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Off-shell generalization of the Gordon's method for a nonlocal potential*

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Gordon's method for constructing wave functions for a nonlocal potential is generalized to obtain solutions of the van Leeuwen–Reiner equation. Exact analytical expression for off-shell Jost solutions and Jost functions are derived for the Mongan case IV potential.

[NUCLEAR REACTIONS Scattering theory; Generalization of the Gordon's method to off-shell scattering; Jost solutions and Jost functions for the Mongan case IV potential.]

The importance of nonlocal potentials in studying nuclear scattering reactions has been emphasized by a number of investigators.[1] Recently, two of us[2] have derived expressions for half-off-shell and fully off-shell T and K matrices for a number of realistic separable nucleon-nucleon interactions by using a wave function approach. In this approach T and K matrices are obtained from an inhomogeneous form of the Schrödinger equation[3] (hereafter referred to as the van Leeuwen–Reiner equation). There appear two momenta k and q, where k is an on-shell momentum related to the energy E by $E=k^2+i\epsilon$, $\epsilon \ll 1$, and q is an off-shell momentum. When $q=k$, the equation reduces to the conventional Schrödinger equation. For a nonlocal potential, the van Leeuwen–Reiner equation is an integrodifferential equation. The determination of the off-shell wave function is accomplished by converting this into an integral equation using the standard Green's function technique. In close analogy with the on-shell case,[4] the integral equation for the off-shell wave function is solved in terms of Fredholm determinants and the form factors of the potential.

Gordon[5] has proposed a method for constructing on-shell wave functions for nonlocal potentials, which treats the Schrödinger equation for separable interactions as an inhomogeneous differential equation. The fact that the solution (unknown) appears in the constant factor multiplying the inhomogeneity term does not present difficulties in constructing the solutions. It will be useful to have in the literature an off-energy-shell generalization of the Gordon's method. We attempt to do this by dealing with an N-term separable potential and solving for the off-shell wave function associated with the Jost boundary condition. A merit of the present treatment is that it does not involve the evaluation of typical contour integrals associated with the Tabakin's[6] procedure. Further, the method can be easily extended to potentials of arbitrary rank. Once the Jost solutions are known, the physical,[7] regular,[8] and principal value[9] wave functions can be expressed in terms of them. The T and K matrix calculations then follow in a natural way. Here we consider only the s-wave scattering and omit, for brevity, the subscript $l=0$. All results presented can easily be extended to higher partial waves. We work in units in which $\hbar^2/2m$ is unity.

We consider the scattering of a particle by a nonlocal potential $V(\vec{r},\vec{s})$. The van Leeuwen–Reiner equation for the Jost solution $f(k,q,r)$ is given by

$$\left(\frac{d^2}{dr^2}+k^2\right)f(k,q,r)-\int_0^\infty V(r,s)f(k,q,s)\,ds = (k^2-q^2)\,e^{iqr}. \quad (1)$$

Here

$$V(r,s)=2\pi rs\int_{-1}^{+1} d(\cos\theta)\,V(\vec{r},\vec{s}). \quad (2)$$

For a rank N separable potential

$$V(r,s)=\sum_{i=1}^{N}\lambda_i v^{(i)}(r)\,v^{(i)}(s),$$

Eq. (1) reduces to

$$\left(\frac{d^2}{dr^2}+k^2\right)f(k,q,r) = (k^2-q^2)\,e^{iqr}+\sum_{i=1}^{N}\lambda_i v^{(i)}(r)\int_0^\infty v^{(i)}(s)f(k,q,s)\,ds. \quad (3)$$

The solution of Eq. (3) has the asymptotic normalization

$$f(k,q,r)\underset{r\to\infty}{\sim} e^{iqr}. \quad (4)$$

When $q=\pm k$, the function $f(k,q,r)$ goes over into the two irregular solutions of the Schrödinger equation, which enter into the theory of ordinary

Jost function,[10] i.e.,

$$f(\pm k, r) = f(k, \pm k, r). \tag{5}$$

The general solution of Eq. (3) can be obtained by adding a solution of the differential equation

$$\left(\frac{d^2}{dr^2} + k^2\right) f_1(k, q, r) = (k^2 - q^2)\, e^{iqr} \tag{6}$$

to a particular integral determined by

$$\left(\frac{d^2}{dr^2} + k^2\right) f_2(k, q, r) = \sum_{i=1}^{N} \lambda_i v^{(i)}(r) \int_0^\infty v^{(i)}(s) f_2(k, q, s)\, ds. \tag{7}$$

A solution of Eq. (6) is given by

$$f_1(k, q, r) = e^{iqr}. \tag{8}$$

In the on-energy-shell limit, Eq. (8) yields the solution for a free particle Schrödinger equation satisfying Jost boundary condition, which enters in Eq. (2.7) of Ref. 5.

Following Gordon,[5] we first solve the equation

$$\left(\frac{d^2}{dr^2} + k^2\right) g_i(k, r) = v^{(i)}(r), \tag{9}$$

and then construct the general solution by using the ansatz

$$f(k, q, r) = e^{iqr} + \sum_{i=1}^{N} g_i(k, r) C_i(k, q). \tag{10}$$

From Eqs. (3), (9), and (10) it can be shown that $C_i(k, q)$ satisfies the matrix equation

$$\sum_j \left(\delta_{ij} - \lambda_i \int_0^\infty v^{(i)}(s) g_j(k, s)\, ds\right) C_j(k, q) = \lambda_i \int_0^\infty v^{(i)}(s)\, e^{iqs}\, ds. \tag{11}$$

Equations (10) and (11) taken together completely define the off-shell Jost solution for a rank N separable potential. For the s-wave case, the Jost function is given by

$$f(k, q) = f(k, q, 0). \tag{12}$$

In order to illustrate the usefulness of the general results presented above, we consider the problem of nucleon-nucleon scattering. For reasons of physical interest we focus our attention on the Mongan case IV potential.[11] Mongan has introduced this potential to fit the ${}^1S_0\ N-N$ phase shifts. In configuration space the Mongan potential can be written in the form

$$V(r, s) = \sum_{i=1}^{2} \lambda_i v^{(i)}(r) v^{(i)}(s) = \lambda_1 e^{-\alpha_1 (r+s)} + \lambda_2 e^{-\alpha_2 (r+s)}. \tag{13}$$

From Eqs. (9) and (13),

$$g_i(k, r) = \frac{1}{\alpha_i^2 + k^2}\, e^{-\alpha_i r}. \tag{14}$$

Substituting Eq. (14) in Eq. (11) and specializing to rank 2 case, we get

$$\sum_{j=1}^{2} (\delta_{ij} - a_{ij}) C_j(k, q) = \frac{\lambda_i}{\alpha_i - iq}, \quad i = 1, 2, \tag{15}$$

where

$$a_{ij} = \frac{\lambda_i}{(\alpha_i + \alpha_j)(\alpha_j^2 + k^2)}. \tag{16}$$

The set of simultaneous equations given by (15) can easily be solved to get the values of $C_1(k, q)$ and $C_2(k, q)$. When these values are substituted in Eq. (10), we get the off-shell Jost solution in the form

$$f(k, q, r) = e^{iqr} + \frac{1}{\det a(k)} [I_1(k, q)\, e^{-\alpha_1 r} + I_2(k, q)\, e^{-\alpha_2 r}], \tag{17}$$

where

$$\det a(k) = 1 - \frac{\lambda_1}{2\alpha_1(\alpha_1^2 + k^2)} - \frac{\lambda_2}{2\alpha_2(\alpha_2^2 + k^2)} + \frac{\lambda_1 \lambda_2 (\alpha_1 - \alpha_2)^2}{4\alpha_1 \alpha_2 (\alpha_1 + \alpha_2)^2 (\alpha_1^2 + k^2)(\alpha_2^2 + k^2)}, \tag{18}$$

$$I_1(k, q) = \frac{\lambda_1(\alpha_1 + iq)}{(\alpha_1^2 + k^2)(\alpha_1^2 + q^2)} - \frac{\lambda_1 \lambda_2 (\alpha_1 + iq)}{2\alpha_2(\alpha_1^2 + k^2)(\alpha_2^2 + k^2)(\alpha_1^2 + q^2)} + \frac{\lambda_1 \lambda_2 (\alpha_2 + iq)}{(\alpha_1 + \alpha_2)(\alpha_1^2 + k^2)(\alpha_2^2 + k^2)(\alpha_2^2 + q^2)}, \tag{19}$$

and

$$I_2(k, q) = \frac{\lambda_2(\alpha_2 + iq)}{(\alpha_2^2 + k^2)(\alpha_2^2 + q^2)} - \frac{\lambda_1 \lambda_2 (\alpha_2 + iq)}{2\alpha_1(\alpha_1^2 + k^2)(\alpha_2^2 + k^2)(\alpha_2^2 + q^2)} + \frac{\lambda_1 \lambda_2 (\alpha_1 + iq)}{(\alpha_1 + \alpha_2)(\alpha_1^2 + k^2)(\alpha_2^2 + k^2)(\alpha_2^2 + q^2)}. \tag{20}$$

The Jost function is given by

$$f(k, q) = 1 + \frac{1}{\det a(k)} [I_1(k, q) + I_2(k, q)]. \tag{21}$$

One can perform a couple of checks on the fairly complicated expressions for the off-shell Jost solution and Jost function derived above.

(i) In the limit $q \to k$, Eqs. (17) and (21) yield the corresponding on-shell results given by Mulligan *et al.*[12]

(ii) From Eq. (13) we see that in the special case $\lambda_1 = \lambda$ and $\lambda_2 = 0$, the Mongan potential goes over to the Yamaguchi potential. Substituting these values in Eq. (21) we get

$$f(k,q) = 1 + \frac{2\lambda\alpha}{(\alpha - iq)[2\alpha(\alpha^2 + k^2) - \lambda]} . \quad (22)$$

The result in Eq. (22) is in agreement with the result for the off-shell Jost function for the Yamaguchi potential obtained by using a generalized form of the Jost-Pais theorem.[8]

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[1]G. W. Greenless and Y. C. Tang, Phys. Lett. 34B, 359 (1971); D. R. Thompson and Y. C. Tang, Phys. Rev. C 4, 306 (1971); G. W. Greenless, W. Makofske, Y. C. Tang, and D. R. Thompson, *ibid.* 6, 2057 (1972); A. Boinghoff, H. Huskan, and A. Weiguny, Phys. Lett. 61B, 9 (1976).

[2]B. Talukdar, U. Das, and S. Mukhopadhyaya, J. Math. Phys. (to be published); B. Talukdar, U. Das, N. Mallick, and M. Sengupta, *ibid.* (to be published).

[3]J. M. J. van Leeuwen and A. S. Reiner, Physica 27, 99 (1961).

[4]M. Coz, L. G. Arnold, and D. Mackellar, Ann. Phys. (N. Y.) 59, 219 (1970).

[5]R. G. Gordon, J. Chem. Phys. 52, 6211 (1970).

[6]F. Tabakin, Ann. Phys. (N.Y.) 30, 51 (1964).

[7]M. G. Fuda and J. S. Whiting, Phys. Rev. C 8, 1255 (1973).

[8]C. S. Warke and M. K. Srivastava, Phys. Rev. C 15, 561 (1977).

[9]B. Talukdar, N. Mallick, and S. Mukhopadhyaya, Phys. Rev. C 15, 1252 (1977).

[10]R. Jost, Helv. Phys. Acta 20, 256 (1947).

[11]T. R. Mongan, Phys. Rev. 175, 1260 (1968); 178, 1597 (1969).

[12]B. Mulligan, L. G. Arnold, B. Bagchi, and T. O. Krause, Phys. Rev. C 13, 2131 (1976).