Separable expansions to local potentials in a quasipotential approach

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A recently proposed method which was successfully used to make separable expansions for the nonrelativistic two-body t matrix is here used to construct separable expansions for the two-body t matrix satisfying relativistic unitarity in a quasipotential approach. The present method, unlike other commonly used methods, does not require the calculations of eigenfunctions and eigenvalues of the kernel of the scattering equations and yields simple form factors. The method is illustrated for a simple nucleon-nucleon potential in the context of the Blankenbecler-Sugar equation.

[NUCLEAR REACTIONS Finite rank approximations to the t matrix of a local potential, relativistic unitarity, Blankenbecler-Sugar equation solved, phase shift calculated.]

I. INTRODUCTION

A recently proposed separable expansion[1] for the nonrelativistic t matrix of a local potential gave good convergence at all energies. The method also yielded[2] good results for the case of the realistic $^3S_1$-$^3D_1$ channel. More recently the method has been compared with the unitary pole expansion[3] (UPE) method in calculating trinucleon bound state observables for the Reid[4] soft core potentials. It has been found[5] the the method of Ref. 1 in general gives better results than the UPE method. In particular[5] in the case of the method of Ref. 1 it was not necessary to resort to t-matrix perturbation theory to achieve convergence for trinucleon observables whereas it was inevitable in the case of the UPE. The convergence of the trinucleon observables was by far superior in the case of the separable expansion of Refs. 1 and 2 than in the case of the UPE. The same conclusions were true[6] in a later trinucleon calculation using the one-boson-exchange potentials.[7]

In the present work we apply the method of Ref. 1 to construct separable expansions for the two-body t matrix satisfying relativistic two-body unitarity. These separable expansions can be used to study three-body scattering at intermediate energies and also to determine relativistic effects in the trinucleon bound state calculation. The necessity of using a relativistic theory in the low-energy nucleon-nucleon problem has long been realized. This is because the strong repulsion in the nucleon-nucleon potential at short distances builds short range correlations into two-nucleon wave functions at all energies, which probably cannot be adequately described by a nonrelativistic theory.

The minimal requirement for a relativistic theory is that the scattering amplitude should satisfy relativistic elastic unitarity relation. This could be achieved in the two-body Blankenbecler-Sugar equation[8] which is a one-dimensional on the mass shell approximation for the fully covariant Bethe-Salpeter equation.[9] So in the present work we shall be limited to the consideration of the Blankenbecler-Sugar equation in the construction of the separable expansion for the relativistic two-body t matrix.

Separable potentials and corresponding t matrices are very interesting in practice[10-15] because of the great simplicity they bring to the three- or four-body problem using either relativistic or nonrelativistic kinematics. Separable potentials have been used in relativistic three-body problems[12] but such one-term separable potentials cannot adequately represent the more realistic potentials such as the boson-exchange potentials.[7] Such realistic potentials are in general local and the present method provides a simple prescription for constructing separable expansions to these potentials, which can then easily be used in three-body[8,13] or four-body[14] Blankenbecler-Sugar type equations satisfying the constraints of relativistic unitarity.

The present separable expansion has all the interesting features of the separable expansion of Ref. 1. As in Ref. 16 the present separable

expansion too can be derived from the Schwinger variational principle.[17] The variational nature of the expansion will guarantee good convergence. Another crucial feature of the expansion is that the separable t matrix $t_N(p,q;s)$ of rank N in any partial wave can be made exact at any momentum q for N sets of momentum p and energy s.[1] This property has been well exploited in Refs. 5 and 6 to achieve good convergence for the trinucleon bound state observables. As in Ref. 1 the present expansion has simple form factors whose analytic properties can be easily studied. It is hard to find the analytic properties of other separable expansions[3,18] which involve numerically constructed eigenfunctions. The analytic behavior of the form factors is important if one wants to solve the three-body problem with this separable expansion by the technique of contour rotation.[19] The present method does not give any unwanted singularities and the usual contour rotation technique can be easily applied.

In Sec. II we briefly describe the method, define the two-body partial wave Blankenbecler-Sugar[8] type equation and report the numerical results. We work in the K-matrix form of the Blankenbecler-Sugar equation. We show how the rank 1 K matrix can be made exact at a particular energy and momentum. We chose to make the half-on-shell K matrix exact at an incident laboratory energy of 200 MeV. Due to the pole dominance idea of Lovelace[11] this substantially improves results of scattering at other energies. Finally in Sec. III we give a brief summary and some concluding remarks.

II. SEPARABLE EXPANSION

The Blankenbecler-Sugar equation[8] for the two-body t matrix $t(s)$ has the form

$$t(s) = V + VG_0(s)t(s) = V + t(s)G_0(s)V, \tag{1}$$

where $s = W^2$ is the square of the total energy in the center-of-mass frame. The S-wave projection of Eq. (1) has the explicit form[20]

$$t(p,p';s) = Mv(p,p') + \int_0^\infty q^2 dq \frac{Mv(p,q)t(q,p';s)}{(q^2+M^2)^{1/2}(k^2-q^2+i\epsilon)}, \tag{2}$$

where M is the mass of each particle in inverse femtometers and $s = k^2 + M^2$. Here k is the magnitude of the center-of-mass momentum and the potential matrix elements are expressed in units of femtometers. Equation (2) is written in units $\hbar = c = 1$. We shall stick to this convention of units throughout the rest of the paper. If we consider scattering of two nucleons the kinetic energy $E_{\rm lab}$ of the incident particle in the laboratory system is related to the center-of-mass momentum k by

$$E_{\rm lab} = 2k^2M^{-1}. \tag{3}$$

If we solve Eq. (2) in the K-matrix formulation, i.e., we consider the principal value of the integral involving $(k^2-q^2)^{-1}$ the phase shift is given by[20]

$$\tan\delta = -\frac{\pi k K(k,k;s)}{2(k^2+M^2)^{1/2}}, \tag{4}$$

where K is the K matrix corresponding to Eq. (2). The potential used in Eq. (2) in the present investigation has the simple Yukawa form[20]

$$v(p,p') = \frac{\lambda}{2\pi pp'} \ln\frac{(p+p')^2+\mu^2}{(p-p')^2+\mu^2}. \tag{5}$$

Next we give a brief summary of the method of Ref. 1. (For a more complete review, see Refs. 1 and 16.) The method depends on making the separable expansion to the potential V in Eq. (1) according to the prescription

$$V_N = \sum_{m,n=1}^{N} V|f_m\rangle \Delta_{mn} \langle f_n|V, \tag{6}$$

where

$$(\Delta^{-1})_{nm} = \langle f_n|V|f_m\rangle, \tag{7}$$

where f_n is a set of suitably chosen real functions. Hence to any order N V_N is Hermitian and the corresponding t matrix we get by solving Eq. (1) with this V_N satisfies the constraints of two-body unitarity. Substituting V_N back into Eq. (1) and solving for $t_N(s)$ we get

$$t_N(s) = \sum_{m,n=1}^{N} V|f_m\rangle D_{mn} \langle f_n|V, \tag{8}$$

where

$$(D^{-1})_{nm} = \langle f_n|(V - VG_0V)f_m\rangle. \tag{9}$$

Equations (8) and (9) define the required separable expansion. As shown in Ref. 16, Eqs. (8) and (9) can also be deduced from the Schwinger variational principle.[17] The variational nature of the expansion will guarantee good convergence.

We shall now give the explicit S-wave forms of the momentum space integrals needed to evaluate the separable t matrix given by Eqs. (8) and (9). Generalization to other partial wave is straightforward. We have in explicit notations

$$\langle p|V|f_n\rangle \equiv \langle f_n|V|p\rangle = \int_0^\infty q^2 dq\, f_n(q) Mv(p,q), \tag{10}$$

where

$$\langle f_n|q\rangle \equiv \langle q|f_n\rangle = f_n(q)\,. \tag{11}$$

We also have

$$(D^{-1})_{nm} = \int_0^\infty q^2dq \int_0^\infty p^2dp\, f_n(q) f_m(p)\left[Mv(q,p) - \int_0^\infty dp' p'^2\, \frac{Mv(q,p')Mv(p',p)}{(p'^2+M^2)^{1/2}(k^2-p'^2+i\epsilon)}\right]. \tag{12}$$

To see how the separable expansion works in practice we carried out S-wave numerical calculations with the potential given by Eq. (5) with $\lambda = -\pi$ fm^{-1}, $\mu = 1.0$ fm^{-1}, and $M = 1.0$ fm^{-1} as in Ref. 20. Next we chose the momentum space representation of the expansion function $f_n(p)$ to be

$$f_n(p) = \frac{1}{(p^2+\alpha^2)^2}\left[C_{n-1}{}^1\left(\frac{p^2-\alpha^2}{p^2+\alpha^2}\right)\right], \qquad n = 1,2,3,\ldots,N\,, \tag{13}$$

where $C_{n-1}{}^1$ is the Gegenbauer polynomial of degree $(n-1)$ and is defined by[21]

$$C_{n-1}{}^1(\cos\phi) = \frac{\sin n\phi}{\sin\phi}\,. \tag{14}$$

Here α is a parameter which can be varied to improve the convergence of the separable expansion. The expression in the square bracket in Eq. (13) is formally equivalent to $(p^2+\alpha^2)^{-n}$. The functions represented by Eq. (13) are orthogonal with respect to some weight function. This approximate orthogonalization of the expansion functions is important in order to prevent the matrix D^{-1} from becoming too singular after numerical evaluation of integrals in Eqs. (10) and (12). As in Ref. 16 it is possible to do these integrals analytically with certain class of functions f_n and potential V. If this is done the orthogonalization of the functions is not so important because of the greater accuracy of the integrals. However, in the present work we chose to do the integrals in Eqs. (10) and (12) numerically with choice (13) of the expansion functions.

As in Ref. 1 we exploit the freedom in the choice of the expansion functions to give good results for the rank-1 approximation. It has been shown in Ref. 1 that we can make the rank-N $t_N(p,p';s)$ exact at a particular s and p for all p' and all $N \geq 1$ if we take a new set of expansion functions $\bar{f}_n$, where

$$V|\bar{f}_1\rangle = t(s)|p\rangle\,, \tag{15}$$

$$|\bar{f}_n\rangle = |f_n\rangle\,, \quad n = 2,3,\ldots,N\,. \tag{16}$$

In particular we can make the rank-N result exact at N sets of s and p. But for simplicity we chose to make the rank-1 result exact at a particular s and p. The pole dominance idea of Lovelace will give a good t matrix at other energies.[11] The expression for $t(s)$ in Eq. (15) is not known precisely. This makes the implementation of Eq. (15) difficult. Instead as in Ref. 1 we take a new set of basic functions $\bar{f}_n$ satisfying Eq. (16) and

$$V|\bar{f}_1\rangle = t_N(s)|p\rangle\,, \tag{17}$$

where N is large enough such that $t_N(s)$ is a good approximation to $t(s)$.

With the help of Eq. (8), Eq. (17) becomes

$$V|\bar{f}_1\rangle = \sum_{m,n=1}^{N} V|f_m\rangle D_{mn}\langle f_n|V|p\rangle\,, \tag{18}$$

so

$$|\bar{f}_1\rangle = \sum_{m=1}^{N} |f_m\rangle C_m(p)\,, \tag{19}$$

where

$$C_m(p) = \sum_{n=1}^{N} D_{mn}\langle f_n|V|p\rangle\,. \tag{20}$$

With this new set of expansion functions defined by Eqs. (19), (20), and (16) we can make the rank-N result exact at an energy s and a momentum p for all $N \geq 1$. This will approximately built in the bound or the antibound state pole of the t matrix and because of the pole dominance idea of Lovelace[11] this will give a good representation of the t matrix at the neighboring energies.

All the equations of this section hold good in the K-matrix formulation when the integral over G_0 is replaced by its principal value. Here in this

TABLE I. Phase shifts in radian for different N.

E_{lab} (MeV)	N = 1	2	4	6	8
100	1.403	1.404	1.422	1.434	1.433
150	1.259	1.259	1.263	1.267	1.268
200	1.515	1.151	1.151	1.153	1.154
250	1.063	1.063	1.065	1.066	1.067
300	0.988	0.988	0.996	0.996	0.998
350	0.923	0.923	0.938	0.939	0.938
400	0.866	0.866	0.888	0.888	0.886
450	0.814	0.814	0.845	0.846	0.849
500	0.767	0.768	0.807	0.810	0.810
550	0.725	0.726	0.773	0.777	0.777

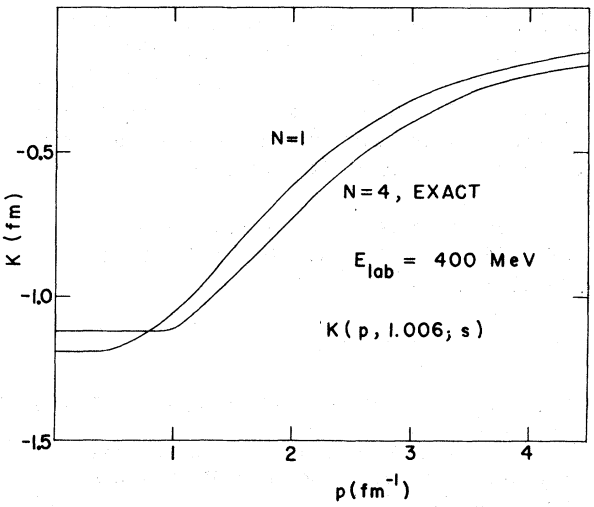

FIG. 1. Half-shell K-matrix elements at an incident kinetic energy $E_{lab}=400$ MeV.

paper we shall work in the K-matrix formulation of Eq. (1). The parameter α in the expansion functions of Eq. (13) is first varied to improve the convergence rate. The value finally adopted after a small experimentaion is $\alpha=1.2$ fm^{-1}.

Next we improve the rank-1 result for the K matrix by a redefinition of the expansion function given by Eq. (19). We implemented Eq. (19) at an incident kinetic energy $E_{lab}=200$ MeV and at a momentum p corresponding to on the energy shell

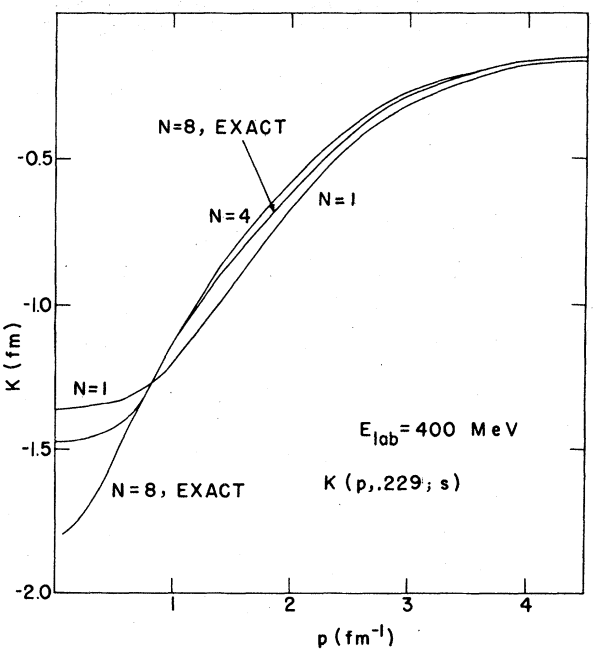

FIG. 2. Off-shell K-matrix elements at an incident kinetic energy $E_{lab}=400$ MeV for momentum $q=0.229$ fm^{-1}.

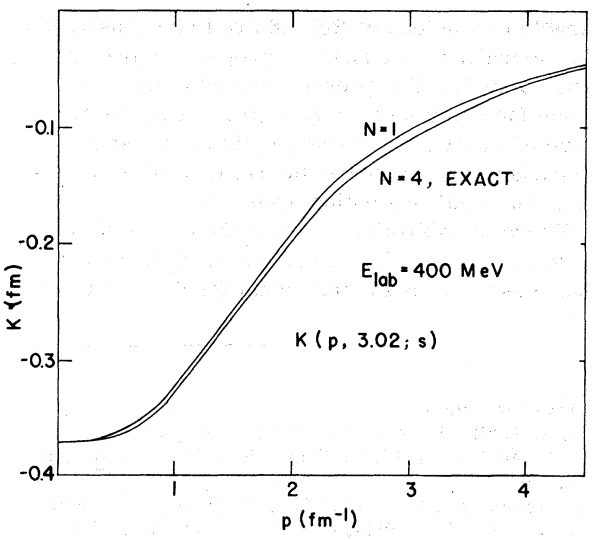

FIG. 3. Off-shell K-matrix elements at an incident kinetic energy $E_{lab}=400$ MeV for momentum $q=3.020$ fm^{-1}.

(on-shell) momentum at $E_{lab}=200$ MeV. We took $N=5$ to redefine $\bar{f}_1$ defined by Eq. (19). This makes the rank-N half-on-shell K matrix "exact" at $E_{lab}=200$ MeV.

In Table I we show on-shell phase shifts at various energies for various values N. (The calculations were carried out in single preceission on an IBM computer.) We also checked that the half-off-shell and fully-off-shell K-matrix elements converge equally rapidly. To have a feeling about how the K-matrix elements converge in practice we plot some graphs for these K-matrix elements at $E_{lab}=400$ MeV. The on-shell momentum corresponding to this energy is 1.006 fm^{-1}. In Fig. 1 we show the half-off-shell K-matrix elements at this energy. Figures 2 and 3 show fully-off-shell K-matrix elements at the same energy for off-shell momentums 0.229 fm^{-1} and 3.020 fm^{-1}, respectively.

III. SUMMARY AND DISCUSSION

In the present work we use the method of Ref. 1 to construct separable expansions for the two-body t matrix, of a local potential, satisfying the constraints of relativistic unitarity. We present numerical results for the K-matrix elements rather than for the t-matrix elements. We test the numerical accuracy of the present method for a Yukawa-type potential in the context of the Blankenbecler-Sugar equation. The method yielded an excellent rank-1 approximation and good convergence at all energies considered.

The present method is similar in spirit to the method of Ref. 18. The form factors of the sep-

arable expansions of Ref. 18 are to be constructed numerically whose analytic properties are difficult to study. The present method yields simple form factors whose analytic properties can be studied easily, and consequently the present separable expansion can be easily used in three- and four-body scattering problems.

There is an arbitrariness in the choice of the functions f_n and also in the choice of numerical parameters once the functional forms have been chosen. However, this arbitrariness can be turned to good advantage, as has been done in this paper, by varying the parameters to obtain the best convergence. We can also chose a functional form as in Ref. 16, which will give simple analytic form factors for the separable expansions.

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