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Impulse approximation in the peripheral region

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We show that the peripheral part of the elastic scattering impulse approximation can be expressed entirely in terms of on-shell information. The impulse approximation is given in terms of an integral over the on-shell elementary amplitude. We find the range of energies required in that amplitude for the peripheral part and discuss the closure approximation to this integral.

[NUCLEAR REACTIONS Scattering theory, low density limit of the impulse approximation. Application to π-nucleus scattering and K-mesonic atoms.]

I. INTRODUCTION

The impulse approximation is the keystone of most treatments of meson-nucleus and nucleon-nucleus scattering. It is the basic building block of all multiple scattering formalisms either directly or through some optical model generated from it. Its use ranges from giving a basis for approximations suitable to very high energy scattering to giving the hadronic part of the mesonic-atom potential. In spite of this wide and often remarkably successful application, the impulse approximation retains many questions and ambiguities. In this paper we show that the peripheral part of the elastic scattering impulse approximation is entirely free from these ambiguities.

The basic difficulty of the impulse approximation applied to elastic hadron-nucleus scattering is that of the relation between free particle scattering, presumed known, and the scattering from bound particles. This transition introduces kinematic and dynamic ambiguities related to such problems as choice of off-shell scattering energy, binding and Fermi motion corrections, choice of relative coordinate, and of other problems of frame transformation. There is a very extensive literature dealing with these issues, much of which is phenomenological rather than being devoted to clarifying fundamental ambiguities. (Most of these general problems of the impulse approximation have been discussed within the context of π-nucleus physics, cf. Ref. 1.) We show here that the peripheral contributions to elastic scattering in the impulse approximation can be expressed entirely in terms of *on-shell* quantities. This removes all the "classical" problems of the impulse approximation. We show that for the leading long range contribution, the scattering energy in that on-shell amplitude is determined and hence we have an explicit and unambiguous expression for the "binding correction." We show how that energy varies as we go to somewhat shorter ranges and determine the region of energies that makes this variation important. At still shorter ranges the effects of model dependent binding corrections (density dependent effects) and explicit off-shell effects in the elementary scattering amplitude become important, although the later corrections decrease in importance with increasing energy. These model dependent corrections set in at about the same range as contributions to the elastic scattering from other more complex processes that are far more difficult to describe. Fortunately, in many phenomenological applications the small impact parameters are characterized by nearly "black disk" scattering and therefore our dynamical ignorance is effectively obscured.

We take a pedagogic approach to our result, in view of the wide interest in the impulse approximation and the wide recognition of its problems. We derive them twice, once in r space, the usual framework for discussions of the impulse approximation, and once in k space, where the standard methods of singularity analysis (cf. Ref. 2) can be used. Our results are intended primarily to provide insight into the nature of the impulse approximation and the range of its validity. Since they form a set of rigorous statements about the peripheral part of hadron-nucleus scattering, they can be used as a constraint on any approximation scheme used for that scattering. Furthermore, since our results divide the parameter space into a peripheral part where we know the amplitude exactly (in terms of on-shell quantities without making the weak binding approximation) and a closer in part, they provide a starting point for phase shift analysis, just as the one pion exchange amplitude provides such a basis in nucleon-nucleon scattering by fixing the high partial waves. This unambiguous knowledge of the high partial waves in the hadron-nucleus problem coupled with the fact that the low waves are nearly totally absorbed may well provide a particularly strong base for phenomenology. In evaluating the hadronic correction in mesonic atoms, unambiguous knowledge of the high partial waves is again useful. Finally, our division of the impulse approximation into an unambiguous long range part and a short range part where the model dependence mixes with other dynamical complications provides insight into the limitations of the impulse description.

In Sec. II we use k-space methods to show that the longest range part of the impulse approximation can be expressed in terms of on-shell information and to derive the interval of pair scattering energies that contribute to this longest range part. We discuss the range of model dependent corrections, but the derivations of these based on singularity analysis is given in Appendix A. In Sec. III we use r-space methods and a partial wave decomposition to show that the leading contribution to the highest partial wave comes from purely on-shell quantities and we derive the pair scattering energy that makes the leading contribution in the closure approximation. Corrections to this energy (still for on-shell quantities) are discussed, but their deriviation is given in Appendix B. Each of Secs. II and III is selfcontained. Readers are invited to enter the argument through the route more familiar to them, but we believe that the two sections taken together give somewhat complementary insight to the issues. Finally, in Sec. IV we give a brief review of our results and conclusions and discuss the application of these results to particular hadron-nucleus systems.

II. k SPACE AND THE PERIPHERAL PART

In this section we use the methods of analytic function theory to show that the longer range part of the impulse approximation for elastic scattering from a complex target can be expressed entirely

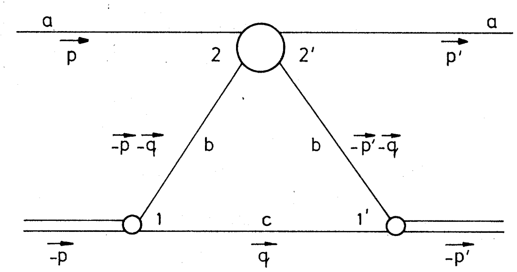

FIG. 1. Graph for the elastic scattering impulse approximation for particle a from $a(bc)$ bound state.

in terms of on-shell quantities and we discuss the domain of validity (in momentum transfer) of that on-shell expression. We express the impulse approximation as an energy integral over the on-shell elementary amplitude and we obtain an explicit expression for the energy interval that contributes to the longest range part in that integral.

The fact that the peripheral part of the impulse approximation is expressible in terms of on-shell information is generally believed, particularly in relativistic S-matrix theory. In nonrelativistic treatments it has been demonstrated explicitly for nucleon-deuteron scattering by Fuda,[3] but we know of no general treatment. Our approach in this section will follow closely the spirit of Fuda.

Consider a general impulse graph (Fig. 1) for the scattering of particle a from a bound state of b and c. We work in the center of mass where the incident momentum is $\vec{p}$ and the scattered momentum $\vec{p}\,'$ ($|\vec{p}|=|\vec{p}\,'|$). The total energy E is given by ($\hbar=1$)

$$E=\frac{p^2}{2m_a}+\frac{p^2}{2(m_b+m_c)}-\beta^2\,, \tag{2.1}$$

where β^2 is the binding energy of the b-c system. The contribution from the graph of Fig. 1 can be written

$$I=\int\frac{v(k_1^{\,2})(k_2^{\,2}|t(\epsilon)|k_2'^{\,2})v(k_1'^{\,2})d^3q}{[E-p^2/2m_a-(\vec{p}+\vec{q})^2/2m_b-q^2/2m_c][E-p'^2/2m_a-(\vec{p}\,'+\vec{q})^2/2m_b-q^2/2m_c]}\,, \tag{2.2}$$

where we have assumed that the b-c bound state is an s wave and that the a-b scattering proceeds purely in s waves. (Higher waves will be considered later.) The $\vec{k}_i$ are the appropriate invariant momenta (essentially relative velocities) for the vertices labeled 1, 1′, 2, and 2′ on the graph. Rather than express I in terms of the bound state wave function, we have used the equivalent representation in terms of a vertex function v and an energy denominator. The only assumption involved in doing this is that the bound state wave function has an exponential tail in r space. The energy argument ϵ of the totally off-shell t matrix in (2.2) is the energy available to the a-b pair in their center of mass and is given by

$$\epsilon=E-\frac{q^2}{2}\left(\frac{1}{m_c}+\frac{1}{m_a+m_b}\right)\,. \tag{2.3}$$

Using simple kinematics and energy conservation (2.1), and the definition of ϵ (2.3), the first energy denominator in (2.2) can be rewritten in two equivalent forms

$$E-\frac{p^2}{2m_a}-\frac{(\vec{p}+\vec{q})^2}{2m_b}-\frac{q^2}{2m_c}=-\beta^2-\frac{m_c m_b}{2(m_c+m_b)}\left(\frac{\vec{p}+\vec{q}}{m_b}+\frac{\vec{q}}{m_c}\right)^2$$

$$=-\beta^2-k_1{}^2 \tag{2.4a}$$

$$=E-\frac{q^2}{2}\left(\frac{1}{m_c}+\frac{1}{m_a+m_b}\right)-\frac{m_a m_b}{2(m_a+m_b)}\left(\frac{\vec{p}+\vec{q}}{m_b}+\frac{\vec{p}}{m_a}\right)^2$$

$$=\epsilon-k_2{}^2\,. \tag{2.4b}$$

There are corresponding expressions for the second denominator with $p\to p'$, and k_1, $k_2\to k'_1$, k'_2. The longest range part of I comes from the vanishing of the two denominators in (2.2). More precisely, I has a branch cut in the momentum transfer t defined by

$$t=-(\vec{p}\,'-\vec{p})^2\,. \tag{2.5}$$

The discontinuity across the part of this cut closest to the physical region of t, and hence contributing to the most peripheral part of I, comes from the vanishing of the denominator in (2.2). In this section we display the branch cut and show that its discontinuity is given entirely in terms of on-shell quantities. In Appendix A we show that the remaining momentum transfer singularities of I coming from the numerator of (2.2) correspond to shorter ranges. It is clear that higher order terms in the multiple scattering series do as well.

To find the contribution to I due to singularities associated with the denominator in (2.2) only, we can evaluate the numerator at the value of k_i and k_i' corresponding to the zero of the denominator. This is straightforward using the kinematic relations of (2.4), and we obtain

$$I_L=\int\frac{d^3q\,v^2(-\beta^2)\langle\epsilon\,|t(\epsilon)|\,\epsilon\rangle}{\{E-p^2/2m_a-(\vec{p}+\vec{q})^2/2m_b-q^2/2m_c\}[E-p^2/2m_a-(\vec{p}\,'+\vec{q})^2/2m_b-q^2/2m_c]}\,, \tag{2.6}$$

where the L on I is to remind us that we now have only the longest range part of I. By virtue of the kinematics in (2.4), we have now expressed I_L entirely in terms of on-shell quantities. The v's are evaluated at the binding energy where they are just the asymptotic normalization constant or coupling constants of the bound state, while the elementary scattering operator t has been put completely on shell. There is still an integral over q which via (2.3) corresponds to variation of ϵ. In fact, ϵ can become negative, but we shall see that the left-hand cuts of t are never reached.

We now display the momentum transfer cut of I_L and calculate the values of q (and therefore ϵ) that give the nearest part of that cut.

We write

$$I_L=v^2(-\beta^2)\int q^2dq\langle\epsilon\,|t(\epsilon)|\,\epsilon\rangle\frac{m_b{}^2}{p^2q^2}\,I'\,, \tag{2.7}$$

where

$$\frac{m_b{}^2}{p^2q^2}I'=\int\frac{d\Omega q}{[E-\frac{1}{2}p^2(1/m_a+1/m_b)-\frac{1}{2}q^2(1/m_b+1/m_c)-\vec{p}\cdot\vec{q}/m_b]} \tag{2.8a}$$

$$\times\frac{1}{[E-\frac{1}{2}p^2(1/m_a+1/m_b)-\frac{1}{2}q^2(1/m_b+1/m_c)-\vec{p}\,'\cdot\vec{q}/m_b]}$$

$$=\frac{m_b{}^2}{p^2q^2}\int\frac{d\Omega q}{(\lambda-\hat{p}\cdot\hat{q})(\lambda-\hat{p}'\cdot\hat{q})}\,. \tag{2.8b}$$

Here $\hat{k}$ is a unit vector $\hat{k}=\vec{k}/|k|$, we have used $|p|=|p'|$, and defined

$$\lambda=\frac{[E-\frac{1}{2}p^2(1/m_a+1/m_b)-\frac{1}{2}q^2(1/m_b+1/m_c)]m_b}{\vec{p}\cdot\vec{q}}\,. \tag{2.9}$$

It is easy to show that $|\lambda|\geq 1$ and, therefore, the denominators of (2.8b) never vanish in the physical region. I' can be evaluated explicitly to yield

$$I'=\frac{4\pi}{(\hat{p}\cdot\hat{p}'-1)}\frac{1}{(1+\beta)^{1/2}}\ln\left(\frac{(1+\beta)^{1/2}-1)}{(1+\beta)^{1/2}+1)}\right)\,, \tag{2.10}$$

where $\beta=-4(\lambda^2-1)p^2/t$ in terms of the λ of (2.9) and the momentum transfer t (2.5). From (2.10) we see that I' has a branch cut from $\beta=-1$ to $\beta=0$, corresponding to a cut in t from $4(\lambda^2-1)p^2$ to ∞. The physical region for t runs from $t=0$

(forward scattering) to $t=-4p^2$. Hence, the closest point of the t cut of I' corresponds to the minimum value (with respect to q) of $4(\lambda^2-1)p^2$. This is $t=(8m_b/m_c)(m_b+m_c)\beta^2$. There is a corresponding value of ϵ at the nearest point given by

$$\epsilon = E - \frac{p^2}{2}\frac{(m_a+m_b+m_c)m_c}{(m_a+m_b)(m_b+m_c)^2} - \beta^2 \frac{(m_a+m_b+m_c)m_b}{(m_a+m_b)(m_b+m_c)}. \tag{2.11}$$

Alternatively, one can express I' as a dispersion integral in t. One obtains, after some algebra,

$$I' = 8\pi p^2 \int_\gamma^\infty \frac{dt'}{[t'(t'-\gamma)]^{1/2}}\frac{1}{t'-t}, \tag{2.12}$$

where

$$\gamma = 4p^2(\lambda^2-1). \tag{2.13}$$

This demonstrates explicitly the existence of the branch cut in t and the fact that the nearest point corresponds to the minimum value of γ. From (2.9) and (2.13) we see that γ varies with q, as shown schematically in Fig. 2. In general, there are contributions to the t cut from higher order processes, off-sehll effects, vertex forms, etc., that correspond to shorter range. They begin at some $t=t_0$. To be consistent, we should only take the contribution from I' corresponding to that part of the t cut running from the minimum value of γ, $\gamma_{\min}=(8m_b/m_c)(m_b+m_c)\beta^2$ to that t_0. This corresponds to limiting the values of q to lie between a_1 and a_2 as shown in Fig. 2. We then obtain for the purely long range part of I_L, called I_{LL},

$$I_{LL} = v^2(-\beta^2)\int_{a_1}^{a_2} 8\pi m_b^2 dq\, t(\epsilon) \times \int_\gamma^{t_0} \frac{dt}{(t'-t)[t'(t'-\gamma)]^{1/2}}. \tag{2.14}$$

From Fig. 2 we see that the variation of γ with q is stationary at $\gamma_{\min}$. Coupling this with the fact that $\gamma_{\min}$ gives the longest range part suggests that we can expect the contributions near $\gamma_{\min}$ to dominate, particularly if the elementary t matrix is slowly varying. The corresponding elementary scattering energy ϵ is given in (2.11). Evaluating the t matrix at this energy corresponds to a "closure" approximation and the value of ϵ in (2.11) corresponds to a specific prescription for the binding correction. If the t matrix is rapidly varying, we must carry out the integral in (2.14), and we see that it is energies ϵ below those of (2.11) that will contribute. Clearly, the a_1 and a_2 are to be chosen in (2.14) so that the integral does not take ϵ into the left-hand cut of the amplitude. It is, of course, possible to encounter bound state

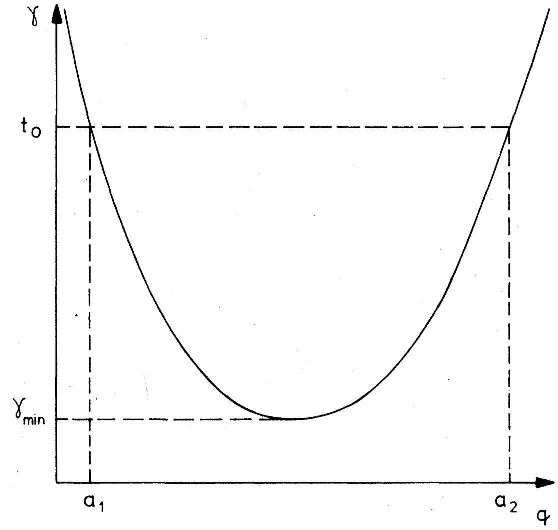

FIG. 2. Schematic representation of the dependence of γ[Eq. (2.13)] vs q.

poles in the two body t matrix, but these are integrable in the usual way.

Let us now consider the problem of higher partial waves in the two body t matrix. We begin with p waves. The off-shell t matrix appearing in I must now be

$$\langle \vec{k}_2 t(\epsilon)|\vec{k}_2'\rangle = \vec{k}_2\cdot\vec{k}_2' f(k_2^2, k_2'^2, \epsilon). \tag{2.15}$$

The principle question is what to do with the factor of $\vec{k}_2\cdot\vec{k}_2'$. On-shell, this factor is $\epsilon\cos\theta$. Hence we write

$$\langle \vec{k}_2|t(\epsilon)|\vec{k}_2'\rangle = \frac{\vec{k}_2\cdot\vec{k}_2'}{\epsilon}\epsilon f(k_2, k_2'^2, \epsilon). \tag{2.16}$$

When we go to the residue at the pole of the denominators in I to find the longest range part, we put $\vec{k}_2^2$ and $\vec{k}_2'^2$ equal to ϵ in f, but leave the factor $\vec{k}_2\cdot\vec{k}_2'/\epsilon$. This is purely a polynomial in $\vec{q}$, $\vec{p}$, and $\vec{p}'$, and will not affect the momentum transfer cut structure, while $\epsilon f(\epsilon,\epsilon,\epsilon)$ is the invariant on-shell p-wave t matrix.

Our prescription for $\cos\theta$ may appear to be yet another arbitrary way of dealing with the angles in this problem, but the reader may verify that it is the only prescription that preserves the analytic structure and reduces to the correct form on-shell. For us, on-shell means $k_2^2=k_2'^2=\epsilon$, and thus we can write

$$\cos\theta = \frac{\vec{k}_2\cdot\vec{k}_2'}{k_2^2} = \frac{\vec{k}_2\cdot\vec{k}_2'}{\epsilon} = 1-\frac{t}{4\mu\epsilon},$$

where the last form expresses $\cos\theta$ in terms of the momentum transfer t and an appropriate reduced mass μ, thus demonstrating explicitly the equivalence of the Kisslinger and Laplacian form in our treatment. (Note our form for $\cos\theta$ permits $\cos\theta > 1$.) For the general case, one expands the off-shell t matrix in I in partial waves and replaces $\cos\theta$ by $\vec{k}_2 \cdot \vec{k}_2'/\epsilon$ in the expansion. For the largest range part, all other k_2^2 and $k_2'^2$ terms in the t matrix should be put on-shell. Since $\cos\theta = \vec{k}_2 \cdot \vec{k}_2'/\epsilon$ is no longer the cosine of an angle, it can be bigger than one, and hence lead to a kinematic enhancement. The method is easily generalized to bound state wave function with $L \neq 0$ as we show in Sec. III. The presence of spin causes only algebraic complications. Finally, if one of the particles is relativistic (as in π-nucleus scattering), the on-shell prescriptions given here again go through and serve to remove ambiguities associated with variable choice for the π-N amplitude since all such choices must agree on-shell.

There remains only the question of further momentum transfer singularities associated with the vertex dependence or off-shell dependence of the numerator in (2.2). These quantities have left-hand cuts in k_i^2 and $k_i'^2$. Further momentum transfer cuts arise from pinches between these left-hand cuts and the denominator singularities of (2.2). Since locating the new singularities is technical, we relegate it to Appendix A and only give the results here. If the nearest point of the k_1^2 cut due to the bound state vertex comes at $k_1^2 = -\mu^2$, the corresponding momentum transfer cut begins at $(2m_b/m_c)(m_b + m_c)(\mu + \beta)^2$. In most problems, $\mu \gg \beta$, and hence this cut is at considerably shorter range than the propagator cut. The expression for the nearest point of the propagator cut associated with off-shell behavior of t is more complex and is given in Appendix A. It moves away from the physical region with increasing energy, so that at high energy, off-shell corrections become relatively less important. The location of these next cuts should be taken to fix t_0 in (2.14).

III. r SPACE AND THE PERIPHERAL PART

In this section we use the r-space representation and a partial wave decomposition of the elastic scattering impulse approximation for hadron-nucleus scattering to show that for large l the impulse amplitude is determined entirely in terms of on-shell information and to evaluate the effective scattering energy that makes the leading long range contribution. We consider the special case of an infinately heavy target nucleus. (The general mass case is dealt with in Sec. II.) The impulse approximation for elastic scattering from momentum $\vec{p}$ to $\vec{p}\,'$ can then be written (we start in k space and pass to r space later)

$$F(\vec{p}\,', \vec{p}) = \sum_{i=1}^{A} \int d^{3A}K d^{3A}K' \psi^*(\vec{K}_1', \ldots, \vec{K}_A') \times \delta_i^{3(A-1)}(\vec{K}', \vec{K}) \times \langle \vec{\kappa}', \vec{K}_i' + \vec{p}\,' | \tau_i(E) | \vec{\kappa}, \vec{K}_i + p \rangle \times \psi(\vec{K}_1, \ldots, \vec{K}_A), \tag{3.1}$$

where ψ is the nuclear ground state wave function and $\delta_i^{3(A-1)}$ is the product of δ functions in all but the ith coordinate. $\vec{\kappa}^{(\prime)}$ are the initial (final) projectile nucleon relative momenta, which in terms of the lab momenta $\vec{p}^{(\prime)}, \vec{K}^{(\prime)}$ of the projectile (mass m) and nucleon (mass M), respectively, are given by

$$\vec{\kappa}^{(\prime)} = \frac{M\vec{p}^{(\prime)} - m\vec{K}^{(\prime)}}{m + M}, \tag{3.2a}$$

where E is the kinetic energy of the projectile

$$E = p^2/2m. \tag{3.2b}$$

(For convenience, we have not included here the binding energy in the definition of E as is done in Sec. II) τ_i is the operator describing the scattering on the bound nucleon i. τ_i satisfies the following Lippmann-Schwinger equation

$$\tau_i(E) = v_i + v_i \frac{1}{E - H_A - T_p} \tau_i(E), \tag{3.3}$$

with the projectile nucleon potential v_i, the nuclear Hamiltonian

$$H_A = \sum_{l=1}^{A} T_l + \tfrac{1}{2} \sum_{l,k} V(l,k) = H_{A-1}^{(i)} + T_i + \sum_{k \neq i} V(i,k), \tag{3.4}$$

and the projectile kinetic energy operator T_p.

For peripheral scattering, only the low-density region of the nucleus is important and therefore the interaction of the struck nucleon with the other target nucleons [$\sum_{k \neq i} V(i,k)$ in Eq. (3.4)] contributes to higher order in the density and can be neglected. With this approximation, τ_i the bound t matrix can be related in a simple way to the free projectile nucleon t matrix, which is defined by

$$t(\epsilon) = v + v \frac{1}{\epsilon - T_r} t(\epsilon), \tag{3.5}$$

(T_r being the kinetic energy operator for the relative motion).

With

$$T_p + H_A \approx T_p + H_{A-1}^{(i)} + T_i = H_{A-1}^{(i)} + T_r + T_c, \tag{3.6}$$

where T_c is the kinetic energy operator for the projectile struck nucleon c.m. motion, we have

$$\tau_i(E) = t(E - H_{A-1}^{(i)} - T_c). \tag{3.7}$$

In order to simplify the calculation, we assume the projectile nucleon interaction to be described by an s-wave separable t matrix

$$\langle \vec{\kappa}' | t(\epsilon) | \vec{\kappa} \rangle = \frac{v(\kappa'^2) v(\kappa^2)}{D(\epsilon)}, \tag{3.8}$$

and assume for the nucleus a shell model description with single particle wave functions ψ_i and binding energies ϵ_i, which are related to the damping factors β_i of ψ_i in the asymptotic region as

$$\epsilon_i = \frac{\beta_i^2}{2M}. \tag{3.9}$$

From now on, i numbers the shell model orbits, and not the nucleons. With these assumptions, Eq. (3.1) reads

$$F(\vec{p}', \vec{p}) = \sum_{i=1}^{A} \int d^3K' d^3K \, \psi_i^\dagger(\vec{K}') v(\kappa'^2) \times \langle \vec{K}' + \vec{p}' | D^{-1}(E - \epsilon_i - T_c) | \vec{K} + \vec{p} \rangle \times \psi_i(\vec{K}) v(\kappa^2). \tag{3.10}$$

Transforming to coordinate space, Eq. (3.10) can be written as

$$F(\vec{p}', \vec{p}) = \sum_{i=1}^{A} \int d^3R' d^3R \, \phi_i^\dagger(\vec{R}') \phi_i(\vec{R}) \times \left\langle \vec{R}' \left| D^{-1}\left(E - \epsilon_i + \frac{\Delta}{2(M+m)}\right) \right| \vec{R} \right\rangle, \tag{3.11}$$

with the wave function $\phi_i(\vec{R})$ for the c.m. motion

$$\phi_i(\vec{p}, \vec{R}) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3K \, v\left[\left(\frac{M\vec{p} - m\vec{K}}{M+m}\right)^2\right] \times e^{-i(\vec{K}+\vec{p})R} \psi_i(\vec{K}) \tag{3.12}$$

$$= \left(\frac{1}{2\pi}\right)^{3/2} v\left[\left(\frac{M\vec{\nabla}_\rho + m\vec{\nabla}_R}{i(M+m)}\right)^2\right] \psi_i(R) \times e^{-i\vec{p}\cdot\vec{\rho}} |_{\vec{\rho}=\vec{R}}. \tag{3.13}$$

For the asymptotic region of the bound state wave function we have

$$\Delta\psi_i(R) \approx \beta_i^2 \psi_i(R), \tag{3.14a}$$

and, therefore, in the peripheral region, $\phi_i(R)$ can be written as

$$\phi_i(R) \approx \left(\frac{1}{2\pi}\right)^{3/2} v\left(\frac{M}{M+m} p^2 - \frac{M}{M+m} \beta_i^2 + \frac{mM}{(M+m)^2} (\vec{\nabla}_R + \vec{\nabla}_\rho)^2\right) \psi_i(R) e^{-i\vec{p}\vec{\rho}} |_{\vec{\rho}=\vec{R}}$$

$$= \left(\frac{1}{2\pi}\right)^{3/2} v\left[2\mu\left(E - \epsilon_i + \frac{\Delta}{2(M+m)}\right)\right] \psi_i(\vec{R}) e^{-i\vec{p}\vec{R}}, \tag{3.14b}$$

where we have used the reduced mass

$$\mu = \frac{Mm}{M+m},$$

We note that the same (energy) dependence appears in both the vertex functions and the D function in (3.10), which means that in the peripheral limit we are considering, the projectile nucleus amplitude is determined by the on-shell projectile-nucleon amplitude.

From (3.11) and (3.14) we obtain the partial wave amplitudes

$$F_l(p) = \int d\Omega_p d\Omega_{p'} \, Y_{lm}(\hat{p}') Y_{lm}^*(\hat{p}) F(\vec{p}', \vec{p})$$
$$= (4\pi) \sum_{i=1}^{A} \int d^3R d^3R' \, \phi_l^{m*}(\vec{R}') \left\langle \vec{R}' \left| D^{-1} \; E - \epsilon_i + \frac{\Delta}{2(M+m)} \right| \vec{R} \right\rangle \phi_l^m(\vec{R}), \tag{3.15}$$

with

$$\phi_l^m(p, R) = \left(\frac{1}{2\pi}\right)^{3/2} v\left[2\mu\left(E - \epsilon_i + \frac{\Delta}{2(M+m)}\right)\right] \times \psi_i(R) j_l(pR) Y_{lm}(\hat{R}).$$

Having established the result that peripheral scattering is determined by the on-shell amplitude, we go further to find the effective scattering energy which makes the leading contribution in the peripheral region. This will bring our calculation in contact with the usual closure or weak binding approximation with an explicit form for the binding corrections. We will also establish a criterion for the validity of this type of closure approximation. It should be noted that this approximation is in no way necessary for our treatment of the peripheral partial waves in terms of on-shell information only.

In the following, we focus our discussion on the energy dependence of the elementary amplitude $[D(\epsilon)]$ and neglect the induced energy dependence via the vertex functions in (3.15), which is particularly justified for projectile-nucleus scattering in the region of a resonance or close to threshold. The method is easily generalized to include the induced energy variations in the vertex functions as well.

The basic idea for evaluating the integral is very simple. The states $|\phi_i^m\rangle$ describing the projectile-nucleon c.m. motion, are not eigenstates of the c.m. kinetic energy operator. However, for a not too rapid energy dependence of D^{-1}, we can approximate the corresponding matrix elements for nucleons in a given shell $i=n_i, L_i, M_i$ as follows:

$$\sum_{M_i}\langle\phi_i^m|D^{-1}(E-\epsilon_i-T_c)|\phi_i^m\rangle = \sum_{M_i}\langle\phi_i^m|\phi_i^m\rangle D^{-1}(E-\epsilon_i-\langle T_c\rangle_i)\,. \quad (3.16)$$

The expectation value of the c.m. kinetic energy operator is determined so that the deviations in (3.16) are of the order of $(T_c-T_c\rangle)^2$ or

$$\langle T_c\rangle_i=\frac{\sum\langle\phi_i^m|T_c|\phi_i^m\rangle}{\sum_{M_i}\langle\phi_i^m|\phi_i^m\rangle}\,. \quad (3.17)$$

This approximation is a kind of doorway-state expansion where the correction terms to the first-order approximation (3.16) are determined by the fluctuations of the kinetic energy operator T_c in the doorway-state $|\phi_i^m\rangle$.

Including the lowest order fluctuations, we have

$$\sum_{M_i}\langle\phi_i^m|D^{-1}(E-\epsilon_i-T_c)|\phi_i^m\rangle \approx\sum_{M_i}\langle\phi_i^m|\phi_i^m\rangle(D^{-1}(\bar{E}_i) +\frac{1}{2}\frac{\partial^2D^{-1}}{\partial E^2}\Big|_{\bar{E}_i}(\langle T_c^2\rangle-\langle T_c\rangle^2)\,, \quad (3.18a)$$

with

$$\bar{E}_i=E-\epsilon_i-\langle T_c\rangle_i \quad (3.18b)$$

and

$$\langle T_c^2\rangle_i=\frac{\sum_{M_i}\langle\phi_i^m|T_c^2|\phi_i^m\rangle}{\sum_{M_i}\langle\phi_i^m|\phi_i^m\rangle}\,. \quad (3.18c)$$

In this approximation, the following expression for the partial wave scattering amplitude [cf. Eqs. (3.15) and (3.18)] is obtained

$$F_l(k)=\sum_i t(E-\epsilon_i-\langle T_c\rangle_i)\langle\phi_i^m|\phi_i^m\rangle \times\left(1+\frac{1}{2}\frac{D^{-1}(\bar{E}_i)''}{D^{-1}(\bar{E}_i)}(\langle T_c^2\rangle_i-\langle T_c\rangle_i^2)\right)\,. \quad (3.19)$$

We now sketch briefly how to evaluate the expectation values of the T_c and T_c^2 [(3.17) and (3.18)] in the CM states $|\phi_i^m\rangle$ corresponding to peripheral scattering. We write (in the r representation)

$$\langle T_c\rangle=\frac{p^2}{2(M+m)}-\frac{\beta_i^2}{2(M+m)}-\frac{\sum_{M_i}\int_0^\infty d\tau\,\psi_i(r)Y^*_{L_iM_i}(\hat{r})j_l(pr)Y_{lm}(\hat{r})[\vec{\nabla}\psi_i(r)Y_{L_iM_i}(\hat{r})]}{\sum_{M_i}\int_0^\infty d\tau|\psi_i(r)Y_{L_iM_i}(\hat{r})j_l(pr)Y_{lm}(\hat{r})|^2(m+M)}[\vec{\nabla}j_l(pr)Y_{lm}(\hat{r})]^*\,, \quad (3.20)$$

where we have used the asymptotic relation (3.14a) for the nucleon wave functions

$$\psi_i(\vec{r})=\psi_i(r)Y_{L_iM_i}(\hat{r})\,. \quad (3.21)$$

It is easily seen that for a closed shell nucleus only the radial derivatives in the integral in (3.20) contribute, and, using the asymptotic form of the radial wave function

$$\psi_i(r)\to c\frac{e^{-\beta_i r}}{r}\,, \quad (3.22)$$

Eq. (3.20) can be rewritten in the following way

$$\langle T_c\rangle_i=\frac{p^2-\beta_i^2}{2(M+m)}-\frac{1}{(M+m)} \times\frac{\int_0^\infty r^2dr\,j_l(pr)e^{-\beta_i r}/r\,j_l'(pr)(e^{-\beta_i r}/r)'}{\int_0^\infty r^2dr\,j_l^2(pr)e^{-2\beta_i r}/r^2}\,,$$

or after integration by parts

$$\langle T_c\rangle_i=\frac{p^2+\beta_i^2}{2(M+m)}+\frac{1}{M+m} \times\frac{\int_0^\infty dr\,(\beta_i/r+1/2rp^2)e^{-2\beta_i r}j_l^2(pr)}{\int_0^\infty dr\,e^{-2\beta_i r}j_l^2(pr)}\,. \quad (3.23)$$

For large l,

$$l\gg(1+\beta_i^2/p^2)^{1/2}\,, \quad (3.24)$$

the integrals in (3.23) can be evaluated analytically. ($l\gg L_i$ has been assumed already in using the exponential wave function in Eq. (3.23) instead of the Hankel function.) The details of this calculation are given in Appendix B. The dominant contributions to the integrals comes from the region

$$r\approx R=\frac{l}{(p^2+\beta_i^2)^{1/2}}\,, \quad (3.25)$$

and, therefore, the integral term in (3.23) contributes only to higher order in the peripheral expansion ($1/l$). The final result is

$$\langle T_c\rangle_i=\frac{1}{2(M+m)}\left(p^2+\beta_i^2+\frac{2\beta_i(p^2+\beta_i^2)^{1/2}}{l}\right)\,, \quad (3.26)$$

and, therefore, the effective scattering energy, i.e., the energy available for the relative projectile-struck nucleon motion, is [cf. Eq. (3.18)]

$$\bar{E}_i = E - \epsilon_i - \langle T_c \rangle_i$$
$$= E - \frac{\beta_i^2}{2M} - \frac{p^2 + \beta_i^2}{2(M+m)} + O(1/l). \qquad (3.27)$$

The interpretation of this result is simple. In the weak binding limit, the only remaining term is $p^2/2(M+m)$, and, therefore, $\bar{E}_i$ is the c.m. energy corresponding to the lab energy E for the initial nucleon at rest. For a finite binding energy ϵ_i, the c.m. energy is further decreased on the one hand by the separation energy and on the other hand by the increase of the radial c.m. momentum from p to $|p + i\beta_i|$ because of the zero point motion of the bound nucleon. This equation agrees with Eq. (2.11) in the appropriate mass limit.

In order to discuss the validity of the closure approximation to the impulse approximation (Eq. 3.15) with the optimal choice (3.27) for the closure energy, we give the results for the fluctuation terms. The details of the calculation are presented in Appendix B and follow the general ideas outlined above.

Two cases are considered corresponding to different physical situations. For high-energy scattering, the conditions for peripheral scattering (R—nucleus radial, p_F—Fermi momentum)

$$l \gg pR \qquad (3.28a)$$

and

$$k \gg p_F \qquad (3.28b)$$

imply

$$l \gg L_i \qquad (3.28c)$$

since

$$L_i \lesssim p_F R\,.$$

In this situation, the fluctuations to lowest order in $(1/l)$ do not depend on the nucleon angular momentum. The result is

$$\langle T_c^2 \rangle_i - \langle T_c \rangle_i^2 = \left(\frac{p^2 + \beta_i^2}{2(M+m)}\right)^2 \frac{2\beta_i}{l(p^2 + \beta_i^2)^{1/2}}\,. \qquad (3.29)$$

For low energy scattering, relevant for the impulse approximation for mesonic atoms calculations, a finite angular momentum of the struck nucleon can influence appreciably both the closure energy and the fluctuations. For zero incident momentum ($p = 0$), the result is

$$\langle T_c \rangle_i = \frac{\beta_i^2}{2(M+m)}\left(\frac{(2l+1)^2}{2[l^2 - (L_i + \frac{1}{2})^2]} - 1\right) \qquad (3.30)$$

and

$$\langle T_c^2 \rangle_i - \langle T_c \rangle_i^2 = \left(\frac{\beta_i^2}{2(M+m)}\right)^2 \frac{(2l+1)^2}{[l^2 - (L_i + \frac{1}{2})^2]} \times\left(\frac{l(2l-1) + L_i(L_i+1)}{2[(l-1)^2 - (L_i + \frac{1}{2})^2]} - \frac{(2l+1)^2}{4[l^2 - (L_i + \frac{1}{2})^2]}\right). \qquad (3.31)$$

IV. DISCUSSION

We have shown that the peripheral part of the elastic scattering impulse approximation can be expressed entirely in terms of on-shell information. We have demonstrated this result both in k space by using analyticity arguments and in r space using Schrödinger dynamics and formal scattering theory. In the k-space treatment we show that the momentum transfer dependence of the elastic scattering impulse approximation is associated with a branch cut in the momentum transfer. The discontinuity across the part of that cut closest to the physical region, and, therefore, determining the peripheral part of the impulse amplitude, is given entirely in terms of the elementary scattering amplitude on-shell although possibly for unphysical values of the scattering energy. The range of momentum transfers determined by purely on-shell information is also established. The r-space derivation uses the fact that the peripheral region is a low density one in which the interaction of the struck nucleon with the target can be neglected and the corresponding bound state wave function taken in its asymptotic form. It should be noted that this is not a weak binding approximation. Many of the classical ambiguities of the impulse approximation arise because it involves an integral over an off-shell elementary scattering amplitude. Our results show that the peripheral contribution can be reformulated in such a way as to completely avoid this problem. This eliminates, for example, the uncertainties associated with the so-called angle or frame transformations in π-nucleus scattering, problems arising from the Lorentz noninvariance of the t matrix and other related problems, all of kinematic origin and all disappearing for the on-shell amplitude.

The other classic difficulty of the impulse approximation involves the choice of elementary scattering energy. Our treatment gives an unambiguous prescription for the range of energies that contribute (in the on-shell amplitude) to the peripheral part of the elastic scattering impulse approximation amplitude. The closure approximation to the integral over this range of energies corresponds to evaluating the elementary amplitude at a fixed energy value. In our formalism a natural choice for this value emerges. In the k-space language the closest point on the momentum transfer branch cut corresponds to a particular choice of elementary scattering energy, namely, the maximum in the integral range, and it is natural to choose this stationary value as fixed energy to be used in the elementary amplitude in the closure approximation. In r space we make an operator expansion valid in the peripheral region to obtain the stationary energy, which agrees with the energy obtained in the k-space treatment. The elementary scattering energy that emerges in this way (2.11) and (3.27) corresponds to a definite "binding correction" that has a particularly simple physical interpretation [see the discussion after, eg. (3.27)]. If the elementary amplitude has a rapid variation with energy, the closure approximation will, of course, be invalid. (Our on-shell treatment of the impulse approximation in no way requires the closure approximation.) Our derivation of the stationary energy value allows a quantitative study of the energy shifts involved and of the validity of the stationary energy approximation.

The binding corrections are obviously most serious if the (on-shell) elementary amplitude is varying rapidly with energy. This is the case close to threshold or in the vicinity of a bound state or resonance. We stress that the binding corrections to the impulse approximation are not necessarily negligible at high energy ($E \gg E_B$) as is often argued. Rather, the relevant quantity for the weak binding limit is

$$\left| E_B \frac{\partial t}{\partial E} \right| \ll |t(E)| .$$

As we see from Eq. (3.27), the effective scattering energy is always shifted towards lower energies by the amount

$$\delta E = \frac{2M + m}{M + m} E_B$$

from the lab energy (in the infinite target mass case). In fact, if the energy integral is explicitly evaluated, we see from (2.14) that the actual energies shifts are even larger, but always in this sense.

For nucleon-nucleus scattering at high energy, the nucleon-nucleon t matrix may indeed be sufficiently slowly varying to permit the closure approximation, but for lower energies it certainly is not.

For example, in p-d scattering there is a pole in the n-p triplet amplitude just below threshold (the deuteron) and a near pole in the singlet amplitude that must be dealt with explicitly. For the cases of π-nucleus scattering at resonance and K^--mesonic atoms, this energy shift is comparable or even larger than the width of either the Δ or the Y^*. For the case of the K^--nucleus interaction the effective energy is close to the resonance because of the energy shift. However, as the evaluation of the fluctuation term below indicates, the expansion of the integral is not valid in this case.

In order to illustrate the physical meaning of the fluctuation term, we assume a linear from for $D(\epsilon)$ (Breit-Wigner)

$$D(\epsilon) \propto \epsilon - R + i\Gamma/2 ,$$

and rewrite Eq. (3.19) in the following way (considering nucleons of one shell only).

$$F_0^l(p) \approx f_0^l(p) \frac{1}{E - R - \epsilon_i - \langle T_c \rangle_i + \frac{1}{2} i \Gamma} \left(1 + \frac{1}{(E - R - \epsilon_i - \langle T_c \rangle_i + \frac{1}{2} i \Gamma)^2} (\langle T_c^2 \rangle_i - \langle T_c \rangle_i^2) \right)$$

$$\approx f_0^l(p) \frac{1}{E - R - \epsilon_i - \langle T_c \rangle_i + \frac{1}{2} i \Gamma - (\langle T_c^2 \rangle_i - \langle T_c \rangle_i^2)/(E - R - \epsilon_i - \langle T_c \rangle_i + \frac{1}{2} i \Gamma)}$$

At resonance ($E - R - \epsilon_i - \langle T_c \rangle_i = 0$), the additional term induces a width due to fluctuation of the kinetic energy operator, i.e., due to the zero-point motion of the nucleons. This is the "Fermi-averaging" of the elementary amplitude. For the peripheral partial waves in π-nucleus scattering at resonance, the induced width is [cf. Eq. (3.29)]

$$\tfrac{1}{2} \Gamma^{FM} = \frac{\langle T_c^2 \rangle - \langle T_c \rangle^2}{\frac{1}{2} \Gamma} = (1/l) 50 \text{ (MeV)}$$

for $T_\pi = 210$ MeV and $\epsilon_i = 15$ MeV. In other words, for peripheral waves ($l > 5$), the range of energies involved in the integral is appreciably smaller than the half-width (55 MeV) of the π-N amplitude. In the peripheral partial waves, the fluctuation expansion seems to converge for this case. In K^--nucleus scattering at threshold, on the other hand, the induced width is larger than the elementary width of the $I = 0$ subthreshold amplitude ($\frac{1}{2}\Gamma$) $\approx$ (15 MeV). In this case, the closure expansion can only give an idea of the range of energies involved.

Using Eqs. (3.30) and (3.31), we obtain as typical values for the average energy shift and fluctuation of the kinetic energy operator in the kaonic $5g$ level for nuclei in the s-d shell ($L_i = 2, \epsilon_i = 8.5$ MeV),

$$\bar{E} = -25.5 \text{ MeV}$$

and

$$(\langle T_c^2 \rangle - \langle T_c \rangle^2)^{1/2} = 31.7 \text{ MeV} ,$$

i.e., we find that the average scattering energy shifted down just close to the Y^* resonance; however, as the value for the fluctuation energy indicates, there is a very wide range of energies actually contributing in the energy integral. As a consequence, the effect of the Y^* resonance will be smeared out. For an accurate evaluation of the impulse approximation, the integral (3.11) has to be calculated numerically.

The form (3.10) of the impulse approximation, which we have shown to be rigorous in the peripheral limit, has been used both in the problem of kaonic atoms[4] and π-nucleus scattering[5] for the construction of an optical potential. Although the impulse approximation in this form guarantees the correct description for peripheral scattering, it is not evident that in the calculation of the energy shifts and widths of the lower levels in the kaonic atoms or in the description of π-nucleus elastic scattering the low-density limit is realized. In Ref. 5 it was found to be necessary to vary the binding energy in order to fit π-nucleus scattering, which indicates the presence of higher order modifications. Intuitively, it is obvious that part of the downward shift [cf. Eq. (3.27)] will be canceled by binding effects of the Y^* or Δ resonance. In the form (3.10) it is assumed implicitly that the projectile nucleon resonance does not interact with the residual nucleus. This is a reasonable approximation only in the low-density or peripheral limit. Attempts to include those higher order binding correcting have been made for both the K-nucleus[6] and the π-nucleus problem.[7]

These remarks show how our results can be made to shed light on the validity of various approximation schemes normally used in conjunction with the impulse approximation. They also serve as a rigorous boundary for any impulse-approximation-based treatment.

Our division of the problem into a peripheral part and a shorter range part quantified either in terms of the next singularities or of density corrections, also helps to separate the relatively simple peripheral waves from the complex dynamical regime of the low partial waves. This same separation can be used as a phenomenological tool as the basis for a phase shift analysis where the peripheral partial waves are calculated from on-shell information and the inner waves are left free. Using the peripheral waves to construct an optical potential may be a convenient first step in such an analysis, but it is not essential. Only if the fluctuation corrections to the impulse approximation are not important will the corresponding optical potential be local.

Our work here has been limited to the elastic scattering impulse approximation. It is clear, however, that the peripheral parts of inelastic processes will also be given in terms of on-shell information, and that a similar analysis can be made there. Even in the elastic scattering treatment, since the elementary amplitudes are not purely real, the peripheral partial waves will reflect inelastic processes. That is, the total partial wave cross sections will contain inelastic processes (nucleus breakup).

Since, for π-nucleus scattering the low partial waves are dominated by pion absorption leading to a black disk amplitude, most of the reaction cross section in which the pion survives must come from peripheral waves. The experimental observation of an upward shift in the resonance energy in this type of reaction (cf. $\pi + {}^{12}\mathrm{C} \to \pi + {}^{11}\mathrm{C} + n$)[18] is then certainly consistent with our picture.

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APPENDIX A: CORRECTIONS TO THE ON-SHELL PART k SPACE

In this Appendix we obtain the position of the branch cuts in momentum transfer due to the finite range of the bound state wave function and due to off-shell effects in the elementary t matrix. We again follow closely the spirit of Fuda.[3] The wave function vertex $v(k)$ appearing in the numerator of (2.2) in general has a left-hand cut. We write

$$v(k^2) = \frac{1}{\pi}\int_{\mu_v^2}^{\infty} \frac{dk_1^2 \rho(k_1^2)}{k^2 + k_1^2}\,, \tag{A1}$$

so that $k^2 = -\mu_v^2$ is the closest point of that cut to the physical region of v ($k^2 > 0$). For the elementary t matrix we write the N/D

$$(k^2|t(\epsilon)|k'^2) = \frac{N(k^2, k'^2, \epsilon)}{D(\epsilon)}\,. \tag{A2}$$

N also has only left-hand cuts and we write

$$N(k^2, k'^2, \epsilon) = \frac{1}{\pi^2}\int_{\mu_N^2}^{\infty} \frac{n(q^2, q'^2, \epsilon)}{(q^2+k^2)(q'^2+k'^2)} dq^2 dq'^2\,, \tag{A3}$$

where the closest point of the N cut is called μ_N^2. If we substitute (A1) and (A3) into the expression for I, (2.2), we obtain an expression with six denominator factors, three of which contain the primed momenta and three the unprimed. Using the kinematic relations (2.4), each triplet of denominators can be decomposed by partial fractions into a sum of three terms, each containing one denominator of interest. I then becomes a sum of nine terms, each with a primed denominator and an unprimed one. One term will involve the original two denominators of (2.2). In that term the spectral integrals are easily done to give the on-shell form of the numerators as in Eq. (2.6). The remaining terms are either mixed terms having one energy denominator from (2.2) and one spectral denominator from (A1) or (A3), or terms with two spectral denominators. It is easy to see that the mixed terms have the next longest range and we now calculate the closest point of their momentum transfer branch cut. In the partial fraction decomposition some of the coefficients of the denominators will depend on the loop momentum q and the energy E, as does the on-shell t matrix in I_L. However, since these factors do not involve the external momenta, they will not lead to any momentum transfer dependence. The t-cut structure can thus be obtained directly from the denominator factors. (This method is equivalent to finding the singularities of reduced graphs.)

A typical mixed term coming from a wave function factor (A1) and an energy denominator is

$$\int_{\mu_v^2}^{\infty} dx^2\rho(x^2)\int \frac{d^3q}{[E-p^2/2m_a-(\vec{p}+\vec{q})^2/2m_b-q^2/2m_c](x^2+k_1'^2)}\,. \tag{A4}$$

The largest range part of this integral comes from the lower limit of the x^2 integration, and hence we consider

$$\int \frac{d^3q}{[E-p^2/2m_a-(\vec{p}+\vec{q})^2/2m_b-q^2/2m_c](\mu_v^2+k_1'^2)} \tag{A5}$$

which, using (2.4) can be written

$$\int \frac{d^3q}{(k_1^2+\beta^2)(k_1'^2+\mu_v^2)}\,, \tag{A6}$$

or in terms of the explicit form for $\vec{k}_1$ and $\vec{k}_1'$ (2.4),

$$\int \frac{d^3Q}{[(\vec{Q}+\vec{K})^2+A_\beta^2][(\vec{Q}+\vec{K}')^2+A_\mu^2]}\,, \tag{A7}$$

where

$$A_\beta^2=\frac{2(m_c+m_b)}{m_c\cdot m_b}\beta^2,\quad A_\mu^2=\frac{2(m_c+m_b)}{m_c\cdot m_b}\mu_v^2$$
$$\vec{K}=\frac{\vec{p}}{m_b},\quad \vec{K}'=\frac{\vec{p}'}{m_b}, \tag{A8}$$

and where we have changed variables $\vec{q}(1/m_a+1/m_b)=\vec{Q}$. In transforming (A6) into (A7), we have dropped overall factors that are irrelevant to the location of the branch point. Using the standard methods for finding the location of singularities of a Feynman graph, we find that (A7) has a branch cut in $(\vec{K}-\vec{K}')^2$ from $-(A_\beta+A_\mu)^2$ to $-\infty$, or, in terms of the momentum transfer, from $t=(2m_b/m_c)(m_b+m_c)(\beta+\mu_v)^2$ to ∞.

Exactly the same analysis could be applied to (2.6) by simply putting $\mu_v=\beta$ in (A6) to show that the branch point due to the propagators of (2.2) alone comes at $t=(8m_b/m_c)(m_b+m_c)\beta^2$. Since, in general, $\mu>\beta$ and often $\mu\gg\beta$, the cut in (2.6) is closer than that from (A6) as we expect.

The nearest branch point in momentum transfer coming from a propagator and the N function (A3) can be found by using the lower limit from (A3) and studying

$$\int \frac{d^3q}{(k_2'^2+\mu_N^2)(k_1^2+\beta^2)}\,. \tag{A9}$$

This can be transformed into an integral of the form (A7) by the substitution

$$\vec{K}'=\vec{p}'\left(\frac{m_a+m_b}{m_am_b}\right),\quad \vec{K}=\vec{p}\,\frac{m_c}{m_b(m_b+m_c)}\,,$$
$$A_\mu^2=2\mu_N^2\left(\frac{m_a+m_b}{m_am_b}\right),\quad A_\beta^2=2\beta^2\frac{m_c}{m_b(m_b+m_c)}\,. \tag{A10}$$

This then leads to a branch cut at

$$\left(\vec{p}\frac{m_c}{m_b+m_c}-\vec{p}'\frac{m_a+m_b}{m_a}\right)^2 = -2m_b\left[\left(\frac{m_a+m_b}{m_a}\right)^{1/2}\mu_N+\left(\frac{m_c}{m_b+m_c}\right)^{1/2}\beta\right]^2. \tag{A11}$$

This is now not a branch cut in t alone but also in the energy. If we write

$$\vec{p}=\vec{K}+\tfrac{1}{2}\vec{\Delta},\quad \vec{p}'=\vec{K}-\tfrac{1}{2}\vec{\Delta},$$
$$p^2=p'^2=K^2+\tfrac{1}{4}\Delta^2=K^2-\tfrac{1}{4}t\,, \tag{A12}$$
$$(\vec{K}\cdot\vec{\Delta}=0 \text{ by energy conservation}),$$

we obtain from (A11) that the branch point in t is at

$$t=\frac{8m_b(\mu_N/\sqrt{\alpha}+\sqrt{\gamma}\,\beta)^2+4p^2(\gamma-1/\alpha)^2}{(\gamma+1/\alpha)^2-(\gamma-1/\alpha)^2}\,, \tag{A13}$$

where we have written $\gamma=m_c/m_c+m_b$ and $\alpha=m_a/m_a+m_b$. Thus the t cut due to off-shell effects moves out with increasing incident kinetic energy (p^2). Hence the leading wave function cuts are more important than off-shell effects in the impulse approximation at high energies. It should be noted, however, that branch cuts coming from higher order terms in the multiple scattering will be in comparable locations to those we have discussed here.

APPENDIX B

This Appendix gives the details of the calculations in coordinate representation, the results of which have been discussed in Sec. III. As in the text, the target is assumed to be sufficiently heavy and to be an L-S closed shell nucleus. The relevant quantities are of the form

$$I_l(0) \equiv \sum_{nLM} \int d\vec{r}\, j_l(pr) Y_{lm}^*(\hat{r}) \psi_{nL}(r) Y_{LM}^*(\hat{r}) \times O j_l(pr) Y_{lm}(\hat{r}) \psi_{nL}(r) Y_{LM}(\hat{r}) , \quad \text{(B1)}$$

where j_l represents the spherical wave of the projectile with momentum p, $\psi_{nL}(r)Y_{LM}(\hat{r})$ is the wave function for an occupied orbit (n, L, M) in the target nucleus, and the operator O is either the unit operator 1 or the Laplacian Δ or the Laplacian squared Δ^2. The average and the fluctuation of the kinetic energy of the projectile-nucleon c.m. motion are then given by

$$\langle T_c \rangle = -\frac{1}{2(M+m)} \frac{I_l(\Delta)}{I_l(1)} ,$$

$$\langle T_c^2 \rangle - \langle T_c \rangle^2 = \frac{1}{4(M+m)} \left[\frac{I_l(\Delta^2)}{I_l(1)} - \left(\frac{I_l(\Delta)}{I_l(1)} \right)^2 \right]. \quad \text{(B2)}$$

The angular part of the integral and the summation over the magnetic quantum number M can be performed using the well known identity

$$Y_{LM}(\hat{r}) Y_{lm}(\hat{r}) = \sum_\lambda (LMlm|\lambda M+m)(L0l0|\lambda 0) \times \left(\frac{(2l+M)(2L+M)}{(2\lambda+1)4\pi} \right)^{1/2} Y_{\lambda M+m}(\hat{r}) \quad \text{(B3)}$$

and the sum rules

$$\sum_\lambda (L0l0|\lambda 0)^2 = 1 ,$$

$$\sum_\lambda (L0l0|\lambda 0)^2 \lambda(\lambda+1) = l(l+1) + L(L+1) , \quad \text{(B4)}$$

$$\sum_\lambda (L0l0|\lambda 0)^2 [\lambda(\lambda+1)]^2 = [l(l+1) + L(L+1)]^2 + l(l+1)L(L+1) .$$

The expression (B1) can thus be reduced to a simpler form:

$$I_l(0) = \sum_{nL} \int^\infty dr\, r^2 j_l(pr) \psi_{nL}(r) O_{L,l} j_l(pr) \psi_{nL}(r) , \quad \text{(B5)}$$

where the operators $O_{L,l}$ corresponding to 1, Δ, and Δ^2 are, respectively,

$$1: \; O_{L,l} = 1 ,$$

$$\Delta: \; O_{L,l} = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1) + L(L+1)}{r^2} , \quad \text{(B6)}$$

$$\Delta^2: \; O_{L,l} = \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1) + L(L+1)}{r^2} \right)^2 + \frac{l(l+1)L(L+1)}{r^4} .$$

Since we are interested in the peripheral partial waves where the impact parameter l/p is much larger than the range of the potential for the orbitals ψ_{nL}, the following asymptotic form can be used for ψ_{nL} in Eq. (B5).

$$\psi_{nL}(r) \sim i^{L+2} \beta_{nL} g_{nL} h_L^{(1)}(i\beta_{nL} r) , \quad \text{(B7)}$$

where β_{nL} is the damping factor of the tail of the wave function ψ_{nL}, related to the separation energy ϵ_{nL} of the orbit by

$$\beta_{nL}^2 = 2M\epsilon_{nL} , \quad \text{(B8)}$$

and g_{nL} is the asymptotic normalization (or coupling constant). The quantity $I_l(0)$ for the peripheral waves is approximated by

$$I_l(0) \approx \sum_{nL} g_{nL}^2 \int_0^\infty dr\, r^2 j_l(pr) [i^{L+2} \beta_{nL} h_L^{(1)}(i\beta_{nL} r)] \times O_{L,l} j_l(pr) [i^{L+2} \beta_{nL} h_L^{(1)}(i\beta_{nL} r)] . \quad \text{(B9)}$$

This expression makes sense only if the integral converges, i.e., $l > L+1$. The following two cases are interesting for the application to actual problems, i.e., (a) $p > \beta_{nL}, l \gg L$ which corresponds to the scattering problems at not too low energy and (b) $p \ll \beta_{nL}, l > L$ which corresponds to the scattering problems at very low energy and the cases of mesic atoms. Let us discuss these two cases separately.

(a) With partial integration and using the differential equation for j_l, one can reduce all the necessary integrals to the form

$$I_L(n, \beta) \equiv \int_0^\infty [j_l(pr)]^2 e^{-2\beta r} r^{n+1} dr , \quad \text{(B10)}$$

with $n \ll l$. For $n = 0$, this integral can be expressed by the Legendre function of second kind, i.e.,

$$I_l(0, \beta) = \frac{1}{2p^2} Q_l \left(\frac{p^2 + 2\beta^2}{p^2} \right) . \quad \text{(B11)}$$

The average and the fluctuation of r are given by

$$\langle r\rangle = -\frac{1}{2}\frac{\partial}{\partial\beta}[\ln I_l(0,\beta)],$$
$$\langle r^2\rangle - \langle r\rangle^2 = \frac{1}{4}\frac{\partial^2}{\partial\beta^2}[\ln I_l(0,\beta)]. \quad \text{(B12)}$$

The asymptotic form of Q_l for large l gives

$$\ln I_l(0,\beta) \sim -(2l+1)\ln[\beta+(p^2+\beta^2)^{1/2}] - \tfrac{1}{2}\ln[\beta(p^2+\beta^2)^{1/2}] + O\left(\frac{1}{l}\right), \quad \text{(B13)}$$

from which one obtains

$$\langle r\rangle = \frac{l+\frac{1}{2}}{(p^2+\beta^2)^{1/2}} + \frac{p^2+2\beta^2}{4\beta(p^2+\beta^2)}, \quad \text{(B14)}$$
$$\langle r^2\rangle - \langle r\rangle^2 = \frac{(l+\frac{1}{2})\beta}{2[(p^2+\beta^2)^{1/2}]^3}.$$

The result for $I_l(n,\beta)$ is

$$I_l(n,\beta) \approx I_l(0,\beta)\left(\langle r\rangle^n + \frac{n(n-1)}{2}[\langle r^2\rangle - \langle r\rangle^2]\langle r\rangle^{n-2}\right) \quad \text{(B15)}$$

to the next to the lowest order in $1/l$.

In order to calculate the integrals appearing in Eq. (B9) to the first order in $1/l$, one can use the following approximation for the spherical Hankel function, i.e.,

$$i^{L+2}\beta h_L^{(1)}(i\beta r) \approx \frac{e^{-\beta r}}{r}\left(1+\frac{L(L+1)}{2\beta r}\right). \quad \text{(B16)}$$

Since the second term gives a common factor for $I_l(1)$, $I_l(\Delta)$, and $I_l(\Delta^2)$, one gets

$$I_l(1) = \sum_{nL} \tilde{g}_{nL}{}^2 I_l(-1,\beta_{nL}),$$
$$I_l(\Delta) = -\sum_{nL}\tilde{g}_{nL}{}^2[(p^2+\beta_{nL}{}^2)I_l(-1,\beta_{nL}) + 2\beta_{nL}I_l(-2,\beta_{nL})], \quad \text{(B17)}$$
$$I_l(\Delta^2) = -\sum_{nL}\tilde{g}_{nL}{}^2[(p^2+\beta_{nL}{}^2)(p^2+5\beta_{nL}{}^2)I_l(-1,\beta_{nL}) + 4\beta_{nL}(p^2+3\beta_{nL}{}^2)I_l(-2,\beta_{nL}) - 4\beta_{nL}{}^2 l(l+1)I_l(-3,\beta_{nL})],$$

with the modified

$$\tilde{g}_{nL}{}^2 = g_{nL}{}^2\left(1+\frac{2L(L+1)(p^2+\beta_{nL}{}^2)^{1/2}}{\beta_{nL}(2l+1)}\right).$$

The least bound orbit becomes dominant for very large l, in which case one obtains simple expressions for the average and the fluctuation of the kinetic energy:

$$\langle T_c\rangle_i = \frac{1}{2(M+m)}(p^2+\beta_i{}^2)\left(1+\frac{1}{l}\cdot\frac{2\beta_i}{(p^2+\beta_i{}^2)^{1/2}}\right), \quad \text{(B18)}$$
$$\langle T_c{}^2\rangle_i - \langle T_c\rangle_i{}^2 = \frac{1}{4(M+m)^2}(p^2+\beta_i{}^2)^2\cdot\frac{1}{l}\cdot\frac{2\beta_i}{(p^2+\beta_i{}^2)^{1/2}}$$

where β_i is β_{nL} for the dominant orbit.

(b) since $p \ll \beta_{nL}$, $j_l(pr)$ in Eq. (B9) can be approximated by

$$j_l(pr) = \frac{(pr)^l}{(2l+1)!!}. \quad \text{(B19)}$$

All the necessary integrals are now reduced to

$$H_L(n,\beta) \equiv (-1)^l\int_0^\infty [h_L{}^{(1)}(i\beta r)]^2 r^{2n+2}dr = \frac{1}{\beta^{2n+3}}\frac{2^{2n-1}[n!]^2\Gamma(n+L+\frac{3}{2})\Gamma(n-L+\frac{1}{2})}{(2n+1)!}. \quad \text{(B20)}$$

$I_l(O)$ is given by

$$I_l(1) = \frac{-p^{2l}}{[(2l+1)!!]^2}\sum_{nL} g_{nL}{}^2(\beta_{nL})^2 H_L(l+1,\beta_{nL}),$$

$$I_l(\Delta) = \frac{p^{2l}}{[(2l+1)!!]^2}\sum_{nL} g_{nL}{}^2(\beta_{nL})^2[(p^2-\beta_{nL}{}^2)H_L(l+1,\beta_{nL}) + l(2l+1)H_L(l,\beta_{nL})],$$
$$+ 2l[(2l+1)(p^2-\beta_{nL}{}^2) - 2l\beta_{nL}{}^2]H_L(l,\beta_{nL})$$

$$I_l(\Delta^2) = \frac{-p^{2l}}{[(2l+1)!!]^2}\sum_{nL} g_{nL}{}^2(\beta_{nL})^2\{(p^2-\beta_{nL}{}^2)H_L(l+1,\beta_{nL}) + 2l[(2l+1)(p^2-\beta_{nL}{}^2) - 2l\beta_{nL}{}^2]H_L(l_1\beta_{nL}) + 2l(l-1)[2l(2l-1) - L(L+1)]H_L(l-1,\beta_{nL})\}. \quad \text{(B21)}$$

If one considers only the dominant orbit (L_i), the average and the fluctuation of the kinetic energy become

$$\langle T_c\rangle_i = \frac{1}{2(M+m)}\left(p^2-\beta_i^2+\frac{(2l+1)^2}{2[l^2-(L_i+\frac{1}{2})^2]}\beta_i^2\right),$$

$$\langle T_c^2\rangle_i-\langle T_c\rangle_i^2=\frac{\beta_i^4}{4(M+m)^2}\frac{(2l+1)^2}{[l^2-(L_i+\frac{1}{2})^2]}\times\left(\frac{l(2l-1)+L_i(L_i+1)}{2[(l-1)^2-(L_i+\frac{1}{2})^2]}-\frac{(2l+1)^2}{4[l^2-(L_i+\frac{1}{2})^2]}\right). \quad \text{(B22)}$$

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