Virtual state of the three nucleon system

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The existence of a virtual state of the three nucleon system is established on the basis of three different analyses. Values for its pole position and residue in the doublet, s-wave, n-d elastic scattering amplitude, are obtained from a fit to the experimental data, from partial wave dispersion relations, and from an exact three-particle, separable potential calculation. The calculations indicate that these parameters are determined mainly by the one-nucleon exchange mechanism and the doublet scattering length a_2. For $a_2 = 0.65$ fm our best calculation gives an energy of 0.482 MeV below the elastic threshold, on the second Riemann sheet, and a residue parameter $C_v^2 = 0.0504$, where C_v^2 is defined in analogy to the triton asymptotic normalization parameter.

[NUCLEAR REACTIONS Three-nucleon virtual state; fits to data; dispersion relations; separable potential calculations.]

I. INTRODUCTION

It is by now well established that the doublet, s-wave effective range quantity $k \cot\delta$ has a pole just below the elastic threshold for n-d scattering. This pole has been incorporated in the phenomenological formula which has been used[1] to fit the low energy data for $k \cot\delta$. Its position and residue have been calculated from dispersion relations as well as exact solutions of three-particle equations with separable interactions.[2] The residue of the pole is defined by

$$k \cot\delta \xrightarrow[k^2 \to k_0^2]{} \frac{\text{residue}}{k^2 - k_0^2}.$$

The calculations show that this residue is negative, which implies that $k \cot\delta$ sweeps through a large range of positive values for $k^2 < k_0^2$. Below the breakup threshold the elastic scattering amplitude can be written in the form

$$\tilde{f}(k^2) = \frac{1}{k \cot\delta - ik}. \tag{1}$$

A virtual state exists if the denominator vanishes for k negative imaginary. Since $k \cot\delta$ is an even function of k, the above remarks suggest that there is a virtual state in the doublet, s wave. Unpublished calculations have found such a state.[3,4] It is important to establish the existence of such a state, as it may play as important a role in the theory of the four nucleon system as the $^1S_0$, two-nucleon virtual state plays in the theory of the three nucleon system.

Here we present calculations for the position of the virtual state pole in $\tilde{f}(k^2)$ and its residue. We define the residue parameter C_v^2 in analogy to the asymptotic normalization parameter of the triton[5] C_t^2. We obtain values for the pole position and residue from the van Oers-Seagrave fit,[1] from N/D solutions to partial wave dispersion relations, and from an exact separable potential calculation. The calculations indicate that these parameters are determined mainly by the one-nucleon exchange mechanism and the doublet scattering length a_2. For[6] $a_2 = 0.65$ fm our best calculation gives an energy of 0.482 MeV below the elastic threshold, on the second Riemann sheet, and $C_v^2 = 0.0504$.

II. VIRTUAL STATE POLE POSITION AND RESIDUE

The van Oers-Seagrave[1] fit to the low energy, doublet, s-wave phase shifts is given by

$$k \cot\delta = -A + Bk^2 - C/(1 + Dk^2), \tag{2}$$

with $A = 0.3105$ fm^{-1}, $B = 0.85$ fm, $C = 3.138$ fm^{-1}, and $D = 478.5$ fm^2. This fit gives for the pole in $k \cot\delta$ and its residue $k^2 = -2.09 \times 10^{-3}$ fm^{-2}, and -6.56×10^{-3} fm^{-3}, respectively, which agree well with the results of Ref. 2 for $a_2 = 0.29$ fm. From (1) and (2), it follows that $\tilde{f}(k^2)$ has a virtual state pole at $k_v = -i0.129$ fm^{-1} or at an energy $B_v = 3\hbar^2 |k_v|^2/(4M) = 0.515$ MeV below the elastic threshold, on the unphysical sheet. Here M is the nucleon mass.

We define the residue parameter by[5,7]

$$\tilde{f}(k^2) \xrightarrow[k^2 \to k_v^2]{} \frac{3iC_v^2}{2(k - k_v)}. \tag{3}$$

From (1)–(3) we find $C_v^2 = 0.0718$ which is remarkably small compared to our value of $C_t^2 = 3.3$ for the triton.[7]

In our dispersion theory calculations we shall include inelasticity effects, so we replace (1) by

$$\tilde{f}(k^2)=(\eta e^{2i\delta}-1)/(2ik)\,, \tag{4}$$

where η, the inelasticity, is one below the breakup threshold and less than one above it. We introduce a dimensionless energy variable and amplitude by means of the relations

$$z=3k^2/(4\alpha^2)\,, \tag{5}$$

$$f(z)=(\eta e^{2i\delta}-1)/(2iz^{1/2})\,, \tag{6}$$

where $\alpha=0.231\,61\ \text{fm}^{-1}$ is the deuteron wave number. In general[2,7-11] this amplitude has a right-hand cut (RHC) due to two- and three-particle unitarity, and a left-hand cut (LHC) associated with the exchange of nucleons and mesons. Two-particlelike N/D equations can be written in terms of the effective amplitude $F(z)$ defined by[7,12]

$$F(z)=\frac{f(z)}{R(z)}+\frac{1}{2iz^{1/2}}\left(\frac{1}{R(z)}-1\right)\,, \tag{7}$$

where

$$R(z)=\exp\left(-\frac{iz^{1/2}}{\pi}\int_1^\infty dy\frac{\ln\eta(y)}{y^{1/2}(y-z)}\right). \tag{8}$$

The function $R(z)$ has RHC's starting at $z=0$ and $z=1$, but no LHC's and is such that $F(z)$ satisfies a two-body-like unitarity relation, namely,

$$\operatorname{Im}F^{-1}(z+i\varepsilon)=-z^{1/2},\quad z\geq 0. \tag{9}$$

on the LHC we have

$$\operatorname{Im}F(z+i\varepsilon)=\operatorname{Im}f(z+i\varepsilon)/R(z),\quad z\leq -\tfrac{1}{3}. \tag{10}$$

We can represent $F(z)$ by

$$F(z)=N(z)/D(z)\,, \tag{11}$$

with $N(z)$ carrying the LHC and $D(z)$ the RHC. It is straightforward to show that[2,7] $D(z)$ can be obtained by solving the equation

$$D(z)=1+U(z^{1/2})-\frac{iz^{1/2}}{\pi}\int_{-b}^{-1/3}dy\frac{D(y)\operatorname{Im}F(y+i\varepsilon)}{y^{1/2}(z^{1/2}+y^{1/2})} \tag{12}$$

and that $N(z)$ can be obtained from

$$N(z)=-z^{-1/2}\operatorname{Im}D(z+i\varepsilon),\quad z>0. \tag{13}$$

Here $U(z^{1/2})$ is given by the same expression as the integral term in (12), except that the limits of integration are from $-\infty$ to $-b$. The point $z=-b$ divides the LHC into a part $(z>-b)$ whose discontinuity will be put in explicitly, and a part $(z<-b)$ whose effect will be parametrized through the function $U(z^{1/2})$. The point $z=-3$ lies at the junction of the one- and two-nucleon exchange cuts.[2,7-11] The function $U(z^{1/2})$ is analytic in the $z^{1/2}$ plane, cut along the negative imaginary axis from $-i\infty$ to $-i\sqrt{b}$.

In Ref. 2 the pole in $k\cot\delta$ was studied by means of the above N/D equations with $\eta=1$ and $b=3$, thereby neglecting inelasticity effects, and including explicitly only the one-nucleon exchange cut. The function $U(z^{1/2})$ was expanded in powers of $z^{1/2}$, and only the term linear in $z^{1/2}$ was retained. In general the constant term does not appear, since $U(0)=0$. The coefficient of the linear term was determined by fitting $f(z)$ at $z=0$ to the doublet scattering length a_2. An approximate analytic solution of the N/D equations was obtained by approximating the Born amplitude by a pole term. The approximate D function is given by Eq. (20) of Ref. 2. If this function is set equal to zero, a quadratic equation in $z^{1/2}$ is obtained, whose solutions are

$$z_\pm^{1/2}=i(2\gamma)^{-1}\{(1-\sqrt{3/5}\,\beta)\pm[(1-\sqrt{3/5}\,\beta)^2+4\sqrt{3/5}\,\gamma]^{1/2}\}, \tag{14}$$

where

$$\beta=2\alpha a_2/\sqrt{3}, \tag{15}$$

and

$$\gamma=\frac{2\sqrt{3/5}\,\beta+2\sqrt{5/3}\,d-\beta d}{2\sqrt{3/5}+d} \tag{16}$$

with

$$d=\frac{8}{5\sqrt{3}\,(1-\alpha\rho)}\,. \tag{17}$$

Here ρ is the deuteron effective range which we take to be[2,7] 1.701 fm. For reasonable values of a_2, γ is positive, and therefore $z_+^{1/2}$ and $z_-^{1/2}$ are positive and negative imaginary, respectively. Clearly $z_+^{1/2}$ and $z_-^{1/2}$ give the triton and virtual state poles in our approximate amplitude. The parameter $d=1.524$ is approximately $2\sqrt{3/5}$. If we approximate d by this value and expand to first order in β, we find

$$z_\pm\approx-\tfrac{3}{10}\,(3\pm\sqrt{5}\,)(1\mp\tfrac{2}{5}\,\sqrt{3}\beta). \tag{18}$$

A few numerical calculations show that (18) is a good approximation to results obtained from (14). Thus our simple theory predicts that the triton and virtual state energies are linear functions of the doublet scattering length. The Phillips plot[5,13] shows that an approximately linear relation exists between calculated values of the triton energy and doublet scattering length. Equations (14) and (18) do not give good quantitative agreement with this plot, however, we shall see that they give fairly accurate results for the virtual state.

From (3) and the results of Ref. (2), it is straightforward to show that the approximate amplitude that led to (14) and (18) gives for the residue parameter

$$C_v^2 = \frac{2(|z_+|^{1/2} - |z_-|^{1/2})(\sqrt{3/5} - |z_-|^{1/2})}{3(|z_+|^{1/2} + |z_-|^{1/2})(\sqrt{3/5} + |z_-|^{1/2})}. \quad (19)$$

For $a_2 = 0.29$ fm [the value from Eq. (2)] we find from (14) and (19) $B_v = 0.542$ MeV and $C_v^2 = 0.0630$, while for the latest experimental value[6] of $a_2 = 0.65$ fm, we find $B_v = 0.578$ MeV and $C_v^2 = 0.0547$. We see that our approximate N/D amplitude gives somewhat different values for the virtual state parameters than (2), and also predicts a non-negligible variation of B_v and C_v^2 with a_2.

In order to display more thoroughly the model dependence of the virtual state parameters, we present a set of calculations based on the N/D equations with various treatments of the RHC and LHC. The numerical method is described in Ref. 7. Initially we consider only $a_2 = 0.65$ fm. If we solve exactly the N/D equations that led to the approximate amplitude considered above we find $B_v = 0.491$ MeV and $C_v^2 = 0.0485$. This model can be improved upon by using the conformal mapping technique of Ref. 7 to obtain a better parametrization of $U(z^{1/2})$. We use Eq. (3.22) of Ref. 7 to map the $z^{1/2}$ plane, cut along the negative imaginary axis from $-i\infty$ to $-i\sqrt{b}$, onto the interior of the unit circle in the w plane, centered on the origin. We then expand $U(z^{1/2})$ as a power series in $w(z^{1/2})$, which converges everywhere in the cut $z^{1/2}$ plane. If we keep the two leading terms of the series and insure that $U(0) = 0$, we have

$$U(z^{1/2}) = c[w(z^{1/2}) - w(0)]. \quad (20)$$

The parameter c is real, and can easily be determined by the doublet scattering length.[7] Using (20) in (12) with $b = 3$ and $\eta = 1$, we find $B_v = 0.478$ MeV and $C_v^2 = 0.0512$. We see that the mapping technique produces changes of 3% and 6% in B_v and C_v^2, compared to the results immediately above. From now on we use the parametrization given by (20).

If we keep everything the same, but put in the inelasticities calculated from the reference potential of Ref. 7, we find $B_v = 0.486$ MeV and $C_v^2 = 0.0496$, which differ by 2% and 3%, respectively, from the results with $\eta = 1$.

The treatment of the LHC can be improved upon by using the reference potential model[7] of the two-nucleon and Δ' cuts with $b = -12.134$. Using the calculated inelasticities, we find $B_v = 0.482$ MeV and $C_v^2 = 0.0504$, or changes of 1% and 2% compared to the results immediately above. In Table I we give the virtual state parameters for a set of doublet scattering lengths. The N/D model used is the one just described. We see that there is a significant variation of B_v and C_v^2 for the range considered.

It is possible to carry out straightforward separable potential calculations for the virtual state parameters if one realizes that the n-d elastic scattering amplitude can be written in the form

$$f(z) = \frac{1}{h(z) - iz^{1/2}}, \quad (21)$$

where $h(z)$ is an even function of $z^{1/2}$. The proof of this follows. On the interval $0 \leq z \leq 1$, $f(z)$ satisfies the two-body unitarity relation (9). This implies that $h(z)$ is a real, analytic function of z, except for a LHC beginning at $z = \frac{1}{3}$, a RHC beginning at $z = 1$, and poles at the zeros of $f(z)$. Such a zero exists and gives rise to the pole in $k\cot\delta$ discussed in the Introduction. If one removes this pole from $h(z)$, a function is obtained which can by expanded as a Taylor series in powers of z, which converges for $|z| < \frac{1}{3}$. This leads to a representation for $h(z)$ which is clearly even in $z^{1/2}$. It is easy to calculate values of $h(z)$ for $z^{1/2}$ positive imaginary. From these $f(z)$ can be obtained for $z^{1/2}$ negative imaginary, which makes it possible to determine the virtual state parameters. The results of such a calculation with the reference potential described in Ref. 7 are $a_2 = 0.986$ fm, $B_v = 0.498$ MeV, and $C_v^2 = 0.0448$. This potential reproduces the experimental triton energy. An N/D calculation with the model used for Table I gives $B_v = 0.503$ MeV and $C_v^2 = 0.0438$ for

TABLE I. Virtual state parameters for various doublet scattering lengths.

a_2 (fm)	B_v (MeV)	C_v^2
−1.2	0.374	0.0932
−0.9	0.391	0.0857
−0.6	0.407	0.0783
−0.3	0.424	0.0711
0	0.442	0.0643
0.3	0.460	0.0577
0.6	0.479	0.0514
0.9	0.498	0.0455
1.2	0.517	0.0399

$a_2 = 0.986$ fm. Thus our most sophisticated N/D model gives results for the virtual state parameters which agree very well with those obtained from the exact solution of three-body equations with separable interactions.

The calculations presented here, as well as others we have performed indicate that the results of Table I are reliable, and furthermore that the virtual state parameters are determined mainly by the one-nucleon exchange mechanism and the doublet scattering length.

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