Asymptotic normalization parameter of the triton

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On the basis of exact separable potential calculations, we show that there is a significant variation of the asymptotic normalization parameter C_t^2 of the triton with its binding energy. We present a partial wave dispersion relation technique for determining this parameter from the triton energy, the doublet n-d scattering length, the doublet, s-wave, n-d inelasticities, and the two-nucleon, on-shell scattering amplitudes. We test the method and find it to be accurate and stable with only low-energy information used as input. For a doublet scattering length of 0.65 fm we obtain $C_t^2 = 3.3 \pm 0.1$, where the error limits are determined from uncertainties in the inelasticities and the analytic continuation of the two-nucleon amplitudes to negative energies.

[NUCLEAR REACTIONS exact separable potential calculations; partial wave dispersion relations for n-d elastic scattering amplitudes.]

I. INTRODUCTION

In recent years a great deal of attention has been focused on the asymptotic normalization parameters (ANP's) of the ^{3}H and ^{3}He bound state wave functions. A summary of the results obtained through January 1974 is contained in the review article of Kim and Tubis.[1]

The ANP's appear in a great many situations, and there are a number of ways for determining them. They play an important role in distorted-wave Born-approximation calculations of direct nuclear reactions,[2] and, in fact, such analyses provide a somewhat indirect determination of them.

The peripheral model[3,4] provides a more direct and less model dependent determination of these parameters from nuclear reaction data. In this model the amplitudes are determined by a small number of singularities located close to the physical region. The ANP's are directly related to the strengths of these singularities. The results obtained with this approach are briefly summarized in Ref. 5.

The peripheral model has also been used[6] in a phase shift analysis of p-^{3}He elastic scattering in order to determine the phase shifts for $L>1$. The ANP of ^{3}He was determined by taking it as an adjustable parameter in fitting the data.

The analytic structure of scattering amplitudes has led to another useful technique for extracting ANP's: the so-called expansion technique in the $\cos\theta$ plane.[7-11] In this method the strengths of the nearest singularities to the physical region in the $\cos\theta$ plane at fixed energy are determined from differential cross section data. These strengths are directly related to the ANP's. Conformal mapping techniques have been used to increase the effectiveness of the method when important background singularities are present[7-9] and Coulomb corrections have been incorporated.[10,11]

Yet another method based on the analytic structure of scattering amplitudes is the use of dispersion relations. Both forward[12,13] and partial wave[14,15] dispersion relations have been used to determine ANP's.

The asymptotic part of the three-body wave function makes an important contribution to the matrix elements that describe the electro- and photodisintegration of ^{3}H and ^{3}He, therefore various analyses[16-20] of these processes have led to values for the ANP's. The photodisintegration calculations[16,19,20] have used wave functions based on separable potential models of the three-nucleon system. The first determinations[17,19] of the ANP's from such wave functions gave values consistently higher than those obtained from other methods[1]; however, the most recent calculation[20] shows that this is not a general feature of the separable potential model.

In the last few years several authors[21-25] have extracted ANP's from three-nucleon wave functions obtained from exact solutions of the Faddeev equations. Calculations have been carried out for local,[21,22,24] as well as nonlocal[23,25] potentials. The results indicate that the ANP of ^{3}H is sensitive to the two-nucleon potential. It is worth noting that various integral expressions have been derived for the ^{3}H ANP,[20,26,27] which make it possible to calculate it from either configuration space or momentum space wave functions.

Here we shall present two types of calculations for the ANP of the triton: exact separable potential calculations and calculations based on partial wave dispersion relations. The purpose of our separable potential calculations is to study the variation of the ANP with the triton binding energy. Our results indicate that it is important to use a potential which gives the experimental binding energy for the triton when calculating its ANP. From the point of view of methodology, it is worth noting that we extract the ANP from the residue of the triton pole in the n-d elastic scattering amplitude rather than from the triton wave function. Our procedure is numerically stable and quite straightforward.

Our partial wave dispersion relation calculations are an attempt to determine the triton ANP from the two-nucleon, on-shell amplitudes, the triton energy, the doublet, n-d scattering length, and the doublet s-wave, n-d inelasticity. It is known[15,28] that the discontinuity across a significant part of the left-hand cut (LHC) in the partial wave, n-d elastic scattering amplitude depends only on the analytic continuation of the on-shell, two-nucleon amplitude to negative energies. Part of the LHC discontinuity in our region of interest also depends on the deuteron wave function, which is off-shell two-nucleon information; however, we find that this contribution to the LHC discontinuity has an insignificant effect on our results. We use a conformal mapping technique to parametrize the effect of the omitted portion of the LHC. This introduces two parameters which we adjust to the triton energy and the doublet scattering length. We take inelasticity effects into account by means of Froissart's[29] method. We present tests of our approach which show that it is accurate, and not very sensitive to the input once the triton energy and doublet scattering length have been specified.

In Sec. II we present the results of our exact separable potential calculations of the ANP of the triton C_t^2 for a range of values of its binding energy. Section III gives our partial wave dispersion relation approach for determining C_t^2. The analytic structure of the doublet, s-wave, elastic n-d scattering amplitude is discussed and analyzed in Sec. III A. In Sec. III B, we derive the N/D equations used to solve the partial wave dispersion relations. Our numerical results are given in Sec. III C. Section IV provides a brief summary and discussion of what we have found. Throughout, we work in units such that $\hbar^2/M$ is one, where M is the nucleon mass.

II. SEPARABLE POTENTIAL CALCULATIONS

The model of the three-nucleon system that we use is that given by the well known Amado-Lovelace equations,[30,31] which corresponds to the use of spin-dependent, central, s-wave separable potentials to describe the two-nucleon interaction. Our notation and normalizations are given in Ref. 32. Throughout, the two-nucleon triplet and singlet states are denoted by 1 and 2, respectively.

For the triplet form factor we assume

$$g_1(p)=\int_{\mu+\alpha}^{\infty} d\beta\, \frac{\sigma(\beta)}{p^2+\beta^2}\,, \tag{2.1}$$

where

$$\sigma(\beta)=\left[\frac{4\alpha}{(1-\alpha\rho)}\right]^{1/2}[-(V_0/\mu)+\zeta\delta(\beta-\beta_1)]. \tag{2.2}$$

The representation (2.1) for the deuteron vertex function has been obtained by a number of authors.[33,34] We shall see in Sec. III A that using this form ensures that the branch point of one of the low energy singularities in the n-d scattering amplitude occurs in the right place. In (2.2) α is the deuteron wave number; i.e., α^2 is the deuteron binding energy in fm^{-2}, ρ is the deuteron effective range, μ is the inverse pion Compton wavelength, V_0 is the strength of the one pion exchange potential in fm^{-2}, and ζ and β_1 are adjustable parameters. The V_0/μ term in (2.2) can be justified[34] on the assumption that the deuteron wave function corresponds to a two-nucleon interaction with a Yukawa tail. Since we are assuming central forces V_0 does not have a well defined theoretical value for the triplet state where the tensor force occurs. For the singlet state it would be 10.463 MeV.[35] We rather arbitrarily choose a value of 20 MeV to account for the fact that the triplet force is stronger than the singlet force, and we take $\mu=0.7\ \text{fm}^{-1}$.

The parameters ζ and β_1 are fitted to[36] $\alpha=0.23161\ \text{fm}^{-1}$ and $\rho=1.701$ fm as follows. The t matrices for our potentials can be written in the form[28,32]

$$t_n(p,q;s)=g_n(p)\Delta_n(s)g_n(q),\quad n=1,2 \tag{2.3}$$

with

$$\Delta_n^{-1}(s)=-\lambda_n^{-1}+\int_0^{\infty} dx\, \frac{x^2 g_n^2(x)}{p^2-s}\,, \tag{2.4}$$

where s is the two-body energy parameter, and λ_n determines the strength of the interaction. In the triplet state, we eliminate λ_1 by demanding that $\Delta_1(s)$ have a pole at $s=-\alpha^2$. This gives

$$\Delta_1^{-1}(s)=(s+\alpha^2)\int_0^{\infty} dx\, \frac{x^2 g_1^2(x)}{(x^2+\alpha^2)(x^2-s)}\,. \tag{2.5}$$

We set

$$\int_0^{\infty} dx \frac{x^2 g_1^2(x)}{(x^2+\alpha^2)^2}=1, \tag{2.6}$$

which is just the bound state normalization condition. The on-shell amplitude is given by

$$t_n(p,p;p^2+i\epsilon)=-\frac{2}{\pi}\frac{1}{p\cot\delta_n-ip}\,, \tag{2.7}$$

where δ_n is the two-nucleon, s-wave phase shift. In the triplet state, we can use effective range theory to write

$$p\cot\delta_1 = -\alpha + \tfrac{1}{2}\rho(p^2+\alpha^2) + \cdots ; \tag{2.8}$$

from (2.3)–(2.8), we obtain

$$t_1(p,p;p^2+i\epsilon) \xrightarrow[p^2\to-\alpha^2]{} \frac{g_1^2(i\alpha)}{p^2+\alpha^2}, \tag{2.9}$$

where

$$g_1^2(i\alpha) = \frac{4\alpha}{\pi(1-\alpha\rho)} . \tag{2.10}$$

Equations (2.6) and (2.10) give us the two conditions we need to determine ζ and β_1.

For the singlet form factor we choose

$$g_2(p) = [1-(p/p_0)^2]/(p^2+\beta_2^2)^n , \tag{2.11}$$

where β_2, p_0, and the integer n are adjustable parameters. This form is chosen for its simplicity and because it gives us the freedom to sweep out a range of triton binding energies, while maintaining a fit to the singlet scattering length a and the effective range r_0. Our procedure is to choose p_0 and n, and then fit λ_2 and β_2 to $a=-23.715$ fm and $r_0=2.73$ fm, using the method of Ref. 37. With $n=2$ and p_0 ranging from 1.27 fm^{-1} to 1.74 fm^{-1}, we find triton binding energies ranging from 6.53 to 8.81 MeV, respectively. The potential that reproduces the experimental triton energy of 8.48 MeV has $p_0=1.64$ fm^{-1}. From now on we shall refer to this interaction as the *reference potential*.

In order to find the ANP of the triton we use the relation between it and the residue of the triton pole in the doublet, s-wave, elastic, n-d scattering amplitude. If δ is the corresponding phase shift, η the inelasticity, and k the neutron wave number in the c.m. frame, then we have for the n-d amplitude[1,17,19,20]

$$\tilde{f}(k^2) = (\eta e^{2i\delta}-1)/(2ik) , \tag{2.12}$$

$$\tilde{f}(k^2) \to -\frac{3\xi C_t^2}{k^2+\xi^2} , \tag{2.13}$$

where

$$E_t = \alpha^2 + \tfrac{3}{4}\xi^2 , \tag{2.14}$$

with E_t the triton energy in fm^{-2} and C_t^2 the ANP of the triton as defined in Ref. 1.

The triton pole occurs in the integral term in the Amado-Lovelace equations [see Eq. (6) of Ref. 32], and sits on the one-nucleon exchange cut which is carried by the Born term (see Sec. III A). We locate the position of the pole and its residue by using the Padé approximant technique of Lavine.[38] Our results are given in Fig. 1.

We see that there is a significant variation of C_t^2 with the triton energy. For the reference potential the value of C_t^2 is 3.66, while for the energy of the Reid potential,[39] $E_t=6.70$ MeV, the value of C_t^2 is 3.21. This is somewhat higher than the value of 2.86 found by Kim and Tubis.[21] If we extrapolate their value to the experimental energy by simply using the ratio of our calculated values, a value of 3.26 is found.

We study the sensitivity of C_t^2 to the vertex functions by refitting the parameters in (2.2) and (2.11), starting with different values for V_0 and n. In order to prevent a proliferation of numbers, we adjust p_0 in each case so that E_t comes out to be 8.48 MeV. Some typical results are given in Table I. The first row is our reference potential.

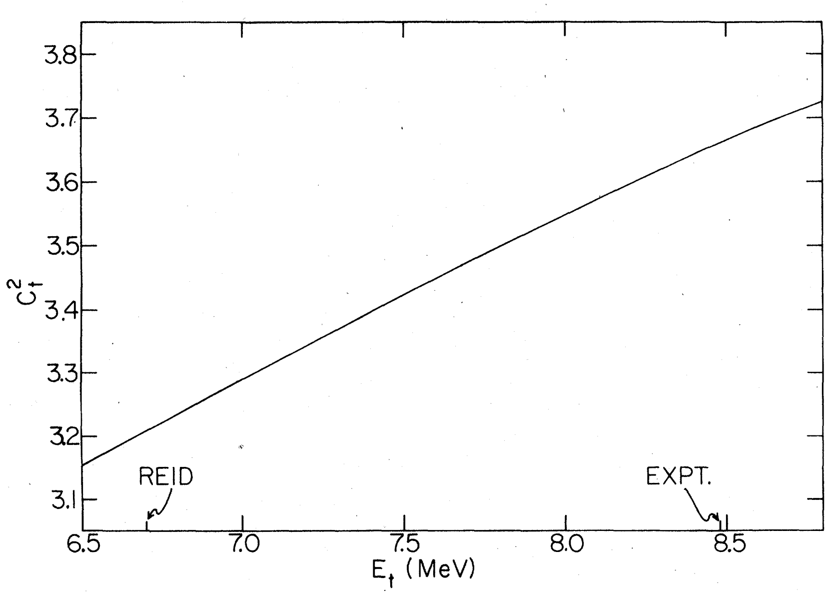

FIG. 1. Asymptotic normalization parameter of the triton as a function of its binding energy.

TABLE I. C_t^2 for various potentials. [a]

V_0 (MeV)	n	p_0 (fm^{-1})	C_t^2
20	2	1.64	3.66
20	3	1.60	3.68
20	5	1.58	3.68
0	2	1.67	3.63

[a] See Sec. II for the meaning of the parameters.

We see that within our framework there is very little sensitivity to the two-body potentials. This should not be interpreted to mean that Fig. 1 represents a universal function relating C_t^2 and E_t. The local potential results[21,24] do not fall precisely on our curve, although they do follow its general trend. Also, the results of Sec. III show that for a given value of E_t, noticeable changes in C_t^2 arise from small variations in the doublet scattering length. Our results should be interpreted as only giving a rough indication of the variation of C_t^2 with E_t.

III. PARTIAL WAVE DISPERSION RELATIONS

Here we shall present a method for extracting C_t^2 from the N/D solution of partial wave dispersion relations for the n-d scattering amplitude. The essential difference between our approach and that used previously by other authors[14,15] is that our amplitude is constructed so as to give the experimental triton energy E_t and doublet scattering length a_2. We have seen in the last section that C_t^2 is sensitive to E_t. We shall see in Sec. III C that using an amplitude with specified values of E_t and a_2 makes the calculated value of C_t^2 insensitive to the discontinuities across the distant parts of the LHC and right-hand cut (RHC).

A. Analytic structure of the n-d amplitude

We begin by discussing the analytic structure of the s-wave, doublet, elastic, n-d scattering amplitude. Following Barton and Phillips,[40] we introduce a dimensionless energy variable and amplitude by means of the relations

$$z=3k^2/(4\alpha^2)\ , \tag{3.1}$$

$$f(z)=(\eta e^{2i\delta}-1)/(2iz^{1/2})\ . \tag{3.2}$$

In general,[14,15,28,40] this amplitude has an RHC due to two- and three-particle unitarity, and an LHC associated with the exchange of nucleons and mesons. Since we are dealing with a real, analytic function the discontinuities are related to the imaginary part in the usual way, i.e.,

$$[f(z+i\epsilon)-f(z-i\epsilon)]/2i=\mathrm{Im}f(z+i\epsilon)\ . \tag{3.3}$$

From (3.2), we have on the RHC

$$\mathrm{Im}f(z+i\epsilon)=z^{1/2}|f(z)|^2+\frac{1-\eta^2}{4z^{1/2}}\ ,\quad z\geq 0 \tag{3.4}$$

where the inelasticity $\eta=1$ for $0\leq z\leq 1$ and $\eta<1$ for $z>1$. We shall be using values of η taken from separable potential calculations as well as from experiment.

The nearest left-hand singularity is the one-nucleon exchange cut, whose discontinuity is given by[36]

$$\mathrm{Im}f(z+i\epsilon)=\pi/[\sqrt{3}\,(1-\alpha\rho)z],\ -3\leq z\leq -1/3. \tag{3.5}$$

In all of our calculations we use the values for α and ρ given in Sec. II.

The two-nucleon exchange cut begins where the one-nucleon exchange cut ends, and has a model independent discontinuity across its leading edge given by the relations[28]

$$\mathrm{Im}f(z+i\epsilon)=\frac{64\pi}{3(1-\alpha\rho)z}\times\sum_{n=1}^{2}J_{1n}^2\int_0^{(|z|^{1/2}-\sqrt{3})/2}du\,f_n(u^2-1+z)\ ,$$

$$-\tfrac{3}{4}(\gamma/\alpha)^2\leq z\leq -3\ , \tag{3.6}$$

$$J_{11}=\tfrac{1}{4},\quad J_{12}=-\tfrac{3}{4}\ ,$$

$$f_n(\omega)=e^{i\delta_n}\sin\delta_n/\omega^{1/2},\quad \omega=s/\alpha^2\ .$$

Here the J_{1n} are spin-isospin recoupling coefficients, the f_n are dimensionless two-nucleon elastic scattering amplitudes with s the two-nucleon c.m. energy, and $-\gamma^2$ is the location of the nearest singularity of the $g_n(p)$ in the p^2-plane (see Sec. II). It is important to note that the recoupling coefficients weight the singlet contribution ($n=2$) nine times as much as the triplet contribution ($n=1$). Also, it can be shown[28] that for the range of z values in (3.6), the range of ω values in the integrand is

$$-\tfrac{3}{4}(\gamma/\alpha)^2-1\leq\omega\leq -4\ . \tag{3.7}$$

The next nearest singularity goes by a variety of names: the anomalous branch point of the nucleon-deuteron vertex function with one nucleon off shell,[40] the π-triangle singularity,[7] and the Δ' singularity.[15] For the sake of brevity, we shall refer to it as the Δ' singularity. Here we shall derive an expression for its discontinuity within the framework of potential theory.

The on-shell Born term for doublet, s-wave, elastic n-d scattering that arises in the three-particle formalism of Alt, Grassberger, and Sandhas[41] is[36]

$$\tilde{f}_B(k^2) = \frac{\pi}{6}\int_{-1}^{1} dx\, \frac{h^2(\frac{5}{4}k^2 + k^2 x)}{\alpha^2 + \frac{5}{4}k^2 + k^2 x}\,, \tag{3.8}$$

where

$$h(p^2) = g_1(p)\ . \tag{3.9}$$

The one-nucleon exchange cut is associated with the vanishing of the denominator in (3.8), whereas the Δ' cut is associated with the singularities of the deuteron vertex function which has the form[33,34] (2.1).

In deriving the discontinutiy across the Δ' cut it is convenient to work with a representation for $h^2(p^2)$. From (2.1) it is clear that $h^2(p^2)$ is analytic in the p^2-plane, cut along the negative real axis beginning at $p^2 = -(\mu+\alpha)^2$. The discontinuity across this cut is easily calculated with the help of the identities

$$h^2(p^2+i\epsilon) - h^2(p^2-i\epsilon) = [h(p^2+i\epsilon) + h(p^2-i\epsilon)] \times [h(p^2+i\epsilon) - h(p^2-i\epsilon)]$$

$$\frac{1}{p^2 \pm i\epsilon + \beta^2} = \mathrm{P}\frac{1}{p^2+\beta^2} \mp \pi i \delta(p^2+\beta^2)\ . \tag{3.10}$$

The result is

$$h^2(p^2+i\epsilon) - h^2(p^2-i\epsilon) = -2\pi i |p|^{-1}\sigma(|p|)\bar{h}(p^2), \tag{3.11}$$

where $\bar{h}(p^2)$ is the principal value form of (2.1). Using the Cauchy integral theorem, we obtain

$$h^2(p^2) = 2\int_{\mu+\alpha}^{\infty} d\beta \frac{\sigma(\beta)\bar{h}(-\beta^2)}{p^2+\beta^2}\ . \tag{3.12}$$

Inserting this into (3.8) and converting to the dimensionless amplitude (3.2), we find

$$\mathrm{Im} f(z+i\epsilon) = -\frac{\pi^2}{2\sqrt{3}\,z}\int_{u_0}^{\sqrt{3}|z|^{1/2}} du\, \frac{\sigma(\alpha u)\bar{h}(-\alpha^2 u^2)}{\alpha^2(1-u^2)}\,,$$

$$z \le -\frac{1}{3}\left(\frac{\mu+\alpha}{\alpha}\right)^2 = -5.39 \tag{3.13}$$

$$u_0 = (\mu+\alpha)/\alpha,\quad z \ge -3[(\mu+\alpha)/\alpha]^2\,,$$

$$u_0 = |z|^{1/2}/\sqrt{3}\,,\quad z \le -3[(\mu+\alpha)/\alpha]^2\ .$$

We see that this discontinuity is model dependent in the sense that it depends on off-shell, two-nucleon information; to be precise, knowledge of the deuteron wave function is required. It is worth noting that the discontinuity vanishes at the branch point.

In order to get some feeling for the relative strengths of the three discontinuities considered, we calculate them for our reference potential (see Sec. II). The results are shown in Fig. 2. The model independent one-nucleon exchange discontinuity falls by an order of magnitude over its range. The two-nucleon exchange discontinuity starts at zero, but rises rapidly. The Δ' discontinuity also starts at zero, but fortunately, never gets very large, at least in the range we are interested in.

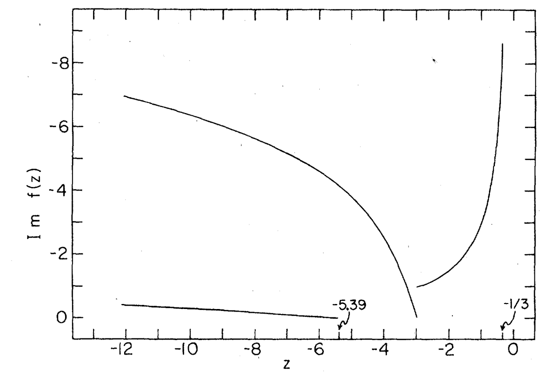

FIG. 2. Left-hand cut discontinuities in the doublet, s-wave, elastic n-d scattering amplitude for the reference potential.

B. N/D formalism

Here we shall derive the equations needed to construct a doublet, s-wave, elastic n-d scattering amplitude with the discontinuities described in Sec. III A, and specified values of E_t and a_2. We start by translating the N/D scheme of Froissart[29] into our notation. We introduce an effective amplitude $F(z)$ through the relation

$$F(z) = \frac{f(z)}{R(z)} + \frac{1}{2iz^{1/2}}\left[\frac{1}{R(z)} - 1\right]\,, \tag{3.14}$$

where

$$R(z) = \exp\left[-\frac{iz^{1/2}}{\pi}\int_1^{\infty} dy \frac{\ln\eta(y)}{y^{1/2}(y-z)}\right]\ . \tag{3.15}$$

The function $R(z)$ has RHC's starting at $z=0$ and $z=1$, but no LHC's and is such that $F(z)$ satisfies a two-body-like unitarity relation, namely

$$\mathrm{Im} F^{-1}(z+i\epsilon) = -z^{1/2},\quad z \ge 0\,. \tag{3.16}$$

On the LHC we have

$$\mathrm{Im} F(z+i\epsilon) = \mathrm{Im} f(z+i\epsilon)/R(z),\quad z \le -\tfrac{1}{3}\ . \tag{3.17}$$

We write

$$F(z) = N(z)/D(z), \tag{3.18}$$

with $N(z)$ carrying the LHC and $D(z)$ the RHC. It is straightforward to show[36] that $D(z)$ can be obtained by solving the equation

$$D(z)=1-\frac{iz^{1/2}}{\pi}\int_{-\infty}^{-1/3}dy\,\frac{D(y)\,\mathrm{Im}F(y+i\epsilon)}{y^{1/2}(z^{1/2}+y^{1/2})}, \tag{3.19}$$

and that $N(z)$ can be obtained from

$$N(z)=-z^{-1/2}\,\mathrm{Im}D(z+i\epsilon),\quad z>0\ . \tag{3.20}$$

From (3.19) it follows that[36] $D(z)$ is an analytic function in the $z^{1/2}$ plane, except for a cut along the negative imaginary axis, beginning at $-i/\sqrt{3}$.

In our approach we put in only part of the LHC explicitly, and parametrize the effect of the omitted portion. We do this by breaking the integral in (3.19) into an integral on the range $-\infty$ to $-b$, say, and one on the range $-b$ to $-\frac{1}{3}$; i.e., we write

$$D(z)=1+U(z^{1/2})-\frac{iz^{1/2}}{\pi}\int_{-b}^{-1/3}dy\,\frac{D(y)\,\mathrm{Im}F(y+i\epsilon)}{y^{1/2}(z^{1/2}+y^{1/2})}\ , \tag{3.21}$$

where $U(z^{1/2})$ stands for the integral on the range $-\infty$ to $-b$. In Ref. 36, where we were only interested in small values of $z\,(|z|\lesssim 1)$, we parametrized $U(z^{1/2})$ by expanding it in powers of $z^{1/2}$. Here we are interested in larger values of $z\,(|z|\lesssim 10)$, so this is not good enough.

In order to get a satisfactory parametrization, we use a conformal mapping technique in which we map the $z^{1/2}$-plane, cut along the negative imaginary axis from $-i\infty$ to $-i\sqrt{b}$, onto the interior of the unit circle in the w-plane, centered on the origin. The mapping we use is given by

$$w(z^{1/2})=\frac{e^{i\pi/4}(z^{1/2}+ib^{1/2})^{1/2}-id}{e^{i\pi/4}(z^{1/2}+ib^{1/2})^{1/2}+id}\ ,\quad d>0 \tag{3.22}$$

where the square root is defined by

$$(z^{1/2}+ib^{1/2})^{1/2}=+\,|z^{1/2}+ib^{1/2}|^{1/2}e^{i\theta/2},\ -\pi/2<\theta<3\pi/2\ . \tag{3.23}$$

The left-hand side of the cut is mapped onto the upper half of the unit circle in the w-plane, while the right-hand side is mapped onto the lower half. Our calculations are carried out along the line

$$0\leq \mathrm{Im}(z^{1/2})\leq b^{1/2}\ , \tag{3.24}$$

which is mapped by (3.22) onto the line

$$\frac{b^{1/4}-d}{b^{1/4}+d}\leq \mathrm{Re}(w)\leq\frac{\sqrt{2}\,b^{1/4}-d}{\sqrt{2}\,b^{1/4}+d}\ . \tag{3.25}$$

We choose d to be the number that makes the two limits in (3.25) equidistant from the origin, namely $d=(2b)^{1/4}$. This minimizes the magnitude of the two limits simultaneously. With this choice the maximum value of $|w|$ on the line given by (3.25) is

$$|w|=\frac{2^{1/4}-1}{2^{1/4}+1}=0.0864\ . \tag{3.26}$$

We expand $U(z^{1/2})$ as a power series in w, which leads to a series that converges everywhere in the cut $z^{1/2}$-plane. The fact that $D(z)$ is a real, analytic function of z implies that $U(z^{1/2})$ has the property

$$U(-z^{1/2})=U^*(z^{1/2}) \tag{3.27}$$

for z positive and real. Since our mapping (3.22) has the same property the coefficients of the power series in w are real. According to (3.19) and (3.21) the coefficients are also constrained by

$$U(0)=0\ . \tag{3.28}$$

We keep the three leading terms in the power series, and using (3.28) write our parametrization in the form

$$U(z^{1/2})=c_1[w(z^{1/2})-w(0)]+c_2[w(z^{1/2})-w(0)]^2\ . \tag{3.29}$$

We now determine c_1 and c_2 so that our amplitude will have a specified triton energy E_t and doublet scattering length a_2. From (3.20), (3.21), and (3.27) we obtain

$$N(z)=\frac{U(-z^{1/2})-U(z^{1/2})}{2iz^{1/2}}+\frac{1}{\pi}\int_{-b}^{-1/3}dy\,\frac{D(y)\,\mathrm{Im}F(y+i\epsilon)}{y-z}\ , \tag{3.30}$$

which in turn gives with the help of (3.18), (3.19), (3.22), and (3.29)

$$F(0)=Bc_1+\frac{1}{\pi}\int_{-b}^{-1/3}dy\,\frac{D(y)\,\mathrm{Im}F(y+i\epsilon)}{y}\ , \tag{3.31}$$

where

$$B=\frac{d}{b^{1/4}(b^{1/4}+d)^2}\ . \tag{3.32}$$

According to (3.1), (3.2), (3.14), and (3.15) we have

$$F(0)=-\frac{2\alpha a_2}{\sqrt{3}}+\frac{1}{2\pi}\int_1^\infty dy\,\frac{\ln\eta(y)}{y^{3/2}}\equiv -A\ , \tag{3.33}$$

which when combined with (3.31) gives us an expression for c_1 that can be used with (3.29) to write (3.21) as an equation for $D(z)$ with c_2 appearing linearly. This parameter can be eliminated by requiring $D(z)$ to have a zero corresponding to the triton energy, i.e.,

$$D(z_t)=0\ , \tag{3.34}$$

where

$$E_t=\alpha^2(1-z_t)\ . \tag{3.35}$$

Carrying out the elimination of c_2, we arrive at our final equation for $D(z)$:

$$D(z)=1-(A/B)[w(z^{1/2})-w(0)]-\{1-(A/B)[w(z_t^{1/2})-w(0)]\}\left[\frac{w(z^{1/2})-w(0)}{w(z_t^{1/2})-w(0)}\right]^2$$
$$-\frac{1}{\pi}\int_{-b}^{-1/3}dy\ D(y)\,\mathrm{Im}F(y+i\epsilon)\left\{\frac{iz^{1/2}}{y^{1/2}(z^{1/2}+y^{1/2})}+\frac{w(z^{1/2})-w(0)}{By}\right.$$
$$\left.-\left[\frac{iz_t^{1/2}}{y^{1/2}(z_t^{1/2}+y^{1/2})}+\frac{w(z_t^{1/2})-w(0)}{By}\right]\left[\frac{w(z^{1/2})-w(0)}{w(z_t^{1/2})-w(0)}\right]^2\right\}. \quad (3.36)$$

C. Numerical results

Our calculations of C_t^2 proceed as follows. A doublet scattering length and a set of inelasticities are used in (3.33) to determine A. This parameter, the triton energy parameter z_t, and the discontinuities described in Sec. III A are put into (3.36). This equation is then solved numerically for $D(z)$, using Gauss-Legendre quadrature rules to convert it to a standard linear matrix equation. In doing this the integral is broken up into three pieces, corresponding to the three discontinuities, so as to have a smooth integrand within each range of integration. The solution for $D(z)$ is then used in (3.30) to obtain a set of values for $N(z)$ in the neighborhood of z_t. It should be noted that in this neighborhood $N(z)$ is complex, except exactly at z_t where it is real. We work with only the real part of $N(z)$ when using Lavine's[38] method to extract the residue of $F(z)$ at z_t. This residue is converted to C_t^2 by means of (2.12), (2.13), (3.1), (3.2), (3.14), and (3.15).

In order to test our method, we use it to extract C_t^2 from the reference potential model of the triton as given in Sec. II, which we recall gave $C_t^2=3.66$. This model gives a doublet scattering length of $a_2=0.986$ fm. We calculate the inelasticities up to a lab energy of 200 MeV or $z=59.9$, and use them to evaluate the integrals in (3.15) and (3.33). We initially take b in (3.36) to be $\frac{3}{4}(\gamma/\alpha)^2$ which according to (3.6) is the lower end of the model independent part of the two-nucleon exchange cut. Recall that $-\gamma^2$ is the location of the nearest singularity of the $g_n(p)$ in the p^2-plane. For the reference potential this turns out to be given by $\gamma=\mu+\alpha$, which leads to $b=12.134$. We take the discontinuities across the LHC to be those shown in Fig. 2. Our N/D method gives a value of $C_t^2=3.65$ which is within 0.3% of the exact value.

Since the value of b is somewhat arbitrary we need to know the sensitivity of our results to it. Table II gives the results which show that even with $b=5.39$, the Δ'-branch point, our method gives a C_t^2 which is within 4% of the exact value. The value at $b=3$ indicates clearly that it is important to include some of the two-nucleon exchange cut so as to get a reasonable value for C_t^2. In order to see the effect of the inelasticity, we carry out a calculation at $b=12.134$ with η set everywhere equal to one. This gives $C_t^2=3.36$ which is within 8% of the value calculated with the theoretical inelasticities, so the inelasticity produces a nonnegligible, but not overwhelming effect.

In order to extract C_t^2 from experiment, we must use experimental values for the triton energy E_t, the doublet scattering length a_2, and the inelasticities. For E_t we use 8.48 MeV. The value of a_2 has moved around a great deal over the years,[1] with the latest value being[42] $a_2=0.65$ fm, so we calculate C_t^2 for a range of values for a_2. We use the inelasticities of Arvieux[43] which are available up to a lab kinetic energy of 46.3 MeV or $z=13.9$. For the LHC discontinuities we use those shown in Fig. 2, which were calculated with the reference potential. The value of b in (3.36) is 12.134. The results are given in Table III. We see that the values of C_t^2 are quite sensitive to a_2. The experimental value of 0.65 fm gives $C_t^2=3.31$, whereas the value for our reference potential, 0.986 fm, gives a value 10% higher.

In order to study the sensitivity of our results to the LHC and RHC discontinuities, we perform a set of calculations with $a_2=0.65$ fm. The most model dependent LH discontinuity is that associated with the two-nucleon exchange cut, and here it is the singlet two-nucleon amplitude that is most important. If we use the Yamaguchi form factor [(2.11) with $p_0=\infty$ and $n=1$] to construct the singlet amplitude, we find a noticeable change in the two-nucleon exchange cut discontinuity, but find that C_t^2 changes only from 3.31 to 3.36, or 1.5%. We

TABLE II. Dependence of C_t^2 on b. [a]

b	C_t^2
12.134	3.65
9.0	3.66
8.0	3.67
7.0	3.70
6.0	3.75
5.39	3.80
3.0	5.36

[a] b is the parameter in (3.36)

TABLE III. C_t^2 for various doublet scattering lengths.

a_2 (fm)	C_t^2
0.1	2.88
0.3	3.02
0.5	3.18
0.65	3.31
0.8	3.45
0.986	3.65
1.2	3.90

find that other models for the triplet and singlet form factors, such as those described at the end of Sec. II, give similar changes in C_t^2, so the uncertainty associated with the LHC is of the order of a percent or two.

In studying the uncertainty in C_t^2 associated with the inelasticity, we use the reference potential model for the LHC. The inelasticities enter the calculations through the integrals in (3.15) and (3.33). In the calculations of Table III, the integrals are simply cut off at $z=13.9$. If we use the theoretical reference potential inelasticities over the range from $z=0$ to $z=59.9$, we find no change from the value of $C_t^2=3.31$, given in Table III.

Another way we can estimate the inelasticity uncertainty is to break up the integral in (3.15) into two pieces and approximate the piece involving the experimentally unavailable inelasticities by its value at $z=0$; i.e., we write

$$\int_1^\infty dy \frac{\ln\eta(y)}{y^{1/2}(y-z)} \approx \int_1^{13.9} dy \frac{\ln\eta(y)}{y^{1/2}(y-z)} + \int_{13.9}^\infty dy \frac{\ln\eta(y)}{y^{3/2}} . \qquad (3.37)$$

For $1 \leq y \leq 13.9$, we use Arvieux's η's.[43] It is easy to see that this approximation gives for $z \leq 0$ values of $R(z)$ that are smaller than the true values, and thereby [see (3.17)] exaggerates the influence of the inelasticity on the LHC discontinuity in $F(z)$. The second integral on the RHS of (3.37) also appears in (3.33), so we include its effect there as well. The results are given in Table IV. In order

TABLE IV. Inelasticity effect on C_t^2.

$\int_{13.9}^\infty dy\, y^{-3/2}\ln\eta(y)$	$\bar\eta$ [a]	C_t^2
0	1	3.31
−0.05	0.911	3.31
−0.1	0.830	3.31
−0.2	0.689	3.31
−0.5	0.394	3.32
−1.0	0.155	3.37
−1.5	0.061	3.58

[a] Defined by (3.39).

to interpret the results it is useful to know that

$$\int_1^{13.9} dy \frac{\ln\eta(y)}{y^{3/2}} = -0.662 , \qquad (3.38)$$

and to calculate an effective inelasticity by

$$\ln\bar\eta \int_{13.9}^\infty dy\, y^{-3/2} = \int_{13.9}^\infty dy\, y^{-3/2} \ln\eta(y) . \qquad (3.39)$$

We see that the integral in Table IV must be somewhat larger in magnitude than that given by (3.38), and $\bar\eta$ must be quite small in order for the value of C_t^2 to deviate significantly from 3.31. Arvieux's inelasticity and the reference potential inelasticity at $z=13.9$ are 0.540 and 0.605, respectively, and rising. It thus appears that a conservative estimate for the inelasticity uncertainty in C_t^2 is a percent or two.

If we assume that the experimental doublet scattering length is 0.65 fm, then we must conclude that based on our calculations $C_t^2=3.31$ with an uncertainty of at most 3–4% associated with the LHC discontinuity and the inelasticity.

IV. SUMMARY AND DISCUSSION

We have shown by means of exact separable potential calculations that there is a significant variation of the ANP of the triton C_t^2 with the triton energy E_t. This strongly suggests that theoretical calculations of C_t^2 should be based on models of the triton which reproduce the experimental value of E_t. Using Fig. 1, we have extrapolated C_t^2 for the Reid potential[21] to the experimental value of E_t, and have obtained $C_t^2=3.26$. This agrees reasonably well with the extrapolated value of 3.48 found by Goldfarb *et al.*[17]

We have developed a partial wave dispersion relation technique for extracting C_t^2 from a knowledge of E_t, the doublet, n-d scattering length a_2, the doublet, s-wave, n-d inelasticity, and the analytic continuation of the on-shell, two-nucleon, s-wave scattering amplitudes to negative energies. Our method has been tested and found to be accurate, given information over a limited energy range. Moreover, it is stable against fairly sizable uncertainties in the inelasticities and the two-nucleon amplitudes. Our extracted result is $C_t^2=3.3\pm0.1$ for $a_2=0.65$ fm, where the error limits are based on the above uncertainties.

We now briefly compare our result with those obtained by other workers. We compare only with ANP's for the triton, and not those of ^{3}He. The DWBA analyses of nuclear reactions[1,2,17,18] give values of C_t^2 in the neighborhood of 3. The average value for the peripheral model analyses is[5] $C_t^2=2.64\pm0.64$. The expansion technique in the $\cos\theta$-plane gives[8] $C_t^2=2.8$ and[9,10] $C_t^2=2.9\pm0.1$.

The most recent value based on the use of forward dispersion relations is[13] $C_t^2 = 2.6 \pm 0.3$. Previous partial wave dispersion relation calculations give[14] $C_t^2 = 3.4$ and[15] $C_t^2 = 2.6 \pm 0.4$.

Goldfarb *et al.*[17] have obtained an average value of $C_t^2 = 2.9 \pm 0.6$ based on a number of sources.

Calculations of C_t^2 based on separable potential models of the triton[16,17,19,20] wave function give values of C_t^2 ranging from 2.6 to 3.8.

Of the various purely theoretical calculations[21-24] of C_t^2, the one based on the Malfliet-Tjon potential[44] has the most realistic value of E_t, namely 8.57 MeV. This potential gives[24] $C_t^2 = 3.8$.

We see that our value of C_t^2 lies somewhat above the middle of the range of values given above. It agrees well with the extrapolated values for the Reid potential. For future reference it should be kept in mind that (see Table III) changes in a_2 from the value of 0.65 fm we have used as the basis for our comparison can significantly change our value of C_t^2.

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