Validity and accuracy of separable potentials in three-body calculations

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(Received 10 May 1978)

The three-body problem is reconsidered using separable potentials for the two-body interactions. Using the separable approximation, the Faddeev equations reduce to coupled integral equations in one continuous variable. The separable two-body interactions used are taken as consisting of two parts to include both attraction and repulsion. Each part of the potential is a spin-dependent central force together with tensor forces. Numerical calculations for the resulting integral equations are carried out to calculate the binding energies of the nuclei ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, and ^{12}C, using separable potentials of the Yamaguchi, Tabakin, Mongan, and Reid forms. The present calculations show the validity of the separable approximation and that the separable potentials extract accurate binding energies.

[NUCLEAR STRUCTURE ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, ^{12}C; three-body model. Calculated binding energies.]

I. INTRODUCTION

The three-body problem has been proved to be one of the most interesting in the study of the nuclear static properties of nuclei. The different approaches used in solving the three-body problem lead to a well-behaved set of three-body integral equations. Faddeev[1,2] and Lovelace[3] formulations are one of the successful approaches to solving the three-body problem. Faddeev successfully introduced an exact solution for the three-body problem. In this solution, Faddeev has shown that a well-behaved set of three-body equations involves the two-body T matrix rather than the potential. So, the Faddeev equations[1,2] remain a well-defined system whatever the potential form is.

Separable potentials are found to be very useful in studying the three-nucleon system because they entail great simplicity in the analysis of the three-body problem. These potentials have been widely used in calculating the three-nucleon binding energy, radius, Coulomb energy, form factor, and also the neutron-deuteron scattering cross section. This simplicity appears by using separable potentials which reduce the Faddeev equations to a set of coupled, one-dimensional integral equations. Thus the Faddeev equations are reduced to a set of coupled integral equations in one continuous variable by using nonlocal separable potentials. While, in the case of using local potentials, the Faddeev equations are reduced to equations in two continuous variables.

We study in the present work the nuclei ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, and ^{12}C. We use the α-particle model by taking the α cluster as a single entity since it is tightly bound without internal structure. Thus each of these nuclei is taken as a bound state of three particles. This brought out the more interesting problem of studying each one of these nuclei as a three-body problem. The two-body interactions used in the present work include both attraction and repulsion potentials. Each of the attraction and repulsion potentials consists of spin-dependent central forces together with tensor forces. The spin-orbit terms are small and are taken in such a way that they fit the corresponding phase shifts. For the nucleon-α interaction, we used a suitable separable form, which was introduced by Mitra *et al.*[4] and applied by Osman[5-7] in extracting the binding energies of light nuclei. For the α-α interactions, we use separable potentials, which are obtained by fitting the α-α scattering length and effective range. In the present work, we use five different forms for the different parts of the two-body interactions. The different forms for the potential functions are the Yamaguchi,[8] the Gaussian, the Tabakin,[9,10] the Mongan,[11,12] and the Reid[13] potentials. Furthermore, the Coulomb forces in the cases of proton-α and α-α interactions are taken into account.[14-16] The effects of both the Coulomb forces and the tensor forces should be included. We follow, in the present work, the Faddeev and Lovelace formulations. Direct numerical calculations are performed for the resulting integral equations. The validity of introducing a separable approximation in the Faddeev equation is studied. Also, by using separable two-body interactions in the resulting integral equations, the binding energies of the finite nuclei are calculated. The ability of the

separable potentials to reproduce the binding energy values of finite nuclei is tested. Comparing our results obtained for the binding energies with the previous calculations, we investigate the accuracy of using separable potentials in the three-body calculations.

In Sec. II, we introduce the Faddeev equations together with the separable approximation. Numerical calculations and results are given in Sec. III. Section IV is left for discussion and conclusions.

II. FADDEEV EQUATIONS AND SEPARABLE APPROXIMATION

Faddeev[1,2] has successfully represented in a matrix form the three-particle system by his equations

$$\begin{pmatrix} T^{(1)}_{(Z)} \\ T^{(2)}_{(Z)} \\ T^{(3)}_{(Z)} \end{pmatrix} = \begin{pmatrix} T_{23}(Z) \\ T_{31}(Z) \\ T_{12}(Z) \end{pmatrix} - \begin{pmatrix} 0 & T_{23}(Z) & T_{23}(Z) \\ T_{31}(Z) & 0 & T_{31}(Z) \\ T_{12}(Z) & T_{12}(Z) & 0 \end{pmatrix} G_0(Z) \begin{pmatrix} T^{(1)}(Z) \\ T^{(2)}(Z) \\ T^{(3)}(Z) \end{pmatrix}, \tag{1}$$

where

$$G_0(Z) = (H_0 - Z)^{-1}, \tag{2}$$

$$T_{23}(Z) = V_{23} - V_{23} G_0(Z) T_{23}(Z), \tag{3}$$

and the total T matrix $T(Z)$ is given by

$$T(Z) = T^{(1)}(Z) + T^{(2)}(Z) + T^{(3)}(Z), \tag{4}$$

with

$$T^{(1)}(Z) = V_{23} - V_{23} G_0(Z) T(Z), \tag{5}$$

and where V_{ij} is the two-body interaction between the particles i and j. Also, the Faddeev equations can be interpreted diagrammatically.[5]

Since the Faddeev equations do not include the potentials, then the solutions we are interested in are the off-energy-shell. In the vicinity of the bound-state pole, the T matrix factorizes in the initial and final momenta and it becomes just the same as that calculated from a separable potential. Thus if the separable potentials are chosen such that they give the correct two-particle wave functions, then the separable potentials correctly give the three-body system properties.

Let us start by using the separable two-body potential introduced by Yamaguchi.[8] This separable potential could be represented in a form

$$V_{ij} = \lambda f(p) f(q). \tag{6}$$

The potential V_{ij} should be strong enough to give a bound state. The potential strength λ and the form factor $f(p)$ are chosen such that the corresponding Schrödinger equation gives a bound state at energy eigenvalue $-\epsilon$ and a bound-state eigenfunction $|\Phi\rangle$.

Then the off-shell T matrix can easily be written in a closed form as

$$\hat{T}_{ij}(p, q; Z) = \frac{f(p) f(q)}{D(Z)}, \tag{7}$$

where

$$f(p) = \langle p | V_{ij} | \Phi \rangle = -(\epsilon + p^2) \langle p | \Phi \rangle \tag{8}$$

and

$$D(Z) = \frac{1}{\lambda} + 4\pi \int_0^\infty dk \frac{k^2 |f(k)|^2}{Z - k^2}. \tag{9}$$

At a point for which the energy $Z = -\epsilon$, satisfying

$$\frac{1}{\lambda} = 4\pi \int_0^\infty dk \frac{k^2 |f(k)|^2}{\epsilon + k^2}, \tag{10}$$

the separable potential will have a bound state.

The T matrices given by Eqs. (1)–(5) could be expressed by T_{ij}, where i and j are taken as 1, 2, and 3 in cyclic permutation. Then using the same notations as introduced by Lovelace we can write for the T matrix

$$\langle \vec{p}_k, \vec{q}_k | T_{ij}(Z) | \vec{p}'_k, \vec{q}'_k \rangle = \delta(\vec{q}_k - \vec{q}'_k) \langle \vec{p}_k | \hat{T}_{ij}(Z - q_k^2) | \vec{p}'_k \rangle. \tag{11}$$

From this equation, we see that if $\hat{T}_{ij}(Z)$ has a bound-state pole at $Z = -\epsilon_k$, the $T_{ij}(Z)$ will have a branch point there, with a cut going from $-\epsilon_k$ to $+\infty$. This is the right-hand cut. We follow the Faddeev formalism in the present case because the square of the kernel in the Faddeev equations is compact. For Z on the right-hand cuts, Faddeev[2] has shown that the fifth power of the kernel is compact.

The two-body interactions used in the present work contain both attraction and repulsion. The interactions are represented as a short-range repulsive potential surrounded by a long-range attractive potential. The two-body interactions used in the present work have potential functions of the Yamaguchi, Gaussian, Tabakin, Mongan, and Reid forms. The explicit forms of these interactions are represented in Refs. 8–13. These representations for the two-body interactions have been used

by us in some calculations for light nuclei, which are introduced with the different parameters in Refs. 5–7. The two-body interactions used consist of both repulsive and attractive forces. Each part of these forces is a separable potential. The T matrix for a separable two-body interaction with repulsive potential is treated by us[17] using a separable approximation.

Thus, using separable two-body interactions, we get the separable expansion for the T matrix. Introducing these separable approximations into the Faddeev equations, we get coupled integral equations in one continuous variable. The resulting integral equations from the Faddeev equations are given in detail by us.[6] Keeping the sum in the separable expansion as a finite number of terms and owing to the fact that the wave functions are square integrable, these equations are ordinary Fredholm equations. Once the equation is of the ordinary Fredholm type, it can be solved by numerical methods. This is done when one can find the energy Z for which the eigenvalue of the kernel is unity. This means that the homogeneous equation possesses a solution. This value of the energy would then correspond to a bound state of the three-body system.

III. NUMERICAL CALCULATIONS AND RESULTS

The three-body binding energies are obtained by a numerical solution of the resulting three-body integral equations. In the present work, we use separable two-body interactions with separable approximation for the T matrices. We use two-body interactions with potential functions of the Yamaguchi, Gaussian, Tabakin, Mongan, and also of the Reid forms. The different values of the parameters of the different two-body interactions are taken to fit the corresponding phase shifts. The values of the parameters obtained for the different two-body interactions, which fit the corresponding phase shifts, are given in Refs. 5–7, 15, and 16. The phase shifts to which they were fitted are taken from Refs. 18–20. Each of the above-mentioned two-body interactions has two terms, which stand for attractive and repulsive potentials, each of which also includes tensor forces. With the above-mentioned potentials, the three-body ground-state energies are calculated with very high accuracy, using only a very limited number of summation terms. The well-behaved Schmidt-Hilbert[21] theory of integral equations is applied in the present calculations due to the features of the potential used. A 36-point Gaussian integration is used in the present numerical calculations of the three-body integral equations. To convert the integral equation into a matrix eigenvalue equation, appropriate weights and abscissas are chosen for the rapid convergence at infinity. The eigenvalues are given as a function of the energy Z. The values of the energy Z for which a matrix eigenvalue takes the value one are the three-body bound-state energies.

The Faddeev-Lovelace formalism is used in the present work for calculating the binding energies of the nuclei ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, and ^{12}C. The structure of these nuclei is described using the α-cluster model. The ^{3}H nucleus is taken as a bound state of a proton and two neutrons. The ^{3}He nucleus is composed of two protons and a neutron. The ^{6}Li nucleus is considered as composed of an α particle, a proton, and a neutron. The ^{9}Be nucleus is taken as composed of two α particles and a neutron. The ^{12}C nucleus is considered as composed of three bound α particles. The above-mentioned method is used in numerical calculations for the nuclear binding energies for these nuclei. In these calculations, we get only the nuclear ground-state energies. Coulomb energies resulting from Coulomb repulsion should be added to the calculated nuclear energies. The actual three-body ground-state energies are obtained by adding the corrections due to the Coulomb energies to our theoretically calculated values. The results obtained for the three-body binding energies for the nuclei ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, and ^{12}C, using the different two-body interactions are listed in Table I. The experimental values[22–24] are introduced in Table I for the purpose of comparison with the theoretically calculated values. From Table I we see that our theoretically calculated values for the ground-state energies for the ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, and ^{12}C nuclei are quite reasonable and in good agreement with the experimentally observed values.

IV. DISCUSSION AND CONCLUSIONS

The separable potentials used in the present calculations contain both attraction and repulsion. With these separable potentials, the separable expansion obtained for the T matrix converges rapidly. This property is a general characteristic of separable potentials. The inclusion of the attraction and repulsion in the separable two-body interactions reduces the number of separable terms needed to reproduce the two-body data, and this in turn simplifies considerably the three-body calculations.

The α-cluster model is suggested here for the nuclei ^{6}Li, ^{9}Be, and ^{12}C. According to this model the α particles are taken as rigid entities.[25] The size effect of the α particle is already included with the potentials considered here, since the re-

TABLE I. Calculated binding energies in MeV.

Nucleus	Yamaguchi	Gaussian	Tabakin	Mongan	Reid	Experimental
^{3}H	8.615	8.189	8.563	8.446	8.512	8.48
^{3}He	8.194	7.618	8.089	7.816	7.972	7.72
^{6}Li	5.614	4.989	4.824	4.587	4.683	4.53
^{9}Be	1.798	1.419	1.674	1.583	1.625	1.57
^{12}C	7.416	6.329	7.388	7.299	7.316	7.28

pulsive parts of the potentials originate from the exclusion principle operating between two composite particles. We have solved numerically the integral equations resulting from the Faddeev equations using the separable two-body interactions of potential functions of the Yamaguchi, Gaussian, Tabakin, Mongan, and Reid forms. In the case of nonseparable potentials for the Reid form, the N-N interaction is taken just of the Reid form. It is taken as two terms, one attractive and one repulsive.

In the present calculations we are concerned with the three-body nuclear binding energies of the nuclei ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, and ^{12}C. The Coulomb energies are roughly estimated and their approximate values are added to the nuclear values obtained, expressing the ground-state energies which are introduced in Table I and are not far from the observed experimental values. However, the Coulomb forces could be treated accurately by treating the pure Coulomb T matrix in the integral equation. Explicit treatment of the pure Coulomb T matrix is found[14] to improve more accurate values for the binding energies.

One of the more important physical observables of the nuclei, i.e., the binding energies, is deduced in the present calculations from the theory which employs the two-body data only. Thus the three-body model structure presented here provides a good description of the composition of the nuclei ^{3}H, ^{3}He, ^{6}Li, ^{9}Be, and ^{12}C. Thus the present calculations lead to a consistent picture of these nuclei. The Faddeev equations in the present calculations are solved with two-body T matrix. Thus the method of the present calculations is also one of the useful methods for solving the three-body problem. Table I shows that our theoretically calculated values for the binding energies are in good agreement with the experimentally observed values.

Thus from the agreement between the present theoretical calculations for the binding energies and the experimental values we can draw some interesting conclusions. The two-body T matrix in the three-body Faddeev equations plays the role which a potential has in the two-body Lippmann-Schwinger equation. In the present calculations the kernel of the three-particle equations resulting from the Faddeev equations depends upon the T matrix and not on the potentials. Thus choosing separable potentials which correctly give the two-particle data, we could obtain the properties of the three-particle system using separable potentials. As long as each of the two-particle subsystems is dominated by a limited number of bound states, then the Faddeev kernel will surely converge to the correct three-particle amplitude as the number of separable terms is increased.

Previous authors[26-32] have verified the accuracy of separable approximations in reproducing the properties of nonseparable (e.g., local) interactions, which is also true of the potentials in the present work. This paper has also demonstrated that the empirical binding energies of several light nuclei can be satisfactorily predicted by using separable potentials in the Faddeev equations for three-body models of these nuclei.

I am very grateful to Professor Abdus Salam and Professor Paolo Budini as well as the International Atomic Energy Agency and UNESCO for hospitality at the International Centre For Theoretical Physics, Trieste, Italy. I am deeply indebted to Professor Luciano Fonda for his kind hospitality at the Centre ICTP, Trieste, Italy. Thanks are also due the Centro di Calcolo dell'-Università di Trieste for the use of the facilities.

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