Method for scattering equations

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A method is proposed for calculating fully off-shell t matrix elements which are solutions of Lippmann-Schwinger-type equations. This is a generalization of the Kowalski version of the Sasakawa theory of scattering. The method relies on the Fredholm reduction of an integral equation whose kernel has a singularity at a fixed point. The method yields a t matrix which satisfies the conditions of on-shell unitarity and can be readily generalized to the case of multichannel scattering problems.

[NUCLEAR REACTIONS: Singular scattering equations, off-shell t-matrix elements, multichannel scattering theory.]

I. INTRODUCTION

Integral equations in scattering theory usually have kernels with a fixed-point singularity. Sasakawa[1] proposed a new method for solving these scattering integral equations. He showed that the solutions of these equations—especially the phase shifts—can be related to the solutions of some auxiliary nonsingular integral equations. The early discussions[2,3] on the Sasakawa theory of scattering used the wave function description of scattering. Austern[2] has used the Sasakawa approach in the problem of inelastic scattering and rearrangement collisions. Coester[3] studied the convergence properties of the Sasakawa equation for a wide class of potentials. Kowalski[4] has shown that the Sasakawa approach can be reformulated to yield a practical method for computing half-off-shell t-matrix elements. Blasczak and Fuda[5] used the method of Kowalski to compute half-off-shell two-body t-matrix elements for local potentials.

Here we propose a method for solving fully off-shell Lippmann-Schwinger-type equations. The method, as in Ref. 4, relies on solving an auxiliary equation whose kernel is free from singularities. The solution of the original equation is related to the solution of the auxiliary equation. This method is a generalization of the method of Ref. 4 to the case of fully off-shell t-matrix elements. The auxiliary nonsingular equation we get is just the off-shell extension of the same equation in Ref. 4. Hence the conclusions of Coester[3] for the existence of iterative solution of this equation holds and we can employ iterative solutions of the auxiliary equation. A recently proposed method by Bolsterli[6] is also a special case of the present method. This is obvious from the work of Ref. 7.

It is also shown that the present method can be easily extended to the case of multichannel problems. This can be used to study the three-body problem under the breakup threshold.

In Sec. II we describe the method. In Sec. III we apply the method to the case of multichannel scattering. Finally in Sec. IV we give a brief summary and some concluding remarks.

II. THE METHOD

The Lippmann-Schwinger equation at an energy E in a particular partial wave L can be written as

$$t^L(E) = V^L + V^L G_0(E) t^L(E) = V^L + t^L(E) G_0(E) V^L . \tag{1}$$

Here V refer to the potential and t is the t matrix. Throughout the rest of the article we shall not show the partial wave index L explicitly. Here G_0 has the explicit form (taking the reduced mass to be $\frac{1}{2}$)

$$G_0(p,q;E) = (k^2 - q^2 + i\epsilon)^{-1}\delta(p-q) , \tag{2}$$

where $E = k^2$. In explicit notations the fully off-shell matrix element of Eq. (1) can be written as

$$t(p,r;E) = V(p,r) + \lambda \int dq\, q^2 V(p,q)(k^2 - q^2 + i\epsilon)^{-1} t(q,r;E) , \tag{3}$$

where the integration limits here and throughout the rest of the article are from 0 to ∞. λ is a constant whose value depends on the normalization convention used in the partial wave decomposition.

Following Kowalski[4] we introduce a function $\gamma(k,q)$ such that

$$\gamma(k,k) = 1 . \tag{4}$$

It should be noted that this definition differs from that in Ref. 3 by an interchange of k and q.

Now Eq. (3) can be rewritten as

$$t(p,r;E)=V(p,r)+\lambda V(p,k)\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\,t(q,r;E)+\lambda\int dq\,q^2A(p,q;E)t(q,r;E)\,, \tag{5}$$

where

$$A(p,q;E)=[V(p,q)-V(p,k)\gamma(k,q)]\times(k^2-q^2+i\epsilon)^{-1} \tag{6}$$

is a nonsingular kernel. Equation (5) still contains a kernel which is singular at $q=k$. This is contained in the second term on the right of Eq. (5). We introduce the following auxiliary equation with a nonsingular kernel

$$\Gamma(p,r;E)=V(p,r)+\lambda\int dq\,q^2A(p,q;E)\Gamma(q,r;E)\,. \tag{7}$$

Next we would like to relate the solution of Eq. (7) to that of Eq. (5). The formal manipulation needed to do this becomes very transparent in the operator form. We write Eq. (5) in operator form as

$$t(E)=V+\overline{V}(E)H_0(E)t(E)+A(E)t(E)\,, \tag{8}$$

where $\overline{V}$ and H_0 are defined by

$$\overline{V}(p,q;E)=V(p,k) \tag{9}$$

and

$$H_0(p,q;E)=\delta(p-q)(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\,. \tag{10}$$

Equation (7) is written in operator form as

$$\Gamma(E)=V+A(E)\Gamma(E)\,. \tag{11}$$

The matrix element of A is defined by Eq. (6). The solution of Eq. (8) is formally written as

$$t(E)=[1-A(E)]^{-1}V+[1-A(E)]^{-1}\times\overline{V}(E)H_0(E)t(E) \tag{12}$$

and that of Eq. (11) is formally written as

$$\Gamma(E)=[1-A(E)]^{-1}V\,. \tag{13}$$

Using Eqs. (12) and (13) $t(E)$ can be written as

$$t(E)=\Gamma(E)+[1-A(E)]^{-1}\,\overline{V}(E)H_0(E)t(E)\,. \tag{14}$$

With the help of Eqs. (9), (10), and (13) the fully off-shell matrix element of Eq. (14) can be written in explicit form as

$$t(p,r;E)=\Gamma(p,r;E)+\Gamma(p,k;E)I(k,r)\,, \tag{15}$$

where

$$I(k,r)=\lambda\int q^2dq(k^2-q^2+i\epsilon)^{-1}\times\gamma(k,q)t(q,r;E)\,. \tag{16}$$

Equations (15) and (16) can be solved for I to give

$$I(k,r)=\frac{\lambda\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\Gamma(q,r;E)}{1-\lambda\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\Gamma(q,k;E)}\,. \tag{17}$$

Equations (6), (7), (15), and (17) are the fundamental equations of the present method. Using these equations we can calculate the fully off-shell t-matrix elements through the solution of the nonsingular integral equation for Γ. It is easy to see that when $r=k$ the present method reduces to the method of Ref. 4. So the present method should be considered as a generalization of the method of Ref. 4. As in Ref. 4 by virtue of realities of γ and Γ and by Eq. (4), t given by Eq. (15) satisfies half-on-shell unitarity. This holds for all real values of Γ. Thus any real approximation to Γ will yield a unitary on-shell t matrix. For the sake of completeness we give the following simpler form for the half-off-shell t matrix:

$$t(p,k;E)=\frac{\Gamma(p,k;E)}{\Gamma(k,k;E)}\,t(k,k;E)\,, \tag{18}$$

where

$$t(k,k;E)=\Gamma(k,k;E)\left[1-\lambda\int q^2dq(k^2-q^2+i\epsilon)^{-1}\times\gamma(k,q)\Gamma(q,k;E)\right]^{-1}. \tag{19}$$

Equations (18) and (19) are the fundamental results of Ref. 4. Equation (18) results if we take $r=k$ in Eq. (15). A slightly complicated expression for the half-off-shell t matrix results if we take $p=k$ in Eq. (15). In this case the half-off-shell t-matrix elements can be written as

$$t(k,r;E)=\Gamma(k,r;E)+\Gamma(k,k;E)I(k,r)\,. \tag{20}$$

If the potential V in Eq. (1) is symmetric the exact t matrix is also symmetric. So Eqs. (18) and (20) will give the same result for the half-off-shell t-matrix elements even though the two forms appear to be different. But in practice approximate solutions for Γ will be used in Eqs. (18) and (20). Then these two forms will lead to different approximate results. In particular, if iterative solution of Eq. (7) is used in Eqs. (18) and (20) these two forms will show different convergence properties. (In this connection it is to be noted that Γ is not symmetric.) The arbitrariness in the choice of the parameter γ can and should be exploited to improve the convergence of the iteration scheme.

III. MULTICHANNEL SCATTERING

The method of the last section can be easily extended to the case of multichannel scattering problems. In operator notation the multichannel scattering problem has the same form as Eq. (1). But now the matrix element of the operators will involve channel indices over and above the momentum labels. Here G_0 has the explicit form

$$G_{0\alpha\beta}(p_\alpha, q_\beta; E)=(E+h_\beta-q_\beta^2+i\epsilon)^{-1}\delta(p_\alpha-q_\beta)\delta_{\alpha\beta}, \quad (21)$$

where E is the total energy in the center of mass frame, h_β is the channel binding energy, and the reduced channel mass is taken to be $\frac{1}{2}$. The on-shell value of the channel momentum k_β is defined by $k_\beta^2=(E+h_\beta)$. Here α, β, σ, etc. refer to various channels. Now in explicit notation the fully off-shell matrix element of Eq. (1) can be written as

$$T_{\beta\alpha}(p_\beta, r_\alpha; E)=V_{\beta\alpha}(p_\beta, r_\alpha)+\sum_\sigma \lambda \int dq_\sigma q_\sigma^2 V_{\beta\sigma}(p_\beta, q_\sigma)(k_\sigma^2-q_\sigma^2+i\epsilon)^{-1}T_{\sigma\alpha}(q_\sigma, r_\alpha; E). \quad (22)$$

Here the momentum label k refers to the on-shell value. As in Sec. II we introduce a function $\gamma_{\alpha\alpha}(k_\alpha, q_\alpha)$ such that

$$\gamma_{\alpha\alpha}(k_\alpha, k_\alpha)=1. \quad (23)$$

Equation (22) can be rewritten as

$$T_{\beta\alpha}(p_\beta, r_\alpha; E)=V_{\beta\alpha}(p_\beta, r_\alpha)+\sum_\sigma V_{\beta\sigma}(p_\beta, k_\sigma)\lambda\int dq_\sigma q_\sigma^2\gamma_{\sigma\sigma}(k_\sigma, q_\sigma)(k_\sigma^2-q_\sigma^2+i\epsilon)^{-1}T_{\sigma\alpha}(q_\sigma, r_\alpha; E)$$
$$+\sum_\sigma \lambda\int dq_\sigma q_\sigma^2 A_{\beta\sigma}(p_\beta, q_\sigma; E)T_{\sigma\alpha}(q_\sigma, r_\alpha; E), \quad (24)$$

where

$$A_{\beta\sigma}(p_\beta, q_\sigma; E)=[V_{\beta\sigma}(p_\beta, q_\sigma)-V_{\beta\sigma}(p_\beta, k_\sigma)\gamma_{\sigma\sigma}(k_\sigma, q_\sigma)](k_\sigma^2-q_\sigma^2+i\epsilon)^{-1} \quad (25)$$

is nonsingular at $q_\sigma=k_\sigma$. It is to be noted that the formulation of this section is distinct from the multichannel formulation of Ref. 4. Equations (23)–(25) are not related to the corresponding equations of Ref. 4. The formulation of this section is a multichannel generalization of the formulation of Sec. II.

We introduce the following auxiliary equation with a nonsingular kernel:

$$\Gamma_{\beta\alpha}(p_\beta, r_\alpha; E)=V_{\beta\alpha}(p_\beta, r_\alpha)+\sum_\sigma \lambda\int dq_\sigma q_\sigma^2 A_{\beta\sigma}(p_\beta, q_\sigma; E)\Gamma_{\sigma\alpha}(q_\sigma, r_\alpha; E). \quad (26)$$

Finally we have to relate the solution of Eq. (26) to that of Eq. (24). As in Sec. II we again introduce the operator forms of these equations because formal manipulation becomes simple in this form. It is easy to see that in operator form Eqs. (8) and (11)–(14) still hold provided we modify Eqs. (9) and (10) in the following way:

$$\overline{V}_{\beta\sigma}(p_\beta, q_\sigma; E)=V_{\beta\sigma}(p_\beta, k_\sigma), \quad (27)$$

$$H_{0\beta\sigma}(p_\beta, q_\sigma; E)=\delta(p_\beta-q_\sigma)(k_\sigma^2-q_\sigma^2+i\epsilon)^{-1}\gamma_{\sigma\sigma}(k_\sigma, q_\sigma). \quad (28)$$

Now with the help of Eqs. (27) and (28) the fully off-shell matrix elements of Eq. (14) can be written in explicit form as

$$T_{\beta\alpha}(p_\beta, r_\alpha; E)=\Gamma_{\beta\alpha}(p_\beta, r_\alpha; E)+\sum_\sigma \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E)I_{\sigma\alpha}(k_\sigma, r_\alpha; E), \quad (29)$$

where

$$I_{\sigma\alpha}(k_\sigma, r_\alpha; E)=\lambda\int dq_\sigma q_\sigma^2(k_\sigma^2-q_\sigma^2+i\epsilon)^{-1}\gamma_{\sigma\sigma}(k_\sigma, q_\sigma)\, T_{\sigma\alpha}(q_\sigma, r_\alpha; E). \quad (30)$$

From Eqs. (29) and (30) it is easy to see that $I_{\beta\alpha}$ are solutions of

$$I_{\beta\alpha}(k_\beta, r_\alpha; E)=d_{\beta\alpha}(k_\beta, r_\alpha; E)+\sum_\sigma d_{\beta\sigma}(k_\beta, k_\sigma; E)I_{\sigma\alpha}(k_\sigma, r_\alpha; E), \quad (31)$$

where

$$d_{\sigma\alpha}(k_\sigma, r_\alpha; E)=\lambda\int dq_\sigma q_\sigma^2(k_\sigma^2-q_\sigma^2+i\epsilon)^{-1}\gamma_{\sigma\sigma}(k_\sigma, q_\sigma)\Gamma_{\sigma\alpha}(q_\sigma, r_\alpha; E). \quad (32)$$

Equations (29) and (31) are the fundamental equations of this section and are one of the possible multichannel generalizations of the method presented in Sec. II. As noted before this generalization is distinct from the half-on-shell multichannel generalization of Ref. 4. As in Sec. II we have two types of half-on-shell formulas from Eq. (29). They correspond to putting $p=k$ or $r=k$ in Eq. (29). Each of these half-on-shell equations is different from that of Ref. 4. As in Sec. II and in Ref. 4 it is easy to see that the t matrix given by Eq. (29) satisfies on-shell unitarity if V and γ are real and real approximations to the solution Γ of Eq. (26) are considered. This is true in the three-particle scattering below the breakup threshold. Hence the present method may be used to give approximations to the three-body scattering problem satisfying constraints of unitarity provided real approximate solutions of Γ are considered.

IV. SUMMARY AND DISCUSSION

Here a method is proposed for solving fully off-shell Lippman-Schwinger-type equations whose kernel has a singularity at a fixed point. The problem is reduced to the solution of an auxiliary equation whose kernel is free from singularities. The method is a generalization of the method presented in Ref. 4 to the case of fully off-shell t-matrix elements. Reference 4 gives a method for half-off-shell t-matrix elements. The present method yields two types of schemes for the half-off-shell t-matrix elements. One of them is identical with the method of Ref. 4 but the other is distinct from the method of Ref. 4. If approximate perturbative solution of the auxiliary nonsingular equation is employed these two methods will lead to different approximations for half-off-shell t-matrix elements. The method satisfies constraints of on-shell unitarity.

The method is extended to the case of off-shell multichannel problems. As in the single channel problem the method yields two schemes for calculating half-off-shell t-matrix elements. But none of these schemes correspond to the method of Ref. 4 for half-off-shell t-matrix elements in multichannel scattering problem.

Various approximation schemes will probably emerge in the future based on the present method. Numerical investigations are currently being carried out using iterative solution for the auxiliary equation for Γ. The freedom in the choice of γ should be exploited to improve the convergence of iteration scheme. It might be possible to generalize the method to the case of the three-body problem above the threshold for breakup into three particles. This will be a problem of future interest.

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