Electromagnetic effects and weak form factors

Barry R. Holstein
Physics Division, National Science Foundation, Washington, D. C. 20550
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A simple "renormalization" of nuclear form factors is shown to account for the major effects of induced Coulomb corrections to weak decays. Application to present experiments indicates that these effects are at the edge of detectability.

[RADIOACTIVITY Induced Coulomb corrections to weak decays.]

I. INTRODUCTION

There have been numerous treatments of Coulomb corrections to weak interactions. Recently there have been a number of discussions[1] of such corrections within the framework of the elementary particle approach to weak processes in nuclei.[2] An interesting feature which emerged from these treatments was the observation by Bottino, Ciocchetti, and Kim (BCK)[3] that in addition to the conventional finite nuclear size corrections of order $Z\alpha ER$ (where Z,R are the nuclear charge, radius respectively while E is the electron or muon energy) there are additional "induced Coulomb" terms of order $Z\alpha/m_N R$ (where m_N is the nucleon mass) which arise from "recoil" form factors (such as weak magnetism or induced tensor) whose kinematic structure depends on the momentum transfer q between initial and final nuclei.[4] We wish in this note to give a more general discussion of such terms than given by BCK. For definiteness, our treatment here will be within the framework of allowed beta decay. However, our conclusions are more general, and similar results will obtain for muon capture, electron capture, inverse β decay, etc. We shall demonstrate that the major effect of the induced Coulomb terms may be accounted for by a simple "renormalization" of weak form factors. In the next section we define notation and derive this renormalization scheme, while in Sec. III we assess the impact of these results on present generation experiments.

II. DEFINITIONS AND DERIVATION

We have chosen here to quote results specifically for the case of allowed nuclear beta decay, on which experiments are being performed with ever increasing precision. A number of recent experiments[5] have sought information about recoil form factors—specifically weak magnetism, induced tensor. Clearly analysis of such experiments requires an equally precise and careful theoretical treatment. In the absence of electromagnetic effects, the theoretical analysis is straightforward and results may be expressed in a simple closed form.[6] Electromagnetic corrections produce an unavoidable complication to this formalism. Their effects can be divided into two categories:

(i) static distortion of the electron or positron wave functions whose presence can be accounted for by use of a Coulomb wave function for the outgoing electron or positron rather than use of a plane wave[7];

(ii) all other electromagnetic corrections—e.g., bremsstrahlung, hadronic photon exchange. One attempts to account for such terms by factoring out a radiative correction term[8]

$$1+\frac{\alpha}{2\pi}g(E,E_{\max}).$$

We shall be dealing here with the effects of the former.

To zeroeth order in recoil the hadronic weak matrix element can be characterized purely by the Fermi and Gamow-Teller terms. Use of a Coulomb wave function modifies the spectrum via an overall multiplicative factor $F(Z,E)$, the Fermi function, which is common to both Fermi and Gamow-Teller decays, plus additional small corrections of size $Z\alpha qR$, $Z^2\alpha^2$ whose precise form depends upon the Fermi or Gamow-Teller character of the transition. We have previously catalogued the effect of such terms on the various spectral functions.[9]

There is also, of course, a Coulomb correction to recoil form factors such as weak magnetism. This correction can be separated into two pieces:

(a) a term whose origin is similar to the leading corrections discussed above. The effect on the spectrum is $\mathcal{O}((q/m_N)Z\alpha qR)\lll 1$ and is hence negligible.[10]

(b) a term whose origin lies in the fact that the electron or positron energy is affected by the presence of the electrostatic potential. These are the "induced Coulomb" effects of $\mathcal{O}(Z\alpha/m_N R)$ which are of interest to us here.[11]

In order to study these terms we begin by defining notation. Consider the nuclear beta transition

$$\alpha \to \beta + e^- + \bar{\nu}_e .$$

For purpose of discussion we consider electron decay. Modifications appropriate for positrons will be included in the final formulas. Let p_1, p_2, p, k denote the respective four-momenta of parent nucleus, daughter nucleus, electron, and neutrino. The parent and daughter masses are M_1 and M_2. We also define

$$P = p_1 + p_2, \quad q = p_1 - p_2 = p + k , \qquad M = \tfrac{1}{2}(M_1 + M_2), \quad \Delta = M_1 - M_2 . \tag{1}$$

First disregard electromagnetic effects. Then the weak beta decay amplitude is assumed to be given by

$$T_0 = \frac{G}{\sqrt{2}} \cos\theta_C \langle \beta_{p_2} | (V_\mu + A_\mu) | \alpha_{p_1} \rangle l^\mu , \tag{2}$$

where $G \approx 10^{-5}/m_N{}^2$ is the conventional weak coupling constant, $\theta_C \approx 15°$ is the Cabibbo angle, and l^μ is the matrix element of the lepton current

$$l^\mu = \bar{u}(p)\gamma^\mu(1+\gamma_5)v(k) . \tag{3}$$

In order to define the hadronic matrix elements, let parent, daughter nuclei have spins J, J' with projections M, M' on some quantization axis. Then we define weak form factors $a, b, \ldots, j_3$ via

$$l^\mu\langle\beta|V_\mu|\alpha\rangle = \delta_{JJ'}\delta_{MM'}\frac{1}{2M} l^\mu (aP_\mu + eq_\mu) + iC^{M'k;M}_{J'1;J}\frac{b}{2M}(\vec{q}\times\vec{l})_k + C^{M'k;M}_{J'2;J}\left[\frac{f}{2M}C^{nn';k}_{11;2}l_n q_{n'} + \frac{g}{(2M)^3}P\cdot l(\tfrac{4}{5}\pi)^{1/2}Y_2^k(\vec{q})\right] ,$$

$$l^\mu\langle\beta|A_\mu|\alpha\rangle = C^{M'k;M}_{J'1;J}\epsilon_{ijk}\epsilon_{ij\lambda\eta}\frac{1}{4M}\left[cl^\lambda P^\eta - dl^\lambda q^\eta + \frac{1}{(2M)^2}hq^\lambda P^\eta q\cdot l\right]$$
$$+ C^{M'k;M}_{J'2;J}C^{nn';k}_{12;2}l_n(\tfrac{4}{5}\pi)^{1/2}Y_2^{n'}(\vec{q})\frac{j_2}{(2M)^2} + C^{M'k;M}_{J'3;J}C^{nn';k}_{12;3}l_n(\tfrac{4}{5}\pi)^{1/2}Y_2^{n'}(\vec{q})\frac{j_3}{(2M)^2} . \tag{4}$$

Here $a, c,$ are the usual Fermi, Gamow-Teller form factors, b and d represent the weak magnetism and induced tensor terms, respectively, while h is the induced pseudoscalar. The structure functions e, f, g, j_2, j_3 are often omitted but can be of considerable importance for some transitions. This parametrization includes all recoil effects to $\mathcal{O}(q/m_N, q^2R^2)$ and should be adequate for arbitrary allowed transitions. A general analysis of β decay spectra in terms of these form factors has already been given.[12]

Now suppose we turn on electromagnetism. The correct generalization of equation 2 has been shown to be[13]

$$T \simeq \frac{G}{\sqrt{2}}\cos\theta_C \int d^3r\, \bar{\psi}_e(\vec{r},\vec{p})\gamma^\mu(1+\gamma_5)v(k)$$
$$\times \int \frac{d^3s}{(2\pi)^3} e^{i\vec{s}\cdot\vec{r}} \frac{1}{2}\left[\langle \beta_{\vec{p}_2+\vec{p}-\vec{s}}|(V_\mu + A_\mu)|\alpha_{\vec{p}_1}\rangle + \langle\beta_{\vec{p}_2}|(V_\mu + A_\mu)|\alpha_{\vec{p}_1-\vec{p}+\vec{s}}\rangle\right], \tag{5}$$

where $\bar{\psi}_e(\vec{r},\vec{p})$ is the solution to the Dirac equation in the presence of the nuclear Coulomb potential which reduces to $\bar{u}(p)e^{-i\vec{p}\cdot\vec{r}}$ as $Z\to 0$. Here states α and β are prescribed to remain on their mass shells, and in addition we must in the hadronic matrix element make the substitution

$$q_0 \to q_0 + e\phi(r) , \tag{6}$$

where $\phi(r)$ is the nuclear Coulomb potential, in order to maintain gauge invariance.[14]

If we now consider a recoil form factor—e.g., weak magnetism b—we have

$$T = \frac{G}{\sqrt{2}}\cos\theta_C \int d^3r \int \frac{d^3s}{(2\pi)^3}\bar{\psi}_e(\vec{r},\vec{p})e^{i\vec{s}\cdot\vec{r}}\gamma^n(1+\gamma_5)v(k)$$
$$\times\left[\cdots - iC^{M'k;M}_{J'1;J}\epsilon_{mnk}\frac{1}{2M}b((\vec{q}-\vec{p}+\vec{s})^2)(\vec{p}-\vec{s}-\vec{q})^m + \cdots\right] . \tag{7}$$

Writing $\vec{s}$ as a derivative on the exponential and integrating by parts we have

$$T=\frac{G}{\sqrt{2}}\cos\theta_C\int d^3r\int\frac{d^3s}{(2\pi)^3}e^{i\vec{s}\cdot\vec{r}}C^{M'k;M}_{J'1;J}$$
$$\times\left[\cdots-i\epsilon_{mnk}\frac{1}{2M}b((\vec{q}-\vec{p}+\vec{s})^2)(p-q-i\nabla)_m+\cdots\right]\bar{\psi}_e(\vec{r},\vec{p})\gamma^n(1+\gamma_5)v(k)\,. \tag{8}$$

Finally, using

$$-i\vec{\nabla}\psi_e(\vec{r},\vec{p})=\vec{p}\psi_e(\vec{r},\vec{p})+\vec{\delta}\psi_e(\vec{r},\vec{p}) \tag{9}$$

and defining the weak magnetic charge density $\rho_b(r)$ via

$$b(\vec{q}^2)=b(0)\int d^3r\rho_b(r)e^{-i\vec{q}\cdot\vec{r}}\,, \tag{10}$$

we find

$$T=\frac{G}{\sqrt{2}}\cos\theta_C\int d^3r\,\{\cdots+iC^{M'k;M}_{J'1;J}\epsilon_{mnk}\rho_b(r)(1/2M)b(0)[\vec{q}\bar{\psi}_e(\vec{r},\vec{p})+\vec{\delta}\bar{\psi}_e(\vec{r},\vec{p})]_m+\cdots\}\gamma_n(1+\gamma_5)v(k)\,. \tag{11}$$

The term in Eq. (11) involving $\vec{q}\bar{\psi}_e(\vec{r},\vec{p})$ gives rise to the correction terms of type (a) discussed above and is not of interest here. It is the terms in $\vec{\delta}\bar{\psi}_e(\vec{r},\vec{p})$ which yield the induced Coulomb effects. In the Appendix we show that

$$\vec{\delta}\psi_e(\vec{r},\vec{p})\approx-[F_{\rm BJ}(Z,E)]^{1/2}\frac{\alpha Z}{2R}\left[\vec{\gamma}\left(1-\frac{1}{5}\frac{r^2}{R^2}\right)-\frac{2}{5}\frac{r^2}{R^2}\vec{\gamma}\cdot\hat{r}\hat{r}\right]\gamma_0u(p)\,, \tag{12}$$

where $F_{\rm BJ}(Z,E)$ is the Behrens-Janecke Fermi function[15] and $u(p)$ is a free electron spinor.

Generalizing this derivation to include the remaining form factors, we find

$$\begin{aligned}T=[F_{\rm BJ}(Z,E)]^{1/2}\Bigg\{T_0+\frac{G}{\sqrt{2}}\cos\theta_C\frac{\alpha Z}{R}\Bigg[&\delta_{JJ'}\delta_{MM'}\frac{1}{2M}l^0e(\sigma-3\tau)\\&+C^{M'k;M}_{J'1;J}\bigg(\frac{1}{2M}l_k[c\tau+d(\tau-\sigma)+2b\tau]+\frac{q_k}{(2M)^2}l^0h(\sigma-3\tau)\\&-\frac{h}{(2M)^2}m_e\tau\bar{u}(p)\gamma_0\gamma_k(1+\gamma_5)v(k)\bigg)\\&-C^{M'k;M}_{J'2;J}C^{nn';k}_{11;2}\,l_n(q-\tfrac{2}{5}p)_{n'}\frac{\tau}{(2M)^2}(3j_2-\sqrt{6}\,g)\Bigg]\Bigg\}\,,\end{aligned} \tag{13}$$

where

$$\sigma=3\tau=\begin{cases}\frac{3}{2}, & \text{point weak charge}\\ \frac{6}{5}, & \text{uniform weak charge}\\ 1, & \text{surface weak charge}\,.\end{cases} \tag{14}$$

Here σ represents contributions from the gauge invariance substitution, while τ gives the effects arising directly from the Coulomb wave function. Comparison with Eq. (4) shows that a major component of the modifications to the decay spectrum due to induced Coulomb effects can be identified by utilization of unmodified spectral functions but with "effective" coupling constants

$$c_{\rm eff}=c\pm\frac{\alpha Z}{2MR}\left[c\tau+d(\tau-\sigma)\pm2b\tau+h\frac{1}{2M}\left(q_0\pm\frac{3}{2}\frac{Z\alpha}{R}\right)(\sigma-3\tau)\right]\,,$$

$$\begin{aligned}d_{\rm eff}&=d\pm\frac{\alpha Z}{2MR}h(\sigma-3\tau)\,,\\ f_{\rm eff}&=f\mp\frac{\alpha Z}{2MR}(\pm3j_2\tau-\sqrt{6}\,g\tau)\,,\\ a_{\rm eff}&=a\pm\frac{\alpha Z}{2MR}e(\sigma-3\tau)\,,\end{aligned} \tag{15}$$

where the upper (lower) sign is for electron (positron) decay.

III. IMPLICATIONS FOR PRESENT EXPERIMENTS

The interesting feature of these modifications is their experimental implication. In the case of $c_{\rm eff}$ there is none of significance. The modification given here reproduces the previously calculated induced Coulomb terms.[11] It amounts to a small renormalization of the dominant Gamow-Teller form factor, which is much smaller than uncertainties in calculated Gamow-Teller matrix elements due to imperfect nuclear wave functions, relativistic effects,[16] meson exchange,[17] etc. Thus the induced Coulomb terms may as well be absorbed into the definition of c and the unmodified spectra utilized. It might appear that there exists a substantial effect on the ft asymmetry in Gamow-Teller mirror decays from induced Coulomb corrections in that

$$\frac{c_{\text{eff}}^2(e^-)}{c_{\text{eff}}^2(e^+)} - 1 \simeq 2\frac{\alpha Z}{MR}(\tau - \sigma)\frac{d}{c}, \tag{16}$$

which is a correction at the percent level for values $d/Ac \sim \mathcal{O}(1)$. However, the full expression for the ft asymmetry also involves interference with the induced tensor

$$\frac{ft^+}{ft^-} - 1 \simeq \frac{c_{\text{eff}}^2(e^-)}{c_{\text{eff}}^2(e^+)} - 1 + \frac{2}{3}\frac{d}{c}\frac{E_0^+ - E_0^-}{M}, \tag{17}$$

where E_0^-, E_0^+ is the maximum energy of the electron, positron. For mirror decays, the end point energy difference is Coulombic

$$E_0^+ - E_0^- \approx \Delta^+ - \Delta^- \approx \frac{12}{5}\frac{\alpha Z}{R}. \tag{18}$$

Thus

$$\frac{ft^+}{ft^-} - 1 \approx 2\frac{\alpha Z}{m_N R}\frac{d}{Ac}(\tau - \sigma + \tfrac{4}{5}). \tag{19}$$

For a uniform weak charge distribution the asymmetry vanishes and in any case

$$\tfrac{2}{3} \lesssim \sigma - \tau \lesssim 1, \tag{20}$$

so that, since $d/Ac \sim \mathcal{O}(1)$,[18] the effect on the ft asymmetry is quite small— $\lesssim 0.1\%$ for $Z = 10$.

In a similar fashion, since $\sigma - 3\tau = 0$, there is no effect on the ft asymmetry from terms in h and no renormalization of the Fermi terms due to effects from the induced scalar e. That both these results are to be expected is easy to see since the plane wave identity

$$q^\mu \bar{u}(p)\gamma_\mu(1+\gamma_5)v(k) = m_e\bar{u}(p)(1+\gamma_5)v(k) \tag{21}$$

remains essentially unmodified by the Coulomb correction

$$\begin{aligned}[q_0 + e\phi(r)]\bar{\psi}_e(\vec{r},\vec{p})\gamma_0(1+\gamma_5)v(k) - (\vec{q} - \vec{p} + i\vec{\nabla})\cdot\bar{\psi}_e(\vec{r},\vec{p})\vec{\gamma}(1+\gamma_5)v(k) \\ = \bar{\psi}_e(\vec{r},\vec{p})\{[E + e\phi(\vec{r})]\gamma_0 - i\vec{\gamma}\cdot\vec{\nabla}\}(1+\gamma_5)v(k) + \bar{\psi}_e(\vec{r},\vec{p})\not{k}(1+\gamma_5)v(k) \\ = m_e\bar{\psi}_e(\vec{r},\vec{p})(1+\gamma_5)v(k)\end{aligned} \tag{22}$$

as pointed out by Blokhintsev and Dolinsky.[14]

Renormalizations of a and f are of interest in that the conserved vector current hypothesis (CVC)[19] predicts values for these couplings. However, we have already seen that the correction to a vanishes because of the identity $\sigma - 3\tau = 0$. Also, for an analog transition CVC requires that

$$e = 0. \tag{23}$$

Thus there is no modification expected for the Fermi term. In the case of f there is a substantial modification expected to the CVC value since one has from impulse approximation estimates[20]

$$\frac{Z\alpha}{m_N R} j_2/f \sim \frac{Z\alpha}{m_N R} g/f \sim Z\alpha m_N R. \tag{24}$$

Since $\gamma_0 u(\vec{p}=0) = u(\vec{p}=0)$, the induced Coulomb term arising from the induced pseudoscalar h contributes to an effective renormalization of the Gamow-Teller coupling in muon capture

$$c_{\text{eff}}(\mu \text{ capture}) = c + \frac{\alpha Z}{2MR}\tau\left(c - 2d + 2b - h\frac{m_\mu}{2M}\right), \tag{25}$$

which is different from the value for the corresponding electron decay

$$c_{\text{eff}}(e^- \text{ decay}) = c + \frac{\alpha Z}{2MR}\tau(c - 2d + 2b). \tag{26}$$

Using the value derived via the partially conserved axial vector current hypothesis (PCAC)[21]

$$\frac{h}{A^2 c} \simeq \left(\frac{2m_N}{m_\pi}\right)^2 \approx 180, \tag{27}$$

this gives

$$c_{\text{eff}}(\mu \text{ capture}) - c_{\text{eff}}(e^- \text{ decay}) \approx -h\tau\frac{\alpha Z}{2MR}\frac{m_\mu}{2M}. \tag{28}$$

This represents about $1\frac{1}{2}\%$ reduction of the "elementary particle"[22] predictions for the muon capture rate on ^{12}C. Such effects will, of course, be more sizable for heavier nuclei.

We also note that induced Coulomb terms are relevant to analyses of experiments whose aim is to study the question of second class currents. Here, the presence of a second class current may be considered to be signaled by a component of the weak form factor which changes sign between electron and corresponding mirror positron decays.[5] Examination of Eq. (15) then reveals that omission of induced Coulomb effects can result in simulation of a second class piece of the form factors.

Presently there are a number of experiments which attempt to probe for these second class effects. Of these there are two which are of relevance here. The first is the $A=8$ experiment of Tribble and Garvey,[23] wherein the β-α correlation is measured for the mirror processes

$$\begin{array}{l} {}^8\mathrm{B} \to {}^8\mathrm{Be}^*(2.90\ \mathrm{MeV}) + e^+ + \nu_e \\ \qquad\qquad \searrow \alpha + \alpha \\ \qquad\qquad \nearrow \\ {}^8\mathrm{Li} \to {}^8\mathrm{Be}^*(2.90\ \mathrm{MeV}) + e^- + \bar{\nu}_e \ . \end{array}$$

Since this is a $2^+ \to 2^+$ transition, the analysis involves nearly all the form factors. The theoretical prediction can be written in terms of[6]

$$\gamma_\pm^{\mathrm{theory}} = \frac{\pm b - d}{Ac} - \frac{1}{\sqrt{14}}\left[\pm 3\frac{f}{Ac} \pm (\tfrac{3}{2})^{1/2}\frac{g}{A^2c}\,\frac{E_0 - E}{m_N} + 3\frac{j_2}{A^2c}\,\frac{E_0 - 2E}{m_N}\right] - \frac{j_3}{A^2c}\,\frac{3}{\sqrt{35}}\,\frac{E}{m_N} \ . \tag{29}$$

Writing

$$\tfrac{1}{2}(\gamma_- - \gamma_+)_{\mathrm{exp}} = \tfrac{1}{2}(\gamma_- - \gamma_+)_{\mathrm{theory}} + \tfrac{1}{2}(\gamma_- - \gamma_+)_{\mathrm{Coulomb}} \ , \tag{30}$$

we find

$$\tfrac{1}{2}(\gamma_- - \gamma_+)_{\mathrm{Coulomb}} = \frac{27}{5\sqrt{14}}\,\frac{\alpha Z}{2m_N R}\,\frac{j_2}{A^2c}\,\tau \ . \tag{31}$$

Using an impulse approximation calculation[24]

$$\frac{j_2}{A^2c} \approx -300 \ , \tag{32}$$

we find

$$\tfrac{1}{2}(\gamma_- - \gamma_+)_{\mathrm{Coulomb}} = -0.23 \tag{33}$$

which simulates a second class signal,

$$\frac{d_{\mathrm{II}}}{Ac}_{\mathrm{simulated}} = +0.23 \ , \tag{34}$$

where d_{II} is the component of the induced tensor which changes sign between electron and corresponding mirror positron decays and is attributed to second class currents. Application to the experimental value[23]

$$\left(\frac{d_{\mathrm{II}}}{Ac}\right)_{\mathrm{exp}} = 0.6 \pm 0.9 \tag{35}$$

probably should not be done directly because the value of j_2 is somewhat uncertain, the number used in obtaining Eq. (32) arising from an impulse approximation calculation.

Finally we mention that careful β-γ correlation measurements have also been undertaken in the $A=20$ system[25]

$$\begin{array}{l} {}^{20}\mathrm{F} \to {}^{20}\mathrm{Ne}^*(1.634\ \mathrm{MeV}) + e^- + \bar{\nu}_e \\ \qquad\qquad \searrow {}^{20}\mathrm{Ne} + \gamma \\ \qquad\qquad \nearrow \\ {}^{20}\mathrm{Na} \to {}^{20}\mathrm{Ne}^*(1.634\ \mathrm{MeV}) + e^+ + \nu_e \ . \end{array}$$

This is also a $2^+ \to 2^+$ transition so that the same combination of form factors as given in Eq. (29) is involved. We find in this case

$$\tfrac{1}{2}(\gamma_- - \gamma_+)_{\mathrm{Coulomb}} = 1.4 \times 10^{-3} j_2/A^2c \ . \tag{36}$$

Here also an impulse approximation calculation yields[26]

$$\frac{j_2}{A^2c} \simeq -300 \ , \tag{37}$$

so that we find

$$\left(\frac{d_{\mathrm{II}}}{Ac}\right)_{\mathrm{simulated}} = +0.42 \ , \tag{38}$$

which is a relatively large effect, but still somewhat smaller than the present experimental limit[25]

$$\left(\frac{d_{\mathrm{II}}}{Ac}\right)_{\mathrm{exp}} = -0.1 \pm 2.0 \ . \tag{39}$$

In conclusion then, we note that induced Coulomb effects, although small, may be important corrections to the analysis in future generation experiments.

It is a pleasure to acknowledge very useful conversations with Professor S. B. Treiman.

APPENDIX

We start with the exact solution to the Dirac equation in the presence of a Coulomb potential:

$$\psi_e(\vec{r}, \vec{p}) = \left[\frac{(2\pi)^3}{m_e p}\right]^{1/2} \times \sum_{\kappa,\mu} i^l\, C_{l\ (1/2);\ j}^{\mu-\rho\ \rho;\ \mu}\, Y_l^{\mu-\rho *}(\hat{p})\, e^{-i\sigma_\kappa}\psi_{\kappa\mu}(\vec{r}) \ , \tag{A1}$$

where

$$\sigma_\kappa = \tfrac{1}{2}\pi[l(\kappa)+1-\gamma_\kappa]+\eta_\kappa - \arg\Gamma(\gamma_\kappa + i\nu)\,,$$

$$\psi_{\kappa\mu}(\vec{r}) = \begin{pmatrix} g_\kappa(r)\chi_{\kappa,\mu}(\hat{r}) \\ if_\kappa(r)\chi_{-\kappa,\mu}(\hat{r})\end{pmatrix},\ \gamma_\kappa = (\kappa^2-\alpha^2Z^2)^{1/2}\,,$$

$$\nu = \frac{\alpha Z E}{p}\,,\quad \exp(2i\eta_\kappa) = \frac{-\kappa + i\alpha Z m_e/p}{\gamma_\kappa + i\nu}\,, \tag{A2}$$

and $f_\kappa(r)$, $g_\kappa(r)$ are the usual small, large component Dirac wave functions. For the present application it is sufficient to retain only the $|\kappa|=1$ terms, giving

$$P_{j=1/2}\psi_e(r,p) = N^*[w(r) + x(r)\gamma^0 + y(r)\vec{\gamma}\cdot\hat{r} + z(r)\vec{\gamma}\cdot\hat{r}\gamma^0]u(p)\,, \tag{A3}$$

where for a uniform electrostatic charge distribution of radius R

$$N = \frac{1}{4\pi}\left[\frac{(2\pi)^3}{m_e p}\right]^{1/2}\exp\{i[\tfrac{1}{2}\pi(1-\gamma_1)+\eta_{-1} - \arg\Gamma(\gamma_1 + i\nu)]\}\,,$$

$$\delta = \tfrac{1}{2}\pi + \eta_1 - \eta_{-1}\,, \tag{A4}$$

$$w(r) = \frac{1}{2}\left(\frac{2m_e}{E+m_e}\right)^{1/2}\left[g_{-1}(r) + \frac{E+m_e}{p}f_1(r)e^{-i\delta}\right]$$

$$\approx [F_{\rm BJ}(Z,E)]^{1/2}\left[j_0(pr) - \alpha Z E R\left(\frac{1}{2}\frac{r^2}{R^2} - \frac{1}{15}\frac{r^4}{R^4}\right)\right],$$

$$x(r) = \frac{1}{2}\left(\frac{2m_e}{E+m_e}\right)^{1/2}\left[g_{-1}(r) - \frac{E+m_e}{p}f_1(r)e^{-i\delta}\right]$$

$$\approx -[F_{\rm BJ}(Z,E)]^{1/2}\alpha Z m_e R\left(\frac{1}{10}\frac{r^2}{R^2} - \frac{1}{140}\frac{r^4}{R^4}\right)$$

$$y(r) = \frac{i}{2}\left(\frac{2m_e}{E+m_e}\right)^{1/2}\left[f_{-1}(r) + \frac{E+m_e}{p}g_1(r)e^{-i\delta}\right]$$

$$\approx 0\,,$$

$$Z(r) = \frac{i}{z}\left(\frac{2m_e}{E+m_e}\right)^{1/2}\left[f_{-1}(r) - \frac{E+m_e}{p}g_1(r)e^{-i\delta}\right]$$

$$\approx -[F_{\rm BJ}(Z,E)]^{1/2}i\alpha Z\frac{r}{2R}\left(1-\frac{1}{5}\frac{r^2}{R^2}\right),$$

and $F_{\rm BJ}(Z,E)$ is the Behrens-Janecke Fermi function.

Then

$$-i\vec{\nabla}\psi_e(\vec{r},\vec{p}) \approx \vec{p}\psi_e(\vec{r},\vec{p}) - i\vec{\nabla}P_{j=1/2}\psi_e(\vec{r},\vec{p})$$

$$\approx \vec{p}\psi_e(\vec{r},\vec{p}) - i\vec{\nabla}z(r)\vec{\gamma}\cdot\hat{r}\gamma_0 u(p)$$

$$\approx \vec{p}\psi_e(\vec{r},\vec{p}) - \frac{\alpha Z}{2R}\left[\vec{\gamma}\left(1-\frac{1}{5}\frac{r^2}{R^2}\right) - \frac{2}{5}\vec{\gamma}\cdot\hat{r}\hat{r}\frac{r^2}{R^2}\right]\gamma_0 u(p)\,, \tag{A5}$$

which is the result quoted in Eq. (12).

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