PHYSICAL REVIEW C VOLUME 19, NUMBER 1 JANUARY 1979

"Optimal" approximation to projectile–bound-nucleon scattering

S. A. Gurvitz,* J.-P. Dedonder,† and R. D. Amado‡
Swiss Institute for Nuclear Research, S I N, 5234 Villigen
(Received 28 August 1978)

An approximation is presented for the projectile-bound nucleon scattering amplitude in terms of a "quasi-free" amplitude that minimizes corrections. This optimal choice permits factorization of the impulse term into a free on-shell amplitude and a form factor. The first nonvanishing correction terms are estimated.

[NUCLEAR REACTIONS, bound nucleon, scattering matrix, minimization of corrections, on-shell projectile-nucleon amplitude, factorization, higher order corrections.]

I. INTRODUCTION

The major ingredient in a multiple scattering theory of projectile nucleus scattering is the projectile-nucleon t matrix for scattering from a bound nucleon.[1] In impulse approximation this is represented as an integral over the free, but off-shell projectile-nucleon amplitude.[2] There are other schemes for expressing the bound amplitude in terms of the free one.[3] In this paper we develop the "optimal" prescription for doing this, that is, we find an approximation for the bound scattering amplitude such that the first order correction terms vanish. Our approximate amplitude is kinematically, but not dynamically a free projectile nucleon amplitude. Nevertheless, when used in on-shell projectile bound state scattering, it reduces to the usual on-shell, free amplitude (in the Breit frame) and factors out of the impulse approximation integral. Such factorized forms have been used before,[4] but we believe ours is the first derivation of them and the first demonstration that they are the first terms in a systematic expansion designed to minimize corrections.

Since our derivation is systematic we can estimate correction terms and find the parameter that controls the validity of the approximation. That parameter turns out to be the ratio of internal nucleon (virtual) velocity to projectile velocity so long as the elementary t matrix does not vary rapidly with energy. This is reasonable since we would expect to be able to factor projectile and nucleon motion so long as the first is fast and the interaction time short. (The interaction time is related to the energy derivative of the elementary t matrix.[1]

In Sec. II we derive our major results and in Sec. III discuss the correction terms. Generalization of our result to nonlocal interactions is treated in the Appendix.

II. DERIVATION

The first step in a theory of projectile-nucleus scattering, considered as a multiple scattering problem, is to construct the t matrix for the scattering of the projectile from one nucleon bound in the field of the others. That t matrix τ satisfies

$$\tau_i = V_i + V_i G \tau_i \,, \tag{1}$$

where V_i is the potential between the projectile and nucleon i and G is the full Green's function

$$G^{-1} = E - H - K_p \,, \tag{2}$$

where E is the total energy, H the full target Hamiltonian, and K_p the projectile kinetic energy. Solving (1) is equivalent to solving the nuclear many body problem; hence, for practical applications we seek an approximation to τ_i. A common approximation is to replace τ_i by t_i a t matrix calculated with a Green's function that does not involve the full nuclear Hamiltonian. For elastic scattering from the target bound state we will show (within the context of a few simplifying assumptions) how best to choose the Green's function defining t_i, that is, what choice makes the first correction to $\tau_i - t_i$ vanish. As a bonus we will find that for our choice the projectile-nucleus amplitude factors into an elementary on-shell projectile-nucleon amplitude and a form factor, that is, there is no integral over target nucleon wave functions to be done. Since our derivation is in terms of the scattering integral equations, we can easily display the higher order nonvanishing correction terms as well as estimate the parameters (or parameter) that must be small to justify their neglect.

Rather than carry out our argument for the general case—with all the corresponding notational confusion—we will present it for a very simple model and state the generalizations to more realistic examples afterwards.

Consider the scattering of a projectile of mass μ from a single nucleon of mass m bound to an infinitely massive core. We take both the nucleon-projectile potential V and the nucleon-core potential $\tilde{V}$ to be local and assume as in τ of Eq. (1) that there is no projectile core interaction. We wish to find an approximation to the τ matrix of Eq. (1) (now without the index i) of the form

$$t_a = V + VG_a t_a\,, \tag{3}$$

where G_a is a Green's function to be chosen so that Eq. (3) is more easily solved than Eq. (1), but t_a is a good approximation to τ. Given (1) and (3) we can write

$$\tau = t_a + t_a(G - G_a)\tau \tag{4}$$

$$= t_a + t_a(G - G_a)t_a + t_a(G - G_a)t_a(G - G_a)t_a + \cdots \tag{5}$$

and

$$G - G_a = G_a hG \tag{6}$$

$$= G_a hG_a + G_a hG_a hG_a + \cdots\,, \tag{7}$$

where

$$h = G_a^{-1} - G^{-1}\,. \tag{8}$$

Putting (7) into (5) we see that the t matrix τ can be expressed as a double power series, the first two terms of which are

$$\tau = t_a + t_a G_a hG_a t_a + \cdots \equiv t_a + \Delta t_a + \cdots\,. \tag{9}$$

We wish to choose G_a so that for on-shell elastic scattering from a target bound state, the second term on the right-hand side of Eq. (9) vanishes.

A natural choice for the Green's function G_a would seem to be G_0, the free projectile-nucleon propagator. With that choice (3) becomes the elementary projectile-nucleon t matrix off and on shell. Unfortunately, that choice does not make the correction Δt_a of Eq. (9) vanish. We still wish to retain some features of the free Green's function G_0 and, therefore, choose G_a so that it is diagonal in the projectile ($\vec{p}$) and nucleon ($\vec{P} - \vec{p}$) momenta, and so as not to depend on the total projectile-nucleon momentum $\vec{P}$ (Fig. 1). We write

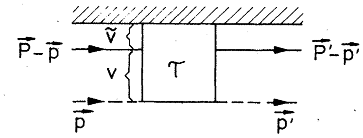

FIG. 1. Graph for the scattering of projectile (dashed line) on the nucleon (solid line) bound by an infinitely heavy core.

$$\langle \vec{p}, \vec{P} | G_a | \vec{p}', \vec{P}' \rangle = \frac{\delta^3(\vec{p} - \vec{p}')\delta^3(\vec{P} - \vec{P}')}{\epsilon - p^2/2\mu}\,, \tag{10}$$

where ϵ is to be determined so that Δt_a is zero. The quantity ϵ may depend on the projectile momentum $\vec{p}$ and on external parameters, but it does not depend on the total projectile-nucleon momentum $\vec{P}$. With this form for the propagator G_a, and with a local potential for the nucleon-projectile interaction V (so that in momentum space V is a function only of $\vec{p} - \vec{p}'$), the scattering matrix t_a will conserve total momentum and be independent of $\vec{P}$ so we can write

$$\langle \vec{p}', \vec{P}' | t_a | \vec{p}, \vec{P} \rangle = \langle \vec{p}' | \hat{t} | \vec{p} \rangle \delta^3(\vec{P} - \vec{P}')\,. \tag{11}$$

Let us now consider the matrix element of Δt defined in Eq. (9) for elastic projectile nucleus scattering from projectile momentum $\vec{p}$ to $\vec{p}'$ with the nucleus in the bound state ψ_0. Using Eqs. (10) and (11) this becomes

$$\langle \vec{p}', \psi_0 | \Delta t_a | \vec{p}, \psi_0 \rangle = \int \Psi_0^*(\vec{P}' - \vec{p}')\, \frac{\langle \vec{p}' | t | \vec{p}_1 \rangle \langle \vec{P}', \vec{p}_1 | (G_a^{-1} - G^{-1}) | \vec{P}, \vec{p}_2 \rangle}{\epsilon_1 - p_1^2/2\mu}\, \frac{\langle \vec{p}_2 | \hat{t} | \vec{p} \rangle}{\epsilon_2 - p_2^2/2\mu}\, \Psi_0(\vec{P} - \vec{p})\, d^3p_1 d^3p_2 d^3P d^3P'\,, \tag{12}$$

where the index on ϵ reminds us that ϵ_i may depend on $\vec{p}_i$. In Eq. (12), $\psi_0(\vec{k})$ is the momentum space wave function of the bound nucleon with binding energy B. It satisfies $H\psi_0 = -B\psi_0$, which reads in momentum space

$$\frac{k^2}{2m}\Psi_0(\vec{k}) + \int \tilde{V}(\vec{k} - \vec{k}')\Psi_0(\vec{k}')d^3k' = -B\Psi_0(\vec{k})\,, \tag{13}$$

where we have used explicitly the fact that the nucleon potential $\tilde{V}$ is local. For the matrix element of h in Eq. (12) we have

$$\langle \vec{P}', \vec{p}_1 | (G_a^{-1} - G^{-1}) | \vec{P}, \vec{p}_2 \rangle = \delta^3(\vec{p}_1 - \vec{p}_2)\left[\left(\epsilon_1 - \frac{p_1^2}{2\mu} - E + \frac{(\vec{P}' - \vec{p}_1)^2}{2m} + \frac{p_1^2}{2\mu}\right)\delta^3(\vec{P} - \vec{P}') + \tilde{V}(\vec{P} - \vec{P}')\right]\,. \tag{14}$$

When we substitute (14) in (12) we can use the Schrödinger equation (13) to do the $\vec{P}$ integration on the $\tilde{V}$

term and the $\delta^3(\vec{P}-\vec{P}')$ to do it in the remaining term to give finally

$$\langle \vec{p}', \Psi_0 | \Delta t_a | \vec{p}, \Psi_0\rangle = \int \Psi_0^*(\vec{P}'-\vec{p}') \frac{\langle \vec{p}' | t | \vec{p}_1\rangle}{\epsilon_1 - p_1^2/2\mu}\left[\epsilon_1 - E + \frac{(\vec{P}'-\vec{p}_1)^2}{2m} - B - \frac{(\vec{P}'-\vec{p})^2}{2m}\right]\frac{\langle \vec{p}_1 | \hat{t} | \vec{p}\rangle}{\epsilon_1 - p_1^2/2\mu}\Psi_0(\vec{P}'-\vec{p})d^3P'd^3p_1 \quad (15)$$

Now introduce the variables

$$\vec{K} = \tfrac{1}{2}(\vec{p}+\vec{p}'), \quad \vec{q} = \vec{p}-\vec{p}', \quad (16)$$

with $\vec{K}\cdot\vec{q}=0$ by energy conservation. If we further use the definite parity of ψ_0, which implies that $\psi_0^*(\vec{A})\psi_0(B) = \psi_0^*(-\vec{A})\psi_0(-\vec{B})$, Eq. (15) can be transformed into

$$\langle \vec{p}', \Psi_0 | \Delta t_a | \vec{p}, \Psi_0\rangle = \int \Psi_0^*(\vec{P}'-\vec{p}') \frac{\langle \vec{p}' | \hat{t} | \vec{p}_1\rangle}{\epsilon_1 - p_1^2/2\mu}\left[\epsilon_1 - E - B - \frac{q^2}{8m} + \frac{(\vec{K}-\vec{p}_1)^2}{2m}\right]\frac{\langle \vec{p}_1 | \hat{t} | \vec{p}\rangle}{\epsilon_1 - p_1^2/2m}\Psi_0(\vec{P}'-\vec{p})d^3P'd^3p_1, \quad (17)$$

which vanishes if we choose

$$\epsilon_1 = E + B + \frac{q^2}{8m} - \frac{(\vec{K}-\vec{p}_1)^2}{2m} = \frac{p^2}{2\mu} + \frac{q^2}{8m} - \frac{(\vec{K}-\vec{p}_1)^2}{2m}, \quad (18)$$

where in this last step we put $E = k^2/2\mu - B$ the on-shell projectile nucleus scattering energy. Equation (18) gives the special form ϵ should have in the propagator G_a of Eq. (10) in order that the scattering operator t_a of Eq. (3) be defined so that the elastic scattering matrix element of Δt_a vanish. The propagator G_a of Eq. (10) with ϵ defined by Eq. (18) is certainly an unfamiliar propagator. It is not invariant in the usual sense because it includes external momenta, but it is a perfectly allowable propagator, and the scattering operator t_a (or $\hat{t}$) calculated with it satisfies all the conditions we have placed upon it along the way.

Let us now see the consequence of using our scattering operator t_a as an approximation for the t matrix τ. We have for the contribution to elastic scattering for the $\tau \sim t_a$ term

$$\langle \vec{p}', \Psi_0 | t_a | \vec{p}, \Psi_0\rangle = \int d^3P d^3P' \Psi_0^*(\vec{P}'-\vec{p}')\delta^3(\vec{P}-\vec{P}')\langle \vec{p}' | \hat{t} | \vec{p}\rangle \Psi_0(\vec{P}-\vec{p})$$

$$= \langle \vec{p}' | \hat{t} | \vec{p}\rangle \int \Psi_0^*(P-p')\Psi_0(\vec{P}-\vec{p})d^3P = \langle \vec{p}' | \hat{t} | \vec{p}\rangle S_0(\vec{p}-\vec{p}'), \quad (19)$$

where $S_0(\vec{q})$ is the ground state form factor for momentum $\vec{q}$. As promised, the amplitude factors; and that comes about from the fact that $\hat{t}$ is independent of the total projectile-nucleon momentum $\vec{P}$.

Let us now look at the equation satisfied by $\langle \vec{p}' | \hat{t} | \vec{p}\rangle$ with the definition of ϵ given by Eq. (18). We will use the variables defined in Eq. (16)

$$\langle \vec{K}-\vec{q}/2 | \hat{t} | \vec{K}+\vec{q}/2\rangle = V(-\vec{q}) + \int \frac{V(\vec{K}-\vec{q}/2-\vec{p}_1)\langle \vec{p}_1 | \hat{t} | \vec{K}+\vec{q}/2\rangle}{(\vec{K}-\vec{q}/2)^2/2\mu + q^2/8m - (\vec{K}-\vec{p}_1)^2/2m - p_1^2/2\mu} d^3p_1. \quad (20)$$

The matrix element $\langle \vec{K}-\vec{q}/2 | \hat{t} | \vec{K}+\vec{q}/2\rangle$ is just the *on-shell* elementary projectile-nucleon scattering amplitude expressed in the Breit frame, i.e., from projectile momentum $\vec{K}+\vec{q}/2$ to $\vec{K}-\vec{q}/2$ and nucleon momentum $-\vec{q}/2$ to $\vec{q}/2$.

In terms of a more familiar invariant t matrix we can write

$$\langle \vec{P}', \Psi_0 | t_a | \vec{P}, \Psi_0\rangle = t(E^{\text{eff}}, \vec{p}'-\eta\vec{K}, \vec{p}-\eta\vec{K})S_0(\vec{q}), \quad (21)$$

where $t(E^{\text{eff}}, \vec{p}'-\eta\vec{K}, \vec{p}-\eta\vec{K})$ is the on-shell invariant projectile-nucleon t matrix for scattering from relative momentum $\vec{p}-\eta\vec{K}$ to $\vec{p}'-\eta\vec{K}$, $[\eta = \mu/(m+\mu)]$ at the effective energy

$$E^{\text{eff}} = \frac{p^2}{2\mu} + \frac{q^2}{8m} - \frac{K^2}{2(m+\mu)}$$

with $\vec{q}$ and $\vec{K}$ defined in Eq. (16).

It is instructive to compare this prescription with recent work that shows how to write the projectile-nucleus amplitude in terms of an on-shell amplitude in the case of peripheral scattering.[5] Using $p^2 = K^2 + q^2/4$ we can rewrite the effective energy in Eq. (21) as

$$E^{\text{eff}} = \frac{p^2}{2\mu} - \frac{p^2}{2(m+\mu)} + \frac{q^2}{8m} + \frac{q^2}{8(m+\mu)}.$$

The leading peripheral contribution comes from the longest range part of the form factor $S_0(\vec{q})$ in Eq. (19). As is shown in Ref. 5 the leading term is obtained by evaluating all terms other than $S_0(\vec{q})$ at the value of q^2 corresponding to the largest range singularity of $S_0(\vec{q})$. If the bound state wave function falls asymptotically like $e^{-\alpha r}$ so that the binding energy is $B=\alpha^2/2m$, $S_0(\vec{q})$ is singular at $q^2=-4\alpha^2$ [the factor 4 comes from the fact that $S_0(\vec{q})$ involves the square of the wave function]. Using this value for q^2 in E^{eff} gives the same result as Ref. 5 for the peripheral region treated in the closure approximation.

It is easy to see that our result may be extended to π-nucleus scattering using the customary, ad hoc, relativistic kinetic energy terms for the pion.

The extension of our method to the case of finite nucleus mass (M) is straightforward. The derivation is simplest in the Breit projectile-nucleus frame. It requires only replacing the nucleon momentum in the nucleon wave function by the relative nucleon-core momenta and the nucleon mass m in the Schrödinger equation (13) by the reduced nucleon-core mass. This changes our result (21) only by replacing $S_0(\vec{q})$ with $S_0(\vec{q}(M-m)/M)$. The t matrix in Eq. (21) now corresponds to on-shell scattering when the struck nucleon has all the momentum of the nucleus and the core remains at rest.

Formally, the first correction term will also vanish for a many-body system so long as the nucleon-nucleon interactions in the target are local, but expansion in that interaction is presumably not valid. If instead we use an optical potential for the bound nucleon dynamics, there will be contributions to the first correction term from the energy dependence of the optical potential even if it is local. These corrections will be small if the energy dependence is weak. Correction terms to Δt coming from multiple scattering on different nucleons are part of the problem of how one goes from Eq. (1) to the full projectile-nucleus t matrix. Nonlocal projectile-nucleons potentials are treated in the Appendix.

III. CORRECTION TERMS

In this section we estimate the correction terms to $\tau - t_a$ with our "optimal" choice of t_a. In particular, by examining the first nonvanishing order we attempt to isolate the dimensionless parameters that control the expansion. (We are aware of the fact that a study of the "next" order is not a proof of convergence, but we hope it will shed some light on the physics of the expansion.)

Consider the double power series for the t matrix τ of Eqs. (5)–(9). Since the propagator G_a is diagonal in the projectile and nucleon momenta and does not depend on the total projectile-nucleon momenta (10), and since the nucleon-core potential $\tilde{V}$ is local, G_a and h commute. Hence, the expansion (9) can be written

$$\tau = t_a[1+(G_a h + G_a{}^2 h^2 + \cdots)G_a t_a + (G_a h + G_a{}^2 h^2 + \cdots)G_a t_a(G_a h_a + G_a{}^2 h_a{}^2 + \cdots)G_a t_a + \cdots]\,. \qquad (22)$$

We have chosen G_a so that for scattering from a nuclear bound state, the contribution of the first correction Eq. (22) ($\sim G_a h$) is zero. The first nonvanishing term [$\sim (G_a h)^2$] is

$$\begin{aligned}\langle \vec{p}'\Psi_0|\Delta_2 t_a|\vec{p},\Psi_0\rangle &= \langle \vec{p}',\Psi_0|t_a h^2 G_a{}^3 t_a + t_a h G_a{}^2 t_a h G_a{}^2 t_a|\vec{p},\Psi_0\rangle \\ &= \int \Psi_0^*(\vec{P}'-\vec{p}')[\langle \vec{p}'|\hat{t}|\vec{p}_1\rangle\langle \vec{p}_1,\vec{P}'|h|\vec{p}_1,\vec{P}''\rangle\langle \vec{p}_1,\vec{P}''|h|\vec{P},\vec{p}_1\rangle\langle \vec{p}_1|G_a{}^3|\vec{p}_1\rangle\langle \vec{p}_1|\hat{t}|\vec{p}\rangle \\ &\quad + \langle \vec{p}'|\hat{t}|\vec{p}_1\rangle\langle \vec{p}_1,\vec{P}'|h|\vec{p}_1,\vec{P}''\rangle\langle \vec{p}_1|G_a{}^2|\vec{p}_1\rangle\langle \vec{p}_1|\hat{t}|\vec{p}_2\rangle \\ &\quad \times\langle \vec{p}_2,\vec{P}''|h|\vec{P},\vec{p}_2\rangle\langle \vec{p}_2|G_a{}^2|\vec{p}_2\rangle\langle \vec{p}_2|\hat{t}|\vec{p}_1\rangle]\,\Psi_0(\vec{P}-\vec{p})d^3p_1 d^3p_2 d^3P'' d^3P' d^3P\,. \qquad (23)\end{aligned}$$

Since G_a and $\hat{t}$ do not depend on the total projectile-nucleon momentum $\vec{P}$, the integration over the $\vec{P}$'s can be done first. Using Eqs. (13) and (14) and applying one operator h to $|\psi_0\rangle$ and one to $\langle\psi_0|$, we find after some algebra

$$\int \Psi_0^*(\vec{P}'-\vec{p}')\langle \vec{P}',\vec{p}_1|h|\vec{p}_1,\vec{P}''\rangle\langle \vec{p}_2,\vec{P}''|h|\vec{P},\vec{p}_2\rangle\Psi_0(\vec{P}-\vec{p})d^3P''d^3P'd^3P = \int \Psi_0^*\left(\vec{Q}-\frac{\vec{q}}{2}\right)\Psi_0\left(\vec{Q}+\frac{\vec{q}}{2}\right) \times \frac{[\vec{Q}\cdot(\vec{p}_1-\vec{p})][Q\cdot(\vec{p}_2-\vec{p}')]}{m^2}\,d^3Q\,, \qquad (24)$$

where the momentum $\vec{q}$ is defined in Eq. (16).

Using Eqs. (10), (18), and (21) we find for (23) (writing $\Delta_2 t_a = \Delta_2' t_a + \Delta_2'' t_a$)

$$\langle \vec{p}', \Psi_0 | \Delta_2' t_a | \vec{p}, \Psi_0 \rangle = \int t(E^{\text{eff}}, \vec{k}, \vec{k}_1) \frac{d^3 k_1}{(E^{\text{eff}} - k_1^2/2\bar{\mu})^3} t(E^{\text{eff}}, \vec{k}_1, \vec{k}') \times \int \Psi_0^* \left(\vec{Q} - \frac{\vec{q}}{2}\right) \Psi_0 \left(\vec{Q} + \frac{\vec{q}}{2}\right) \frac{[\vec{Q}\cdot(\vec{k}_1 - \vec{k})][\vec{Q}\cdot(\vec{k}_1 - \vec{k}')]}{m^2} d^3 Q \tag{25}$$

and

$$\langle \vec{p}', \Psi_0 | \Delta_2'' t | \vec{p}, \Psi_0 \rangle = \int t(E^{\text{eff}}, \vec{k}, \vec{k}_1) \frac{d^3 k_1}{(E^{\text{eff}} - k_1^2/2\bar{\mu})^2} t(E^{\text{eff}}, \vec{k}_1, \vec{k}_2) \frac{d^3 k_2}{(E^{\text{eff}} - k_2^2/2\bar{\mu})^2} \times t(E^{\text{eff}}, \vec{k}_2, \vec{k}') \int \Psi_0^* \left(\vec{Q} - \frac{\vec{q}}{2}\right) \Psi_0 \left(\vec{Q} + \frac{\vec{q}}{2}\right) \frac{[\vec{Q}\cdot(\vec{k}_1 - \vec{k})][\vec{Q}\cdot(\vec{k}_2 - \vec{k}')]}{m^2} d^3 Q , \tag{26}$$

where

$$\vec{k} = \vec{p} - \eta \vec{K}, \quad \vec{k}' = \vec{p}' - \eta \vec{K}, \quad E^{\text{eff}} = k^2/2\bar{\mu} = k'^2/2\bar{\mu}, \quad \bar{\mu} = m\mu/(m+\mu).$$

These exact expressions for the first nonvanishing correction to $\tau \sim t_a$ do not explicitly contain the nucleon potential $\tilde{V}$, but rather require only the off-shell behavior of our amplitude t_a and knowledge of the wave function ψ_0. It is not possible to go beyond Eqs. (25) and (26) without some input, but rather than make detailed assumptions we will estimate the order of magnitude of these terms to extract the expansion parameters. First, we study the part that depends on the bound state wave functions. We have integrals of the form

$$A_{ij}(\vec{q}) = \int \Psi_0^* \left(\vec{Q} - \frac{\vec{q}}{2}\right) \Psi_0 \left(\vec{Q} + \frac{\vec{q}}{2}\right) \frac{Q_i Q_j}{m^2} d^3 Q = \frac{S_2(\vec{q})\delta_{ij}}{3m^2} , \tag{27}$$

where we have used the symmetry of the wave functions to obtain the δ_{ij} and introduced the quantity

$$S_2(\vec{q}) = \int \Psi_0^*(\vec{Q} - \vec{q}/2)\Psi_0(\vec{Q} + \vec{q}/2) Q^2 d^3 Q . \tag{28}$$

$S_2(\vec{q})$ is a second moment of the form factor $S_0(\vec{q})$, and we can estimate it as

$$S_2(\vec{q}) \cong \langle Q^2 \rangle S_0(\vec{q}) , \tag{29}$$

which is exact for harmonic oscillator wave functions. The form (29) is useful since it expresses the correction in terms of $S_0(\vec{q})$, and it is then easy to compare with the leading term, given in Eq. (21). In this comparison, the quantity that enters is

$$\frac{S_2(\vec{q})}{3m^2 S_0(\vec{q})} \cong \frac{\langle Q^2 \rangle}{3m^2} \cong V_N^2 , \tag{30}$$

where V_N is the rms velocity of the bound nucleon. Inserting Eq. (25) and (26) and dividing by the leading term given in Eq. (21), we obtain

$$\frac{\Delta_2' t_a}{t(E^{\text{eff}}, \vec{k}, \vec{k}') S_0(\vec{q})} = \frac{1}{t(E^{\text{eff}}, \vec{k}, \vec{k}')} \int t(E^{\text{eff}}, \vec{k}, \vec{k}_1) \frac{d^3 k_1}{(k^2/2\bar{\mu} - k_1^2/2\bar{\mu})^3} t(E^{\text{eff}}, \vec{k}_1, \vec{k}')(\vec{k}_1 - \vec{k})\cdot(\vec{k}_1 - \vec{k}') \frac{S_2(\vec{q})}{3m^2 S_0(\vec{q})} , \tag{31}$$

$$\frac{\Delta_2'' t_a}{t(E^{\text{eff}}, \vec{k}, \vec{k}') S_0(\vec{q})} = \frac{1}{t(E^{\text{eff}}, \vec{k}, \vec{k}')} \int t(E^{\text{eff}}, \vec{k}, \vec{k}_1) \frac{d^3 k_1}{(k^2/2\bar{\mu} - k_1^2/2\bar{\mu})^2} t(E^{\text{eff}}, \vec{k}_1, \vec{k}_2) \frac{d^3 k_2}{(k^2/2\bar{\mu} - k_2^2/2\bar{\mu})^2} \times t(E^{\text{eff}}, \vec{k}_2, \vec{k}')(\vec{k}_1 - \vec{k})\cdot(\vec{k}_2 - \vec{k}') \frac{S_2(\vec{q})}{3m^2 S_0(\vec{q})} . \tag{32}$$

The integrals remaining in Eqs. (31) and (32) depend only on the projectile-nucleon parameters and are more difficult to estimate. In the evaluation of the integral (31) the following identity is useful[6]:

$$\frac{dt(E, \vec{k}, \vec{k}')}{dE} = \int \frac{t(E, \vec{k}, \vec{k}'')t(E, \vec{k}'', \vec{k}')}{(E - k''^2/2\bar{\mu})^2} d^3 k'' . \tag{33}$$

We can write (using 27) after some algebra

$$\frac{\Delta_2' t_a}{t(E^{\text{eff}}, \vec{k}, \vec{k}') S_0(q)} \cong \frac{S_2(q)}{3m^2 S_0(q)} 2\bar{\mu} \left(\frac{K}{k} b - 1\right) \frac{d}{dE} \ln[t(E, \vec{k}, \vec{k}')] \Big|_{E = E^{\text{eff}} = k^2/2\bar{\mu} = k'^2/2\bar{\mu}} , \tag{34}$$

where $\vec{K}=\frac{1}{2}(\vec{k}+\vec{k}')$ and b is a numerical factor that enters from angular integration. For backward scattering ($\vec{K}=0$) Eq. (34) is exact. For nonbackward scattering (34) gives the order of magnitude of the correction. The derivative in Eq. (34) can be expressed in terms of the scattering time delay $T_0[T_0=(1/t)(dt/dE)]$ (Ref. 1), and using Eq. (30) we can estimate the magnitude of correction

$$\frac{\Delta_2' t_a}{t(E^{\text{eff}},\vec{k},\vec{k}')S_0(\vec{q})}\cong V_N^2\bar{\mu}T_0. \qquad (35)$$

The derivative of t is a dynamical quantity. If t is relatively slowly varying we can approximate T_0 as $1/E$. In that case we get

$$\frac{\Delta_2' t_a}{t(E^{\text{eff}},\vec{k},\vec{k}')S_0(\vec{q})}\cong\frac{V_N^2}{V_x^2}, \qquad (36)$$

which is the satisfying result that the approximation parameter is the relative nucleon to projectile velocity. It enters in the square since the first order term has been made to vanish. It is precisely when this parameter is small that the factorization should be possible. However, this depends on the derivative of t being small. If it is large, as it will be near a resonance, that implies a large time delay T_0, and the parameter of expansion (35) may be large. In that case we can no longer use our approximation (21) for the τ matrix.

The second term (32) is even more difficult to estimate. After some algebra it can be written

$$\frac{\Delta_2'' t_a}{t(E^{\text{eff}},\vec{k},\vec{k}')S_0(\vec{q})}\cong\frac{V_N^2(\mu/k)^2}{t(E^{\text{eff}},\vec{k},\vec{k}')}\int t(E^{\text{eff}},\vec{k},\vec{k}_1)\frac{d^3k_1}{E^{\text{eff}}-k_1^2/2\bar{\mu}}\,t(E^{\text{eff}},\vec{k}_1,\vec{k}_2)\frac{d^3k_2}{E^{\text{eff}}-k_2^2/2\bar{\mu}}\,t(E^{\text{eff}},\vec{k}_2,\vec{k}')C_2, \qquad (37)$$

where C_2 is a factor (or order one) coming from angular integrations. If the t matrices in (37) are slowly varying, we may use

$$\frac{1}{t}\int tG_0td^3k_1\sim 1$$

based on the optical theorem and again get (35) as our estimate. If the t matrix is not slowly varying, again the approximation may fail.

The higher order corrections can be found in the same way as Eqs. (25) and (26). The third order correction depends explicitly on the second derivative of the nucleon potential $\tilde{V}$ in the coordinate representation. Assuming that the nuclear potential is not a strongly varying function, we can hope that our expansion parameter controls the higher order terms as well.

ACKNOWLEDGMENTS

We are grateful to our colleagues F. Lenz and F. Cannata for spirited discussions. One of us (R. D. A.) thanks F. Lentz for arranging a pleasant visit at the Swiss Institute for Nuclear Research. This work was supported in part by the National Science Foundation.

APPENDIX

We consider now the case of a nonlocal separable S-wave projectile-nucleon potential. Galilean invariance implies that it depends only on the relative projectile-nucleon momenta

$$V(\vec{p}-\eta\vec{P},\vec{p}'-\eta\vec{P}')=\delta^3(\vec{P}-\vec{P})\lambda g(\vec{p}-\eta\vec{P})g(\vec{p}'-\eta\vec{P}'), \qquad (A1)$$

where λ is the strength of the interaction.

The potential now depends explicitly on the total projectile nucleon momentum $\vec{P}$, which in turn means that the scattering operator $\hat{t}$ defined by Eq. (11) depends on this momentum $\vec{P}$. Equation (3) becomes

$$\langle\vec{p}|\hat{t}(\vec{P})|\vec{p}'\rangle=\lambda g(\vec{p}-\eta\vec{P})g(\vec{p}'-\eta\vec{P})+\lambda g(\vec{p}-\eta\vec{P})\int g(\vec{p}''-\eta\vec{P})\frac{d^3p''}{\epsilon-p''^2/2\mu}\times\langle\vec{p}''|\hat{t}(\vec{P})|\vec{p}\rangle. \qquad (A2)$$

Using for ϵ the expression (18) found in the case of a local projectile-nucleon potential, we have

$$\epsilon-\frac{p''^2}{2\mu}=\frac{p^2}{2\mu}-\frac{K^2}{2(m+\mu)}+\frac{q^2}{8m}-\frac{(\vec{p}''-\eta\vec{K})^2}{2\bar{\mu}}. \qquad (A3)$$

The solution of Eq. (A2) then reads

$$\langle\vec{p}|\hat{t}(\vec{P})|\vec{p}'\rangle=\lambda\frac{g(\vec{p}-\eta\vec{P})g(\vec{p}'-\eta\vec{P})}{1-\lambda\int d^3p''\dfrac{g^2(\vec{p}''-\eta\vec{P})}{\dfrac{p^2}{2\mu}-\dfrac{K^2}{2(m+\mu)}+\dfrac{q^2}{8m}-\dfrac{(\vec{p}''-\eta\vec{K})^2}{2\bar{\mu}}}}. \qquad (A4)$$

If we now introduce this matrix element in Eq.

(12), we note that the matrix element of Δt_a is not zero because of the explicit dependence of the scattering operator $\hat{t}$ on the total projectile nucleon momentum $\vec{P}$. However, the nucleon wave function ψ_0 has a small extension in momentum space (of the order of k_F, where k_F is Fermi momentum), we may hence assume that the difference $\Delta\vec{P} = \vec{P} - \vec{K}$ is small and of the same order. The projectile nucleon form factor depends on the relative momentum $\vec{p} - \eta\vec{P} = \vec{p} - \eta\vec{K} - \eta\Delta\vec{P}$, and we suppose that in this form factor we can neglect the dependence on $\eta|\Delta\vec{P}| \sim (\mu/m)k_F$. This is equivalent to assuming that the range of the projectile nucleon interaction is very small compared to $(m/\mu)k_F^{-1}$. In Eq. (A4) we therefore simply replace in the form factors g the momentum $\vec{P}$ by $\vec{K}$. In that case all the results derived in the main text hold.

These results can easily be extended to the case of a superposition of separable potentials. One can also easily calculate corrections due to the range of the projectile-nucleon interaction (i.e., due to neglecting the dependence on $\eta\Delta\vec{P}$ in the form factor g).

---

*On leave from the Weizmann Institute of Science, Rehovot, Israel.

†On leave from the Institut de Physique Nucléaire, Division de Physique Théorique (Laboratoire associé du CNRS), F-91406 Orsay Cedex, France.

‡Permanent address: Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104.

[1]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964).

[2]G. F. Chew, Phys. Rev. 80, 196 (1950); G. F. Chew and G. C. Wick, *ibid.* 85, 636 (1952); G. F. Chew and M. L. Goldberger, *ibid.* 87, 778 (1952).

[3]A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) 8, 551 (1959).

[4]The earliest use can be found in O. Chamberlain and M. O. Stern, Phys. Rev. 94, 666 (1954). Some more recent references are L. Bertocchi and A. Capella, Nuovo Cimento 51A, 369 (1967); G. Alberi and L. Bertocchi, *ibid.* 63A, 285 (1969); C. Schmit, Nucl. Phys. A197, 449 (1972); K. Gabathuler and C. Wilkin, *ibid.* B70, 215 (1974); S. A. Gurvitz, Y. Alexander, and A. S. Rinat, Ann. Phys. (N.Y.) 93, 152 (1975); 98, 346 (1976); S. J. Wallace, Phys. Rev. C 12, 179 (1975); J. P. Maillet, J. P. Dedonder, and C. Schmit, Nucl. Phys. A271, 253 (1976).

[5]R. D. Amado, F. Lenz, and K. Yazaki, Phys. Rev. C 18, 918 (1978).

[6]L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Israel Program for Scientific Translations, Jerusalem, 1965).