues.

Now it is easy to relate Eqs. (2.58) and (2.61) by matrix multiplier because one has the factorization property

$$\begin{pmatrix} 1 & 0 & -G_1\overline{V}_3 \\ -G_2\overline{V}_1 & 1 & 0 \\ 0 & -G_3\overline{V}_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -G_1V_1 & -G_1V_1 \\ -G_2V_2 & 1 & -G_2V_2 \\ -G_3V_3 & -G_3V_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & G_0V_1 & -G_0V_2 \\ -G_0V_3 & 1 & G_0V_2 \\ G_0V_3 & -G_0V_1 & 1 \end{pmatrix} . \tag{2.62}$$

The first matrix to the right is the Faddeev matrix and the second is the spurious multiplier matrix.

Hence Eq. (2.58) will have the spurious solutions corresponding to the following eigenvalue problem:

$$\begin{pmatrix} 1 & G_0V_1 & -G_0V_2 \\ -G_0V_3 & 1 & G_0V_2 \\ G_0V_3 & -G_0V_1 & 1 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = 0\,, \qquad (2.63)$$

over and above the solutions defined by the Schrödinger equation. Now we may compare the three spuriosity conditions (2.47), (2.55), and (2.63). Obviously the sets (2.47) and (2.55) are simply related by $\psi_{\mu 1} = G_\mu U^{\mu 1}$. Furthermore (2.63) and (2.55) are just the transposed eigenvalue problems of each other. The three spuriosity conditions are equivalent so far as the spurious energies are concerned. So we shall study just Eq. (2.63).

It is easy to see by adding the three equations in the set of equations (2.63) that the χ_μ's will have the property $\sum_\mu \chi_\mu = 0$ which is just the second possibility mentioned after Eq. (2.60). We now relate the existence of $\chi_\mu \neq 0$ to Eq. (2.11). If we take $V_3 = 0$ we have from Eq. (2.63)

$$\begin{aligned} \chi_2 &= -G_0V_2\chi_3, \\ \chi_3 &= G_0V_1\chi_2, \\ \chi_1 &= G_0V_1\chi_2 + G_0V_2\chi_2 = -\chi_2 - \chi_3. \end{aligned} \qquad (2.64)$$

Thus Eqs. (2.64) will allow discrete spurious solutions since it is identical with Eqs. (2.11) after a change of sign of one pair interaction.

In conclusion, Eq. (27) of Ref. 15 is not uniquely related to the Schrödinger equation. The same is true for Eq. (2.45). The homogeneous system of the channel coupling array scheme of KLT (Ref. 8) allows for nonphysical spurious solutions.

If we start from the choice given by Eq. (2.44) which is uniquely related to the Schrödinger equation, one may find the following coupling scheme which is again free of spurious solutions:

$$\begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & G_1V_2 & G_1V_3 \\ G_2V_1 & 0 & G_2V_3 \\ G_3\overline{V}_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{pmatrix}. \qquad (2.65)$$

This choice has been mentioned by Sandhas.[19] It is easy to read off the multiplier with respect to Eq. (2.44) and the corresponding spurious eigenvalue problem is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -G_3V_2G_0G_1^{-1} & G_3V_2G_0G_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{pmatrix} = 0. \qquad (2.66)$$

Clearly there are only the trivial solutions $\psi_{11} = \psi_{21} = \psi_{31} = 0$. But as soon as one condenses additional pairs in Eq. (2.44) spurious multipliers arise. For example consider

$$\begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & G_1V_2 & G_1V_3 \\ 0 & 0 & G_2\overline{V}_2 \\ G_3\overline{V}_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{pmatrix}. \qquad (2.67)$$

It is easy to read off the multiplier with respect to Eq. (2.44) and the spurious eigenvalues to Eq. (2.67) will arise from

$$\begin{pmatrix} 1 & 0 & 0 \\ G_2V_1G_0G_1^{-1} & 1 & -G_2V_1G_0G_3^{-1} \\ -G_3V_2G_0G_1^{-1} & G_3V_2G_0G_2^{-1} & 1 \end{pmatrix} \times \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{pmatrix} = 0, \qquad (2.68)$$

which allows again for discrete spurious solutions

$$\psi_{11} = 0, \quad \psi_{21} \neq 0, \quad \psi_{31} \neq 0\,.$$

Again in the scattering region this spuriosity will be harmless as the spurious multiplier cancels from both sides. In the scattering region the solution of Eq. (2.52) for the KLT transition operators will be exactly the same as the solution of Eq. (2.53) for the AGS transition operators and this clarifies the doubts raised in Ref. 20 about any possible disagreement between the solutions of Eqs. (2.52) and (2.53). The discussion in this section also invalidates a proof by Bencze and Tandy[15] about the nonexistence of spurious solutions to a general class of n-body scattering equations as we shall see in the next section.

III. FOUR-BODY PROBLEM

In the three-body problem one has three two-body fragmentation channels as well as three pair interactions. The corresponding numbers are 7 and 6 in the four-body problem. However, the number of two-body connected four-body equations of Yakubovskii with a CAI kernel is 18. This corresponds to the most detailed decomposition of the state vector into all possible pair subclusters of the seven two-body fragmentations. Nevertheless it is desirable to have a set of equations for the seven transition operators, which connect the two-body fragmentation channels. Such a set with a three-body connected CAI kernel has been given by Sloan[5] and later generalized by Bencze and Redish[6] to the case of an arbitrary number of particles. There is also a set of six coupled equations[4] for the four-body t matrix. Both sets can be derived by the method of cluster decomposition[11] or by an intimately related method—insertion of various Lippmann-Schwinger equations into each other.

We use the following notation. In addition to the pair indices μ, ν, λ, α, β corresponding to three-body fragmentations we use indices σ, ρ, τ to denote two-body fragmentations. They are of the type $\sigma=(abc)$, d, or $\sigma=(ab),(cd)$ where a, b, c, and d represent particles. The interactions which are internal or external to the channel σ are denoted by V_σ or V^σ, respectively. Thus $V_\sigma = V_{ab} + V_{bc} + V_{ca}$ or $V_\sigma = V_{ab} + V_{cd}$ and $V^\sigma = V_{ad} + V_{bd} + V_{cd}$ or $V^\sigma = V_{bc} + V_{ca} + V_{ad} + V_{bd}$. Here ab denotes the pair of particles a and b and will in general be represented by one of the indices μ, ν, λ, α, and β in the following. We further have $V_\sigma^\mu = V_\sigma - V_\mu$ where the pair interaction V_μ is contained in V_σ.

It is easily verified that

$$\sum_{\nu\neq\mu} V_\nu = \sum_{\sigma\supset\mu} V_\sigma^\mu , \tag{3.1}$$

where $\sigma\supset\mu$ denotes all the two-body fragmentation channels for which V_σ includes the pair interaction V_μ.

Now we have the following two types of Lippmann-Schwinger equations for the scattering states $\Psi_\tau^{(+)}$ belonging to an initial two-body channel τ:

$$\Psi_\tau^{(+)} - G_\mu \sum_{\sigma\supset\mu} V_\sigma^\mu \Psi_\tau^{(+)} = i\epsilon G_\mu \phi_\tau \tag{3.2}$$

and

$$\Psi_\tau^{(+)} - G_\sigma V^\sigma \Psi_\tau^{(+)} = i\epsilon G_\sigma \phi_\tau , \tag{3.3}$$

where the channel state ϕ_τ satisfies $(E-H_0-V_\tau)\phi_\tau = 0$. The seven equations of the second type [Eq. (3.3)] which contain the resolvent operator $G_\sigma = (z-H_0-V_\sigma)^{-1}$ are obvious generalizations of the three basic three-body Lippmann-Schwinger equations [Eq. (2.31)] to the case of four particles.

If we insert Eq. (3.3) into Eq. (3.2) we get

$$\Psi_\tau^{(+)} - G_\mu \sum_{\sigma\supset\mu} V_\sigma^\mu G_\sigma V^\sigma \Psi_\tau^{(+)} = \Big(1 + \sum_{\sigma\supset\mu} G_\sigma V_\sigma^\mu\Big) i\epsilon G_\mu \phi_\tau . \tag{3.4}$$

Equation (3.4) has a Weinberg-type three-body connected kernel. Although the kernel is still noncompact it will allow us to derive the set of equations with CAI kernels of Refs. 4 and 5. It is easy to see that Eq. (3.4) results if we apply the spurious multiplier

$$L_\mu = \Big(1 + \sum_{\sigma\supset\mu} G_\sigma V_\sigma^\mu\Big) \tag{3.5}$$

on Eq. (3.2), and we have the factorization property

$$\Big(1 - G_\mu \sum_{\sigma\supset\mu} V_\sigma^\mu G_\sigma V^\sigma\Big) = \Big(1 + \sum_{\sigma\supset\mu} G_\sigma V_\sigma^\mu\Big) \times \Big(1 - G_\mu \sum_{\tau\supset\mu} V_\tau^\mu\Big) . \tag{3.6}$$

Since Eq. (3.4) appears at an intermediate step to derive the more interesting physical equations with CAI kernels, we shall not study the discrete spurious solutions of this equation in detail. However, the spurious eigenvalue problem corresponding to Eq. (3.4) is

$$L_\mu \chi_\mu = G_\mu \Big(E - H + \sum_{\sigma\supset\mu} V_\sigma^\mu G_\sigma V_\sigma^\mu\Big) \chi_\mu = 0. \tag{3.7}$$

Now we introduce the four-body t matrices T_μ defined by $T_\mu \phi_\tau = V_\mu \Psi_\tau^{(+)}$. Multiplying Eq. (3.4) by V_μ we obtain a set of six coupled equations for T_μ given by

$$T_\mu \phi_\tau - V_\mu G_\mu \sum_{\sigma\supset\mu} V_\sigma^\mu G_\sigma \sum_{\lambda\not\subset\sigma} T_\lambda \phi_\tau = V_\mu \Big(1 + \sum_{\sigma\supset\mu} G_\sigma V_\sigma^\mu\Big) \times i\epsilon G_\mu \phi_\tau , \tag{3.8}$$

where $\lambda\not\subset\sigma$ denotes all the pair interactions V_λ that are not included in the channel interaction V_σ. Equation (3.8) can be compared to the following set of equations for T_μ derived from Eq. (3.2):

$$T_\mu \phi_\tau - V_\mu G_\mu \sum_{\nu\neq\mu} T_\nu \phi_\tau = i\epsilon V_\mu G_\mu \phi_\tau . \tag{3.9}$$

Though Eq. (3.9) has a noncompact kernel, it is in unique correspondence with the Schrödinger equation as is clear from the very construction. We shall see later that the homogeneous part of Eq. (3.8) may have spurious poles in the first sheet of the complex energy plane. From discussions in Sec. II and in Ref. 16 it is clear that on the real energy axis above the lowest scattering threshold only the physical solution survives. This spuriosity is a kind of Federbush disease[9] and it has been shown by Amado[16] that it is not a fatal disease and it will cancel in the physical scattering region unless the spurious poles occur at the physical scattering energy under consideration. Since both Eqs. (3.8) and (3.9) determine the same function $T_\mu\phi_\tau$ at the physical scattering energies they must be related by a matrix multiplier. It has been shown in Appendix A that it is indeed so and the multiplier $B_{\mu\nu}$ is given by

$$B_{\mu\nu}=\delta_{\mu\nu}\left(1+V_\mu\sum_{\sigma\supset\mu}G_\sigma V_\sigma^\mu G_\sigma\right)+\bar{\delta}_{\mu\nu}V_\mu G_{\sigma(\mu,\nu)}(1-V_\nu G_0). \tag{3.10}$$

Here $G_{\sigma(\mu,\nu)}$ is the resolvent operator for channel σ which includes both the pair interactions V_μ and V_ν. It is straightforward to verify that the application of $B_{\mu\nu}$ onto Eq. (3.9) yields Eq. (3.8).

As in the three-body problem the spurious multipliers L_μ of Eq. (3.5) for the state vector have disappeared and are replaced by the matrix multiplier $B_{\mu\nu}$ defined by Eq. (3.10). After we take the limit $\epsilon\to 0$ in Eq. (3.8) we get

$$T_\mu\phi_\tau-V_\mu G_\mu\sum_{\sigma\supset\mu}V_\sigma^\mu G_\sigma\sum_{\lambda\not\subset\sigma}T_\lambda\phi_\tau=\delta(\mu\subset\tau)V_\mu\phi_\tau, \tag{3.11}$$

where $\delta(\mu\subset\tau)$ is 1 whenever V_μ is included in V_τ and is zero otherwise. Equation (3.11) can be compared with the set of six coupled equations of Ref. 4. It is shown in Appendix B the two sets of equations have the same kernel. But they have different inhomogeneous terms. The inhomogeneous terms are different because the present definition of T_μ is slightly different from that in Ref. 4. We chose the present definition of T_μ because then the Sloan equations[5] for the transition operators follow naturally from Eq. (3.8).

Now Eq. (3.11) defines $T_\mu(z)$ which will have spurious poles at complex energies in the physical sheet. The positions of the spurious poles are determined by the matrix eigenvalue problem

$$\sum_\nu B_{\mu\nu}\chi_\nu=0. \tag{3.12}$$

The solution to Eq. (3.11), however, coincides with the physical operators at real energies above the lowest scattering threshold. It is easy to see that the eigenvalue problem of Eq. (3.12) can be put in the following form:

$$\xi_\mu+\sum_{\nu\neq\mu}G_\nu V_\nu\xi_\nu+\sum_{\sigma\supset\mu}G_\sigma\sum_{\nu\subset\sigma}V_\sigma^\nu G_0 V_\nu\xi_\nu=0, \tag{3.13}$$

with $\chi_\mu=V_\mu\xi_\mu$. By using the Lippmann-Schwinger equations (3.2) and (3.3) for the physical state vector we failed to show that Eq. (3.13) is satisfied by the physical state vector $\Psi_\alpha^{(+)}$. But we could not construct a definite proof showing that $\Psi_\alpha^{(+)}$ does not satisfy Eq. (3.13). We see no reasons to believe that Eq. (3.13) does not have nontrivial spurious solutions $\xi_\nu\neq 0$.

Now we are prepared to deduce the set of seven coupled equations for the transition operators which relate the two-body fragmentation channels. We introduce the physical transition operators $U^{\sigma\tau}$ defined by[21]

$$U^{\sigma\tau}\phi_\tau\equiv\sum_{\lambda\not\subset\sigma}V_\lambda\Psi_\tau^{(+)}=\sum_{\lambda\not\subset\sigma}T_\lambda\phi_\tau\,. \tag{3.14}$$

If we sum (3.11) over $\mu\not\subset\rho$ we get the following set of seven coupled equations:

$$U^{\rho\tau}-\sum_{\mu\not\subset\rho}V_\mu G_\mu\sum_{\sigma\supset\mu}V_\sigma^\mu G_\sigma U^{\sigma\tau}=\bar{\delta}_{\rho\tau}(E-H_0-V_{\rho\tau}). \tag{3.15}$$

Equation (3.15) is the Sloan equation.[5] (See Appendix B for detail). The interaction $V_{\rho\tau}$ in Eq. (3.15) is the sum of all pair interactions common to channels ρ and τ.

Let us consider now the properties of the solutions of Eq. (3.15). As in the case of the set of six coupled equations we would like to have a basic set which uniquely corresponds to the underlying Schrödinger equation and to which Eq. (3.15) can be related by a matrix multiplier. This is easily achieved. First Eq. (3.14) is summed over $\sigma\supset\mu$ to give

$$\sum_{\sigma\supset\mu}U^{\sigma\tau}\phi_\tau=\sum_{\sigma\supset\mu}\sum_{\lambda\not\subset\sigma}V_\lambda\Psi_\tau^{(+)}. \tag{3.16}$$

For a particular μ the right-hand side of Eq. (3.16) can be evaluated, and Eq. (3.16) becomes

$$\sum_{\sigma\supset\mu}U^{\sigma\tau}\phi_\tau=2\sum_{\lambda\neq\mu}V_\lambda\Psi_\tau^{(+)}. \tag{3.17}$$

Multiplying Eq. (3.2) by V_μ, summing over $\mu\not\subset\rho$ and using Eqs. (3.14) and (3.17) we get

$$U^{\rho\tau}\phi_\tau-\tfrac{1}{2}\sum_{\mu\not\subset\rho}V_\mu G_\mu\sum_{\sigma\supset\mu}U^{\sigma\tau}\phi_\tau=\sum_{\mu\not\subset\rho}i\epsilon V_\mu G_\mu\phi_\tau. \tag{3.18}$$

Though Eq. (3.18) has a noncompact kernel, keeping $\epsilon\neq 0$ it defines uniquely the physical transition operators. To demonstrate this explicitly we define

$$\tfrac{1}{2}G_\lambda\sum_{\rho\supset\lambda}U^{\rho\tau}\phi_\tau=\psi_\lambda-i\epsilon G_\lambda\phi_\tau\,. \tag{3.19}$$

Then multiplying Eq. (3.18) by $\frac{1}{2}G_\lambda$ and summing over $\rho\supset\lambda$ we obtain

$$\psi_\lambda-i\epsilon G_\lambda\phi_\tau-\tfrac{1}{2}G_\lambda\sum_{\rho\supset\lambda}\sum_{\mu\not\subset\rho}V_\mu(\psi_\mu-i\epsilon G_\mu\phi_\tau)=\tfrac{1}{2}G_\lambda\sum_{\rho\supset\lambda}\sum_{\mu\not\subset\rho}i\epsilon V_\mu G_\mu\phi_\tau \tag{3.20}$$

or

$$\psi_\lambda - \tfrac{1}{2}G_\lambda \sum_{\rho\supset\lambda}\sum_{\mu\not\subset\rho} V_\mu \psi_\mu = i\epsilon G_\lambda \phi_\tau . \tag{3.21}$$

If we operate now by $(1 - G_0V_\lambda)$ on both sides of Eq. (3.21) we get

$$\psi_\lambda - \sum_\mu G_0 V_\mu \psi_\mu = i\epsilon G_0 \phi_\tau \ . \tag{3.22}$$

Thus ψ_λ is independent of λ and fulfills

$$\psi - G_0 V\psi = i\epsilon G_0 \phi_\tau \ , \tag{3.23}$$

which has the unique physical solution

$$\psi = i\epsilon G \phi_\tau = \Psi_\tau^{(+)} . \tag{3.24}$$

Now with the help of Eq. (3.19), Eq. (3.18) becomes

$$U^{\rho\tau}\phi_\tau = \sum_{\mu\not\subset\rho} V_\mu(\Psi_\tau^{(+)} - i\epsilon G_\mu \phi_\tau) + \sum_{\mu\not\subset\rho} i\epsilon V_\mu G_\mu \phi_\tau \tag{3.25}$$

or

$$U^{\rho\tau}\phi_\tau = \sum_{\mu\not\subset\rho} V_\mu \Psi_\tau^{(+)} \ ,$$

which are the physical transition operators.

The only solutions possible to Eq. (3.18) are the physical scattering amplitudes defined by Eq. (3.14). So Eq. (3.18) uniquely corresponds to the underlying Schrödinger equation and does not have any spurious solutions. But this equation has a noncompact kernel and has disconnected pieces in the kernel after any number of iterations whereas the Sloan equation[5]—Eq. (3.15)—has a CAI kernel and is connected after one iteration. Equations (3.15) and (3.18) should be compared with Eqs. (3.11) and (3.9). Equation (3.9) has no spurious solutions but has a noncompact kernel whereas Eq. (3.11) has a CAI kernel but may have spurious solutions.

Now from discussions in Secs. II and III it is clear that Eq. (3.15) may have spurious solutions but the effect of these spuriosities will not show up in the scattering region. The physical transition operators fulfill the Sloan equation[5] as is obvious by the very construction of Eq. (3.15) where we inserted two Lippmann-Schwinger equations into each other. Hence in the physical scattering region (real energy axis above the lowest scattering threshold) both Eqs. (3.15) and (3.18) are satisfied by the physical transition operators defined by Eq. (3.14) and hence they must be related by a multiplier. Thus we are led to compare Eqs. (3.18) and (3.15) which in schematic notation are written as

$$U - KU = \mathring{U} \tag{3.26}$$

and

$$U - K_S U = \mathring{\bar{U}} \ , \tag{3.27}$$

where K and $\mathring{U}$ are the kernel and the homogeneous terms of Eq. (3.18), respectively, and K_S and $\mathring{\bar{U}}$ are the kernel and the inhomogeneous terms of the Sloan equation—Eq. (3.15). Equation (3.26) has the unique physical solution

$$U = (1 - K)^{-1}\mathring{U} \ , \tag{3.28}$$

which also fulfills Eq. (3.27) on the real energy axis above the lowest scattering threshold. Hence we have

$$(1 - K_S)(1 - K)^{-1}\mathring{U} = \mathring{\bar{U}} \equiv C\mathring{U} . \tag{3.29}$$

Comparing Eqs. (3.27) and (3.29) we have the factorization property

$$(1 - K_S) = C(1 - K). \tag{3.30}$$

Unfortunately we could not find a closed expression for C, but perturbation expansion definitely rules out $C = 1$. There is no reason to exclude nontrivial solutions of $CU = 0$ which will give rise to spurious solutions. However, as in the case of the set of six coupled equations these spuriosities will not show up in the solution of the Sloan set of equations at real energies above the lowest scattering threshold. Hence the Sloan set[5] or the BR set[6] can be safely used to calculate the transition probabilities unless these spurious energies occur at the scattering energies. However, special precautions may be needed numerically to avoid admixture of these spurious states into the solution of the problem if one of these spurious energies comes very close to the physical scattering energies.

IV. SUMMARY AND CONCLUSION

We studied spuriosities in few-body scattering equations with two- and three-body connected kernels. The cluster decomposition,[11] usually applied in deriving equations with three-body connected kernels, first introduced by Weinberg,[3] can obviously be replaced by an intimately connected procedure—insertion of various Lippmann-Schwinger equations, corresponding to different subsystem resolvent operators, into each other. The later procedure reveals also the spurious multipliers by which the Lippmann-Schwinger equations with G_0 or G_μ have to be multiplied to arrive at equations with kernels of connectivity higher than two. Only when the inhomogeneous term is simplified by taking the limit $\epsilon \to 0$ will the solutions have spurious poles in the physical sheet corresponding to the discrete eigenvalues of the multipliers. However, these spurious admixtures vanish exactly for real scattering energies and the solutions to the Weinberg-type equations are exactly the physical ones. This has been explicitly demonstrated in Sec. II A for the three-body scattering state.

In the four-body case the corresponding study could have been carried through for the four-body connected Weinberg kernel. Instead we set up equations for the scattering states with a three-body connected kernel, which are spurious too, and served as an intermediate step for the derivation of sets of coupled equations for t matrices and transition operators.

Next we derive the equations for transition operators and t matrices from these equations for the state vectors. The new set of equations does not have the old spuriosities of the state vectors, but surprisingly enough it has developed spuriosities at a new set of energies. In the three-body case we exhibited this for the transition operators and t-matrices in Sec. II B. In both cases the coupled sets of equations with three-body connected kernels turned out to be just the once iterated Faddeev or AGS equations. Thus the spurious matrix multipliers are $(1+K)$ where K is either the Faddeev or the AGS (Ref. 12) kernel. Since it is an overall factor it cancels in the solution, which will be exactly the physical one.

We applied the same considerations to the system of four particles and we found that the spurious multipliers related to the state vectors do not show up in the equations for t matrices and transition amplitudes. We study the set of six coupled equations of Ref. 4 and the Sloan equations[5] and show that they contain spurious multipliers. These multipliers relate these equations to two basic sets of equations which correspond uniquely to the Schrödinger equation. In case of the equation of Ref. 4 we give an explicit form of the spurious multiplier. We found no reason to assume that this multiplier does not have nontrivial discrete spurious solutions. The Sloan equation also has the same structure and the same conclusions follow. However, the spurious multiplier again cancels in the physical scattering region.

It has been shown in Ref. 22 that the BR set of equations[6] can be derived from a general channel coupling class of connected kernel equations. Bencze and Tandy[15] showed that the general channel coupling class of connected kernel equations is free from spuriosities and hence concluded that the BR set of equations is also free from spurious solutions. However to get the BR set of equations from the general channel coupling class of connected kernel equations one has to insert explicit expressions for transition operators connecting three-body fragmentation channels in Eq. (9) of Ref. 15. This is not an identical rewriting of the equation but equivalent to inserting two types of Lippmann-Schwinger equations into each other. Such a step, as we have seen, introduces spuriosity. Hence the proof by Bencze and Tandy does not apply to the BR equations. Moreover, the proof by itself is wrong as we have demonstrated in Sec. II for the case of three particles. There we studied the channel array coupling scheme[8] for three particles. Contrary to proofs given by Hahn, Kouri, and Levin,[14] and by Bencze and Tandy[15] the KLT channel array coupling scheme which is also the starting point of an independent formulation by Hahn[7] is shown to admit spurious solutions.

In conclusion we would like to say that we found spuriosities in all the few-body formulations except the ones by Faddeev and Yakubovskii and by one of the present authors (WG). In contrary to a proof by Hahn, Kouri, and Levin[14] and by Bencze and Tandy[15] the KLT channel array coupling scheme also admits spuriosities. It may be hazardous to use these equations in bound-state problems. Nevertheless the spuriosities will have no effect in the scattering region. It may be worthwhile to explore the typical distribution of spurious poles in the physical sheet for these equations and to study whether special caution is needed to safely exclude any admixture of spurious states into the numerical solution if a spurious pole comes very close to real scattering energies. Such studies are underway.

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APPENDIX A

In this Appendix we derive an expression for the spurious multiplier related to Eq. (3.8). To do that we rewrite the right-hand side of Eq. (3.8),

$$R_\mu = i\epsilon V_\mu G_\mu \phi_\tau + V_\mu \sum_{\sigma\supset\mu} G_\sigma V_\sigma^\mu i\epsilon G_\mu \phi_\tau \tag{A1}$$

with the help of Eq. (3.1) as

$$R_\mu = i\epsilon V_\mu G_\mu \phi_\tau + V_\mu \sum_{\sigma\supset\mu} G_\sigma \sum_{\substack{\nu\subset\sigma \\ \nu\neq\mu}} V_\nu (G_0 + G_\mu V_\mu G_0) i\epsilon \phi_\tau \quad \text{(A2)}$$

or

$$R_\mu = i\epsilon V_\mu G_\mu \phi_\tau + V_\mu \sum_{\sigma\supset\mu} G_\sigma \sum_{\substack{\nu\subset\sigma \\ \nu\neq\mu}} V_\nu G_\mu i\epsilon V_\mu G_0 \phi_\tau + V_\mu \sum_{\sigma\supset\mu} G_\sigma \sum_{\substack{\nu\subset\sigma \\ \nu\neq\mu}} V_\nu G_0 i\epsilon\phi_\tau. \quad \text{(A3)}$$

Now we use the obvious identities.

$$V_\mu G_\mu = (1 + V_\mu G_\mu) V_\mu G_0 \quad \text{(A4)}$$

and

$$(1 - V_\mu G_0)(1 + V_\mu G_\mu) = 1 \quad \text{(A5)}$$

to rewrite (A3) as

$$\begin{aligned} R_\mu &= i\epsilon V_\mu G_\mu \phi_\tau + V_\mu \sum_{\sigma\supset\mu} G_\sigma \sum_{\substack{\nu\subset\sigma \\ \nu\neq\mu}} V_\nu G_\mu (1 - V_\mu G_0) \times i\epsilon V_\mu G_\mu \phi_\tau \\ &\quad + V_\mu \sum_{\sigma\supset\mu} G_\sigma \sum_{\substack{\nu\subset\sigma \\ \nu\neq\mu}} (1 - V_\nu G_0) i\epsilon V_\nu G_\nu \phi_\tau \\ &\equiv \sum_\nu B_{\mu\nu} i\epsilon V_\nu G_\nu \phi_\tau. \end{aligned} \quad \text{(A6)}$$

Comparing the right-hand sides of Eqs. (3.9) and (A6) which is equivalent to Eq. (3.8) we can read off the desired multiplier $B_{\mu\nu}$.

APPENDIX B

In this Appendix we show that Eqs. (3.11) and (3.15) correspond to the scattering equations proposed in Refs. 4 and 5. In this Appendix we follow the definition of the three-body t matrix of Ref. 5. The three-body t matrices $M^\sigma_{\mu\lambda}$ which are intimately related to the kernels of Eqs. (3.11) and (3.15) are defined by

$$M^\sigma_{\mu\lambda} = V_\mu \delta_{\mu\lambda} + V_\mu G_\sigma V_\lambda. \quad \text{(B1)}$$

Equation (B1) is the same as Eq. (2.8) in Ref. 5. We can rewrite Eq. (B1) as

$$M^\sigma_{\mu\lambda} = V_\mu \delta_{\mu\lambda} + V_\mu (G_\mu + G_\mu V^\mu_\sigma G_\sigma) V_\lambda$$

or

$$M^\sigma_{\mu\lambda} = V_\mu \delta_{\mu\lambda} + \delta_{\mu\lambda} V_\mu G_\mu V_\mu + \bar{\delta}_{\mu\lambda} V_\mu G_\mu V_\lambda + V_\mu G_\mu V^\mu_\sigma G_\sigma V_\lambda. \quad \text{(B2)}$$

Now the connected part of $M^\sigma_{\mu\lambda}$ is defined by

$$\bar{M}^\sigma_{\mu\lambda} = \bar{\delta}_{\mu\lambda} V_\mu G_\mu V_\lambda + V_\mu G_\mu V^\mu_\sigma G_\sigma V_\lambda. \quad \text{(B3)}$$

Summing Eq. (B3) over all $\lambda \subset \sigma$ we get

$$\bar{M}^\sigma_{\mu\text{-}} = V_\mu G_\mu V^\mu_\sigma + V_\mu G_\mu V^\mu_\sigma G_\sigma V_\sigma$$

or

$$\bar{M}^\sigma_{\mu\text{-}} = V_\mu G_\mu V^\mu_\sigma G_\sigma G_0^{-1}. \quad \text{(B4)}$$

Here $\bar{M}^\sigma_\mu\text{-}G_0$ is the kernel of the set of equations in Ref. 4 and this has been nicely demonstrated in Ref. 5.

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