PHYSICAL REVIEW C VOLUME 19, NUMBER 6 JUNE 1979

Comparison of two-dimensional boson and variable-moment-of-inertia models

Steven A. Moszkowski
Department of Physics, University of California, Los Angeles, California 90024
(Received 26 December 1978)

We discuss a simple two-dimensional model containing interacting s and d bosons. This model reproduces the general features of actual collective nuclear spectra, e.g., the transition between vibrational and rotational spectra, and it also gives results close to those of the variable moment of inertia model.

[NUCLEAR STRUCTURE Interacting boson model, variable moment of inertia model.]

I. INTRODUCTION

In the last few years, some features of collective excitation spectra in nuclei have been accounted for by means of the interacting boson model (IBA). A rather different description, the variable-moment-of-inertia (VMI) model has also been quite successful, indeed, perhaps even slightly more so than the boson model. The point of this paper is to investigate the boson model in somewhat more detail, to study, for example, the dependence of the spectrum on the number of bosons. Also, we wish to see what, if any, kind of underlying connection can be established between the IBA and VMI models.

For this purpose we find it useful to work with a two-dimensional version of these models. We believe that while this does not fit the experimental data correctly, i.e., $L(L+1)$ is replaced by L^2, whatever *connection* there is between the two models is easier uncovered in two dimensions than in three, nevertheless we believe that the physics is very similar in the two cases [except for replacing L by $(L(L+1))^{1/2}$ etc.]. Thus, we hope that the conclusions drawn from the two-dimensional models are also at least approximately valid in three dimensions.

Section II deals with the transition between vibrational and rotational spectra in a very simple way. In Sec. III we compare the results of boson (IBA) and variable-moment-of-inertia (VMI) models. In Sec. IV we show that a more generalized version of the boson model can reproduce the VMI results, at least in the weak coupling limit.

II. DESCRIPTION OF COLLECTIVE SPECTRA IN TERMS OF QUADRUPOLE BOSON-BOSON INTERACTIONS

In this section we study the collective spectra by introducing boson-boson interactions. This is analogous to the introduction of the Elliott model. The model used here is the two-dimensional version of a model considered by Arima and Iachello.[1] (See also Refs. 2 and 3.) We assume a quadrupole coupling strength g between the bosons (which also implies a single boson energy)

$$\mathcal{E}_0 = -g\,, \tag{1a}$$

$$\mathcal{E}_2 = -\tfrac{1}{2}g\,, \tag{1b}$$

in addition to the assumed splitting ϵ between s and d boson energies. The two-particle energy matrix is shown in Table I.

In the weak coupling limit, expansion of the N boson ground state energy in powers of g gives

$$\mathcal{E}_0 = -N\epsilon - Ng - \frac{N(N-1)}{2}\frac{g^2}{\epsilon}\cdots\,. \tag{2}$$

It is convenient to express these results in terms of the dimensionless coupling parameter

$$X = 2Ng/\epsilon\,. \tag{3}$$

Then

$$\mathcal{E}_0 = -N\epsilon\left[1 - \frac{X}{2N} - \left(1-\frac{1}{N}\right)\frac{X^2}{8N^2}\cdots\right]. \tag{4}$$

The excitation energy of the first excited $L=2$ state is

$$\frac{E_2}{\epsilon} = 1 - \frac{X}{2}\left[1-\frac{3}{2N}\right] - \frac{X^2}{8}\left[1-\frac{5}{N}+\frac{4}{N^2}\right]\cdots\,. \tag{5}$$

In the limit $N\to\infty$ this is consistent with the result

$$E_2/\epsilon = (1-X)^{1/2} \tag{6}$$

obtained by applying the random-phase approximation (RPA). The lowest $L=4$ state is at twice the energy of the 2^+ state in the vibrational limit $X=0$. Moving away from that limit, we find

$$E_4/E_2 = 2\left(1 + \frac{1}{2N}X + \frac{1}{4N}X^2\cdots\right), \tag{7}$$

TABLE I. Energy matrix of the two boson system in the intermediate coupling model discussed in Sec. II.

$L=0$		
	ss	$d\bar{d}$
ss	$-2\epsilon-2g$	$-\sqrt{2}g$
$d\bar{d}$	$-\sqrt{2}g$	$-g$
$L=2$		
	sd	
sd	$-\epsilon-\frac{5}{2}g$	
$L=4$		
	dd	
dd	$-g$	

and similarly for the first excited 0 state, denoted here by 0′,

$$E_{0'}/E_2=2\left(1+\frac{1}{2N}X+\frac{1}{2N}X^2\cdots\right). \tag{8}$$

In the limit $N\to\infty$ both states are at exactly twice the energy of the 2^+ state even if X is appreciable—as long as $X<1$.

Finally, let us consider the rotational (strong coupling limit $X\gg1$). Here the ground state energy is given by

$$\mathcal{E}_0=-g\left[\frac{N(N+1)}{2}+\frac{2N^3}{2N-1}\frac{1}{X}\cdots\right]. \tag{9}$$

In the limit $X\to\infty$, the states form exact rotational bands. The rotational energies are

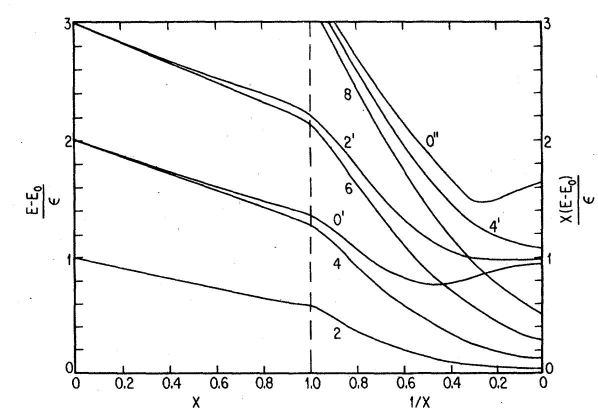

FIG. 1. Excitation energies of collective states vs dimensionless coupling strength [Eq. (3)] in the (s,d) boson model with quadrupole-quadrupole interactions for case $N=8$.

TABLE II. Energy ratios E_4/E_2 and $E_{0'}/E_2$ versus X, the dimensionless coupling strengths for $N=8$ and 16, according to the model discussed in Sec. II.

	$N=8$		$N=16$	
X	4^+	0^+	4^+	0^+
0	2.000	2.00	2.000	2.00
0.4	2.070	2.083	2.038	2.047
0.8	2.207	2.293	2.144	2.219
1.2	2.464	2.812	2.499	3.055
1.6	2.849	3.927	3.272	6.676
2.0	3.373	6.476	3.855	15.69
3.2	3.842	12.19	3.970	28.60
6.4	3.989	20.54	3.996	43.75
∞	4.000	30.00	4.000	62.00

$$E_L=g\left[\frac{1}{8}+\frac{N}{2(2N-1)}\frac{1}{X}\right]L^2+\cdots\,. \tag{10}$$

The rotational spacing increases as X becomes smaller. We also find that there is a further correction term proportional to L^4 in the energy. This leads to a breakdown in the rotational spacing. Thus, we find

$$\frac{E_4}{E_2}=4\left[1-\frac{3}{N^2X^2}\right]+\cdots\,. \tag{11}$$

If $N\to\infty$ or $X\to\infty$, this correction disappears.

Finally, we show some specific results of our calculations. Figure 1 shows the excitation energies of the vibrational and rotational levels as a function of X for the case $N=8$. The rather sudden change from vibrational to rotational spectra as we vary X is apparent. Table II shows the energy ratios E_4/E_2 and $E_{0'}/E_2$ versus X, for $N=8$ and 16. Again, for the 4^+ state we note the transition from vibrational to rotational energies as X exceeds 1, as expected. The transition becomes more sudden with increasing N. Also shown is the energy of the 0^+ state. In the limit $X\to\infty$, the energy ratio is

$$E_{\mathrm{vib}}^{0^+}/E_{\mathrm{rot}}^{2^+}=4N-2\,. \tag{12}$$

Our results seem to agree quite well with those of the variable-moment-of-inertia model,[4] which is based on rather different assumptions. This point is discussed below.

III. COMPARISON OF THE BOSON AND VARIABLE-MOMENT-OF-INERTIA (VMI) MODEL

The two-dimensional VMI model is the same as the three-dimensional one[4] except that $[J(J+1)]^{1/2}$ is replaced by L. The relation between E and L is obtained as follows.

The moment of inertia is a quadratic function of frequency:

$$\mathscr{I} = \mathscr{I}_0(1 + (\omega/\omega_0)^2) . \tag{13}$$

Then

$$L = \mathscr{I}\omega = \mathscr{I}_0\omega\left(\frac{\omega}{\omega_0} + (\omega/\omega_0)^3\right), \tag{14}$$

$$E_L = \int \omega\, dL = \mathscr{I}_0\omega_0^2(\tfrac{1}{2}(\omega/\omega_0)^2 + \tfrac{3}{4}(\omega/\omega_0)^4). \tag{15}$$

Eliminating ω we obtain a parametric relation between E and L. In the limit $\omega \ll \omega_0$, we find for the energies of the yrast states

$$E_L = \frac{L^2}{2\mathscr{I}_0} - \frac{L^4}{4\mathscr{I}_0{}^3\omega_0{}^2} \cdots , \tag{16}$$

the same kind of correction as in the rotation-vibration model. On the other hand, if $\omega \gg \omega_0$ we obtain

$$E_L = \tfrac{3}{4}\mathscr{I}_0\omega_0{}^2(L/\mathscr{I}_0\omega_0)^{4/3}, \tag{17}$$

which implies for the energy ratio of the lowest two excited states

$$R_4 = E_4/E_2 = 2^{4/3} = 2.52 . \tag{18}$$

The corresponding R_4 in 3 dimensions is

$$(\tfrac{10}{3})^{2/3} = 2.23 . \tag{19}$$

We can get even lower values of R_4 by allowing $\mathscr{I}_0$ to be negative, writing

$$\mathscr{I}_0 = -\mathscr{I}^0 \tag{20}$$

$$= \mathscr{I}^0(-1 + (\omega/\omega_0)^2), \tag{21}$$

$$L = \mathscr{I}^0\omega_0\left(-\frac{\omega}{\omega_0} + (\omega/\omega_0)^3\right), \tag{22}$$

$$E_L = \mathscr{I}^0\omega_0{}^2(-\tfrac{1}{2}(\omega/\omega_0)^2 + \tfrac{3}{4}(\omega/\omega_0)^4). \tag{23}$$

In the limit $\omega \gg \omega_0$, we again get

$$R_4 = 2^{4/3} . \tag{24}$$

But in the opposite limit, which is now $\omega \sim \omega_0$, we find by a simple calculation

$$\frac{E_L}{\mathscr{I}^0\omega_0{}^2} = \frac{L}{\mathscr{I}^0\omega_0} + \tfrac{1}{4}(L/\mathscr{I}^0\omega_0)^2 - \tfrac{1}{8}(L/\mathscr{I}^0\omega_0)^3 \cdots . \tag{25}$$

This can be written in the form

$$E_L = a_1L + a_2L^2 + a_3L^3 \cdots . \tag{26}$$

It is convenient to define the parameter ρ by

$$\rho = -\frac{a_1a_3}{a_2{}^2}. \tag{27}$$

The VMI model gives

$$\rho_{\text{VMI}} = 2 . \tag{28}$$

If $L \ll \mathscr{I}^0\omega_0$, we get a harmonic spectrum, i.e.,

$$R_4 = 2. \tag{29}$$

In the corresponding three-dimensional case,

$$R_4 = (\tfrac{10}{3})^{1/2} = 1.82 . \tag{30}$$

In fact, the three-dimensional VMI *never* gives an exact vibrational spectrum, though it comes close to it for the proper choice of the parameters. By comparison, the boson model with quadrupole forces gives (in the weak coupling limit $X \to 0$)

$$E_L = \tfrac{1}{2}\epsilon L + \tfrac{1}{4}gL^2 - \frac{g^2}{16\epsilon}L^3 \cdots , \tag{31}$$

which is of the form (27) with

$$\rho_{\text{IBA}} = \tfrac{1}{2} . \tag{32}$$

Thus the two models give different results, if expanded to order L^3. In general, if the parameters of the two models are chosen so as to give the same E_2 and E_4, i.e., R_4, the boson model with quadrupole interactions gives slightly larger R_8 than the VMI model.

Table III shows a comparison of the two models all the way from the vibrational to the rotational limits. Specifically we tabulate $\sqrt{R_8}$ as a function of R_4. We chose this particular quantity since it happens to be identical to R_4 in both extreme limits and never deviates very much in the intermediate coupling regions.

The results are quoted for the case of 8 bosons. For 16 bosons the results are almost identical (within 0.01).

TABLE III. Energy ratio $\sqrt{R_8}$ as a function of R_4 for various values of R_4 obtained with the three-dimensional boson model with quadrupole forces for $N=8$ (Sec. II) and with the two-dimensional VMI model (Sec. III).

R_4		$\sqrt{R_8}$ Boson	$\sqrt{R_8}$ VMI
$2+\epsilon$	$\epsilon \ll 1$	$2+\frac{3}{2}\epsilon$	$2+\frac{3}{2}\epsilon$
2.25		2.31	2.28
2.50		2.58	2.50
3.00		3.05	2.90
3.50		3.48	3.36
3.75		3.71	3.62
$4-\epsilon$	$\epsilon \ll 1$	$4-\frac{5}{2}\epsilon$	$4-\frac{5}{2}\epsilon$

IV. GENERAL BOSON MODEL

For an arbitrary interaction between the bosons (i.e., not necessarily quadrupole) the energy of the yrast states expanded in powers of L is (in the limit of $N\to\infty$)

$$E=\left[\epsilon+N(w_{20}-w_{00})-\frac{N^2(v_{00\to 2\bar{2}})^2}{4\epsilon}\right]\frac{L}{2}+\left[-\frac{(w_{22}-2w_{20}+w_{00})}{2}+\frac{N(v_{00\to 2\bar{2}})^2}{2\epsilon}\right]\frac{L^2}{4}$$
$$+\left[-\frac{(v_{00\to 2\bar{2}})^2}{4}\right]\left[1+\frac{2N}{\epsilon}(w_{22}-2w_{20}+w_{00})\right]\frac{L^3}{8}\cdots . \quad (34)$$

Here w_{00} represents the matrix element of the interaction between two s bosons. w_{20} is the symmetrized matrix element for d and s bosons, and w_{22} refers to the interaction between two d bosons with the same m. $w_{2\bar{2}}$, where the m's are antiparallel, does not enter into Eq. (33).

Finally, $w_{00\to 2\bar{2}}$ refers to the off-diagonal matrix element of the interaction. According to the RPA, the phonon energy is

$$[\epsilon(\epsilon+Nv)]^{1/2} ,$$

where v is proportional to the interaction matrix element and independent of N.

The boson model reproduces this dependence (up to order N^2) provided

$$(v_{00\to 2\bar{2}})^2=2(w_{20}-w_{00})^2 . \quad (34)$$

We can again parametrize the energy in the form (27). Then in the weak coupling limit

$$X\ll 1 \quad (35)$$

we find

$$\rho_{\rm IBA}=\rho_0\left(1+\frac{2N}{\epsilon}(w_{22}-2w_{20}+w_{00})(1-\rho_0)\right)\cdots , \quad (36)$$

where

$$\rho_0\equiv\frac{(v_{00\to 2\bar{2}})^2}{(w_{22}-2w_{20}+w_{00})^2} . \quad (37)$$

For the special case of the QQ interaction

$$v_{00\to 2\bar{2}}=-\sqrt{2}g, \quad (38a)$$

$$w_{22}=w_{00}=-g, \quad (38b)$$

$$w_{20}=-2g, \quad (38c)$$

we obtain

$$\rho=\tfrac{1}{2}(1+X). \quad (39)$$

However, the value of ρ depends on the nature of the interaction. For example, if in the QQ interaction we change w_{22} from $-g$ to $-2g$ then the phonon condition (34) is still satisfied, but the value of ρ is changed to

$$\rho=2(1-X), \quad (40)$$

which happens to be the same as in the VMI model, at least in the weak coupling limit.

This assumption implies that w_{22} is more attractive relative to, say, w_{20}, than for a quadrupole interaction. This in turn suggests a stronger attraction between two *nucleons* in overlapping orbits than what is implied by a quadrupole force, i.e., which is plausible in view of the short range nature of the effective nucleon-nucleon interaction, though this point has not been investigated in detail.

V. CONCLUSION

We have considered a simple version of the boson model containing only $L=0$ and $L=2$ bosons and a quadrupole interaction between bosons. This model illustrates in a simple way the transition from vibrational to rotational spectra. The results are quite similar to those predicted by a two-dimensional variable-moment-of-inertia model.

The description used here we hope gives some insight into the excitation energies of collective states. It remains now to extend this description to the more realistic, but more complicated, three-dimensional case, though we expect the near equivalence of the boson and VMI results to hold here too.

The author is very grateful to Dr. F. Iachello, Professor T. Tamura, Professor A. Goodman, and Dr. G. S. Goldhaber for stimulating discussions, and thanks the computing center of UCLA and Brookhaven National Laboratory for making computing time available. This work was partially supported by the National Science Foundation.

[1]A. Arima and F. Iachello, Phys. Rev. Lett. 35, 1069 (1975).
[2]S. A. Moszkowski, Phys. Rev. 110, 403 (1958).
[3]K. Taruishi, Prog. Theor. Phys. 39, 53 (1968).
[4]G. S. Goldhaber, C. B. Dover, and A. L. Goodman, Annu. Nucl. Sci. 26, 239 (1976).