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# Orthogonality in medium energy nuclear reactions

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The kinematic range over which orthogonality appreciably changes single-nucleon ejection amplitudes is investigated. No simple prescription is found which can be applied to plane wave approximations to achieve the required modifications.

[ NUCLEAR REACTIONS Orthogonality in nucleon emission amplitudes. ]

It has recently been pointed out[1] that orthogonality may play a more important role in certain nuclear reactions at medium energies than has been generally realized. In the case of a scalar knockout interaction, for example, the amplitude must vanish at zero momentum transfer from the probe that induces the excitation as in $(\gamma,p)$, $(\pi,p)$, $(e,e'p)$, and so on. This feature is usually lost in the conventional calculational schemes such as plane or distorted wave impulse approximations (PWIA or DWIA). In order to study where the effect is significant and to search for a simple prescription for incorporating it in standard PWIA or DWIA, we have examined some exact models and also realistic numerical cases.

Several exact forms for the transition matrix element which automatically respect orthogonality may be obtained from the standard DWIA amplitude

$$A_{\vec{p}}(\vec{k})=\langle f|\mathcal{O}|i\rangle=\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\mathcal{O}\psi(\vec{x}), \qquad (1)$$

where $\mathcal{O}$ is in general a nonscalar operator that involves the momentum transfer for $\vec{k}$ and induces the transition from the state $\psi$ of a bound nucleon to the continuum wave $\phi_{\vec{p}}^{(-)}$ for momentum $\vec{p}$. One of them is generated by exploiting a commutator relation

$$A_{\vec{p}}(\vec{k})=(E_f-E_i)^{-1}\langle f|[H,\mathcal{O}]|i\rangle =(\vec{p}^2+\kappa^2)^{-1}\{\langle f|[-\vec{\nabla}^2,\mathcal{O}]|i\rangle +\langle f|[2mV,\mathcal{O}]|i\rangle\}, \qquad (2)$$

which displays orthogonality, even for approximate wave functions, for $\mathcal{O}\equiv 1$. In Eq. (2), $E_i=-\kappa^2/2m$ is the energy of the bound nucleon of mass $m$, $E_f=E_{\vec{p}}=\vec{p}^2/2m$ is the final nucleon energy, $H$ is the nucleon Hamiltonian, and $V$ is its potential. The commutator $[V,\mathcal{O}]$ may be important for spin, isospin, or velocity-dependent cases but will be ignored here. Then Eq. (2) may be written as

$$A_{\vec{p}}(\vec{k})=-(\vec{p}^2+\kappa^2)^{-1}\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x}) \times[(\vec{\nabla}^2\mathcal{O})+2(\vec{\nabla}\mathcal{O})\cdot\vec{\nabla}]\psi(\vec{x}) =(\vec{p}^2+\kappa^2)^{-1}\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x}) \times[\vec{\nabla}\cdot(\vec{\nabla}\mathcal{O})-(\vec{\nabla}\mathcal{O})\cdot\vec{\nabla}]\psi(\vec{x}), \qquad (3)$$

where the second version follows upon integrating by parts. Again, for $\mathcal{O}\equiv 1$, orthogonality is immediate.

For the case $\mathcal{O}=\exp(i\vec{k}\cdot\vec{x})$, we sketch the derivation of an alternative to Eq. (2) which explicitly vanishes for $k\to 0$ as required by orthogonality. In momentum space the amplitude takes the form

$$A_{\vec{p}}(\vec{k})=\int\frac{d^3p'}{(2\pi)^3}\,\phi_{\vec{p}}^{(-)\dagger}(\vec{p}')\psi(\vec{p}'-\vec{k}), \qquad (4)$$

where

$$\phi_{\vec{p}}^{(-)}(\vec{p}')=(2\pi)^3\delta(\vec{p}-\vec{p}')+(E_{\vec{p}}-E_{\vec{p}'}-i\epsilon)^{-1}\int\frac{d^3p''}{(2\pi)^3}\langle\vec{p}'|V|\vec{p}''\rangle\phi_{\vec{p}}^{(-)}(\vec{p}''). \qquad (5)$$

Substituting Eq. (5) into (4) and taking the limit $A_{\vec{p}}(0)=0$ yields

 

$$\psi(\vec{p}\,'-\vec{k})=-\int\frac{d^3p''}{(2\pi)^3}(E_{\vec{p}'-\vec{k}}-E_{\vec{p}''}+i\epsilon)^{-1}\langle\vec{p}\,'-\vec{k}|t(E_{\vec{p}'-\vec{k}})|\vec{p}\,''\rangle\,\psi(\vec{p}\,''). \tag{6}$$

The matrix element of the scattering amplitude $t$ satisfies

$$(2\pi)^3\delta(\vec{p}-\vec{p}\,'')\langle\vec{p}\,''|t(E_{\vec{p}})|\vec{p}\,'\rangle=\langle\vec{p}\,''|V|\vec{p}\,'\rangle\phi_{\vec{p}}^{(-)\dagger}(\vec{p}\,''), \tag{7}$$

which has been used to obtain (6). Inserting Eqs. (5) and (6) in (4) for general $\vec{k}$,

$$A_{\vec{p}}(\vec{k})=-\int\frac{d^3p'}{(2\pi)^3}[(E_{\vec{p}-\vec{k}}-E_{\vec{p}''}+i\epsilon)^{-1}\langle\vec{p}-\vec{k}|t(E_{\vec{p}-\vec{k}})|\vec{p}\,''\rangle-(E_{\vec{p}}-E_{\vec{p}''+\vec{k}}+i\epsilon)^{-1}\langle\vec{p}|t(E_{\vec{p}})|\vec{p}\,''+\vec{k}\rangle]\,\psi(\vec{p}\,''), \tag{8}$$

which explicitly vanishes at $\vec{k}=0$. We note that using the eikonal form[2] for the combination $G_0t$ in Eq. (8) the amplitude is given by $(\vec{p}\cdot\vec{k}/p^2)A_{\vec{p}}^{\rm PWIA}(\vec{k})$ as in Ref. 1.

If we evaluate Eq. (3) with $\mathcal{O}=\exp(i\vec{k}\cdot\vec{x})$ and $\phi_{\vec{p}}^{(-)}(\vec{x})=\exp(i\vec{p}\cdot\vec{x})$, we find, for $k\ll p$,

$$A_{\vec{p}}(\vec{k})=2\frac{\vec{k}\cdot\vec{p}}{p^2+\kappa^2}\,\psi(\vec{p}-\vec{k}), \tag{9}$$

in contrast to the eikonal result in Ref. 1, and in the previous paragraph, where the factor of 2 is lacking. In the eikonal case it is removed by the action of the potential in $\phi^{(-)}$. This suggests that the modification of PWIA and very likely of DWIA is highly model dependent, which, in fact, we find to be the case for the models we have studied.

We now examine two models where exact results can be obtained analytically. The first one is the Coulomb potential where $\psi(x)\propto\exp(-\kappa x)$ is the nonrelativistic hydrogenic 1S bound state and $\phi^{(-)}$ the corresponding continuum state. The exact expression in coordinate space is

$$A_{\vec{p}}(\vec{k})=\int d^3x\,\exp(-i\,\vec{p}\cdot\vec{x})F(-in,1,i(px+\vec{p}\cdot\vec{x}))\exp(i\vec{k}\cdot\vec{x})\exp(-\kappa x), \tag{10}$$

where $F$ is the confluent hypergeometric function, and $n=-e^2m/p$, $\kappa=e^2m$, and we have suppressed all normalizations. Using standard methods[3] involving an integral representation for $F$, we arrive at the exact result

$$A_{\vec{p}}(\vec{k})=-4\pi\frac{d}{d\kappa}\left.\frac{[k^2-(p+i\kappa)^2]^{in}}{[(\vec{p}-\vec{k})^2+\kappa^2]^{1+in}}\right|_{n=-\kappa/p} \tag{11}$$

It is seen that $A_p(\vec{k}=0)=0$ and the plane wave case is obtained from Eq. (11) with $n=0$. Then, for $k\ll p$,

$$\frac{A_{\vec{p}}(\vec{k})}{A_{\vec{p}}^{\rm PWIA}(\vec{k})}\simeq 2\frac{\vec{p}\cdot\vec{k}}{\vec{p}^2+\kappa^2}e^{-\pi n}\left(\frac{p+i\kappa}{p-i\kappa}\right)^{in}(1+in)\Big|_{n=-\kappa/p}\xrightarrow[n\to 0]{}2\frac{\vec{p}\cdot\vec{k}}{\vec{p}^2+\kappa^2}, \tag{12}$$

to be compared with Eq. (9).

The other exact model we use is a square well potential in the limit of zero range and infinite depth, for which it can be shown[4] that

$$\psi(x)=(\kappa/2\pi)^{1/2}e^{-\kappa x}/x,\quad \phi_p^{(-)\dagger}(\vec{x})=e^{-i\vec{p}\cdot\vec{x}}+[e^{i\delta_0(p)}\sin(px+\delta_0)-\sin(px)]/px,\quad \tan\delta_0(p)=-p/\kappa, \tag{13}$$

and for which

$$A_{\vec{p}}(\vec{k})=4\pi(\kappa/2\pi)^{1/2}\left[\frac{1}{(\vec{p}-\vec{k})^2+\kappa^2}+\frac{e^{2i\delta_0}}{4pk}\ln\left(\frac{p^2-k^2+\kappa^2-2i\kappa k}{(p-k)^2+\kappa^2}\right)-\frac{1}{4pk}\ln\left(\frac{p^2-k^2+\kappa^2-2i\kappa k}{(p+k)^2+\kappa^2}\right)\right]. \tag{14}$$

Here again $A_p(\vec{k}=0)=0$ and

$$A_{\vec{p}}(\vec{k})\simeq 2\frac{\vec{k}\cdot\vec{p}}{p^2+\kappa^2}A_{\vec{p}}^{\rm PWIA}(\vec{k}),\quad k\ll p \tag{15}$$

to be compared to Eqs. (9) and (12). We note that neither exact model has the property of a realistic nuclear potential, namely finite well depth and range and a surface thickness.

We therefore turn to an analysis based on the numerical solution of a real Woods-Saxon potential with well depth −40 MeV, range 3 fm, and thickness 0.5 fm. Careful use of standard numerical procedures is adequate to study the effects of orthogonality, but these may be brought into sharper focus by methods[3] tailored to a discrete mesh. We find for the parameter $\alpha(p)$ defined by

$$\frac{A_{\vec{p}}(\vec{k})}{A_{\vec{p}}^{\rm PWIA}(\vec{k})}=\alpha(p)\frac{\vec{k}\cdot\vec{p}}{\vec{p}^2}+O(k^2), \tag{16}$$

the absolute values 8.6, 2.9, and 1.3 for $p=2.0$, 2.8, and 3.5 $fm^{-1}$ or very roughly $\alpha(p) \sim 63\ fm^{-3}/p^3$. Thus, in the medium energy range, $\alpha(p)$ is not constant as suggested in Eq. (9) or in the eikonal case. (For different values of the potential parameters with or without spin-orbit potential or imaginary part the situation looks similar.) The effects of orthogonality appear to persist out to 1 to 2 $fm^{-1}$, although a precise comparison is difficult because at these momenta the plane wave approximation is poor. The range for which linearity in $k$ holds for the ratio in Eq. (16) is limited to $\sim 0.15\ fm^{-1}$.

We conclude that the strong model dependence of the orthogonality effects has prevented us from finding a simple, general method to incorporate orthogonality. The difficulty will become still more acute when an optical potential with an imaginary part is used to incorporate absorption in the continuum state.

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