Coulomb energies in S-shell nuclei and hypernuclei

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We estimate the Coulomb energy component of the ^{3}He-^{3}H binding energy difference, using the latest elastic electron scattering form factor data, to be 639 ± 19 keV, in agreement with previous estimates. We estimate the Coulomb energy of ^{4}He to be 757 ± 23 keV. We discuss the Coulomb energy contribution to the Λ separation energy difference $B_\Lambda({}^4_\Lambda\mathrm{He})$-$B_\Lambda({}^4_\Lambda\mathrm{H})$.

[NUCLEAR STRUCTURE Coulomb energy, ^{3}He, ^{3}H, ^{4}He, ${}^4_\Lambda$He, Λ-separation energy.]

I. INTRODUCTION

The Coulomb energy problem has long been of significance in nuclear physics, because it had been hoped that where the basic interaction is understood one might obtain unambiguous knowledge of the nuclear wave function. Such appears not to be the case. Physicists have long struggled to reconcile the binding energy difference of even the simplest system of mirror nuclear pairs (^{3}He-^{3}H) with their attempts to calculate the Coulomb energy difference.[1-3] Failure of these efforts has demonstrated the existence of a nonnegligible charge asymmetry in the baryon-baryon strong interaction[4,5] and V_{pp}-V_{nn} is not "understood" as V_{Coulomb}. In the case of the mirror pairs this has led to a search for a reliable determination of the true Coulomb energy difference, so that the effects of charge asymmetry might be studied.[6-9]

Friar first pointed out that one can extract directly from the experimental elastic electron scattering form factors for ^{3}He and ^{3}H the Coulomb energy contribution from the most significant components of the nuclear wave function.[10] In this case one obtained a Coulomb energy difference $\Delta E_C = 0.64 \pm 0.01$ MeV compared to the observed $B(^3\mathrm{H}) - B(^3\mathrm{He}) = 0.764$ MeV. This estimate of the Coulomb energy confirmed the realistic model calculations of that quantity which had properly accounted for the finite charge distribution of the proton. This success suggests other obvious applications of the techniques.

One such interesting possibility is the determination of the Coulomb energy of ^{4}He. First, because most ^{4}He ground-state calculations omit the Coulomb interaction, one would like to know just how much more bound than 28.3 MeV is the uncharged ^{4}He. Second, one would like to compare the true Coulomb energies of ^{3}He and ^{4}He in order to study the effects of compression upon the "^{3}He core." Finally, one would like to know how much $E_C^{^4\mathrm{He}}$ and $B(^3\mathrm{H}) - B(^3\mathrm{He})$ differ, since the latter quantity is the difference in the threshold energies for the $n + {}^3\mathrm{He}$ and $p + {}^3\mathrm{H}$ two-body breakup channels of ^{4}He.

Another interesting Coulomb problem concerns the difference in the separation energies for the mirror hypernuclear pair (${}^4_\Lambda$He-${}^4_\Lambda$H) (Ref. 11)

$$\Delta B_\Lambda = B_\Lambda({}^4_\Lambda\mathrm{He}) - B_\Lambda({}^4_\Lambda\mathrm{H}),$$

where

$$B_\Lambda({}^4_\Lambda\mathrm{H}) = B({}^4_\Lambda\mathrm{H}) - B(^3\mathrm{H}),$$

$$B_\Lambda({}^4_\Lambda\mathrm{He}) = B({}^4_\Lambda\mathrm{He}) - B(^3\mathrm{He}).$$

Because of the ^{3}He core compression within ${}^4_\Lambda$He, one anticipates that ΔB_Λ^C (here ΔB_Λ^C is the value of ΔB_Λ that would result with charge symmetric forces plus a Coulomb interaction between the protons) is negative, whereas the experimental value of ΔB_Λ (=0.34 MeV) is positive, indicating a sizeable charge asymmetry in the Λ-N interaction.[12] Estimates of ΔB_Λ^C vary from small values of 0.01 MeV to significantly larger values of up to 0.25 MeV.[12-15] Thus one would like to understand the purely Coulomb contribution to the binding energy difference, as in the case of ^{3}He-^{3}H, so that the charge asymmetry effects can be studied.

We reexamine here the ^{3}He-^{3}H isodoublet Coulomb energy difference utilizing the techniques discussed in Ref. 10 for direct extraction of ΔE_C from the experimental form factors. We extend the technique to the ^{4}He Coulomb energy problem in order to generate an estimate of $E_C^{^4\mathrm{He}}$. Finally we examine the ΔB_Λ^C problem for the ${}^4_\Lambda$He-${}^4_\Lambda$H isodoublet from the perspective of our results for ^{3}He and ^{4}He in an effort to shed some light upon reasonable limits for that controversial quantity.

II. COULOMB ENERGY CALCULATIONS

An accurate Coulomb energy calculation requires a wave function which has the correct experimentally determined size. Naively, this energy E_C is given by $E_C \sim e^2/R$, where R is the nuclear size. Too small a radius produces too large a Coulomb energy and vice versa. For light nuclei, most common phenomenological potentials produce too large a radius (they underbind the nucleus) which leads to too little Coulomb energy. One way of bypassing this difficulty was suggested several years ago.[10,16,17] If the wave function of a system possesses an expansion in terms of hyperspherical components which is dominated by the lowest-order terms, an identity may be written for E_C in terms of F, the elastic electron scattering (Coulomb) form factor. By using the *observed* form factors, an essentially model-independent measure of E_C may be obtained. This is not to say that the identity is exact. One is *assuming* that the wave function has a certain form; however, the subsequent analysis is "exact."

We illustrate this by examining the form for the wave function of the totally symmetric S state for the three- and four-body systems which are assumed in this technique to be $\psi(\vec{r}_1'^2+\vec{r}_2'^2+\vec{r}_3'^2)$ and $\psi(\vec{r}_1'^2+\vec{r}_2'^2+\vec{r}_3'^2+\vec{r}_4'^2)$, where $\vec{r}_i'$ is the coordinate of the ith nucleon relative to the nuclear center of mass, and ψ is an arbitrary function. It is clear that the form factor $F(q^2)$, which is the matrix element of $\exp(i\vec{q}\cdot\vec{r}_i')$, and $C(q^2)$, which is the matrix element of $\exp[i\vec{q}\cdot(\vec{r}_1'-\vec{r}_2')]$, are intimately related. One finds that $C(q^2)=F[2q^2A/(A-1)]$ for a system of A nucleons. Knowledge of the two-body operator C is all that is needed to calculate E_C, which is essentially the matrix element of $1/|\vec{r}_1'-\vec{r}_2'|$.

Mixed-symmetry states may also be incorporated. This adds no particular complication in the three-body system where form factor information exists for two separate species, ^{3}H and ^{3}He. These form factors may be written in the form

$$2F^{^3\mathrm{He}}=(\tfrac{3}{2}G_E^S+\tfrac{1}{2}G_E^V)F_S+G_E^V F_V\,, \tag{1a}$$

$$F^{^3\mathrm{H}}=(\tfrac{3}{2}G_E^S-\tfrac{1}{2}G_E^V)F_S-G_E^V F_V\,, \tag{1b}$$

where $G_E^V=G_E^p-G_E^n$ and $G_E^S=G_E^p+G_E^n$ are determined from the proton and neutron electric (Sachs) form factors G_E^p and G_E^n. The isoscalar and isovector nuclear body form factors, F_S and F_V, are matrix elements of $\exp(i\vec{q}\cdot\vec{r}_1')$ with respect to specific components of the wave function. In particular, the isovector combination F_V vanishes in the absence of mixed symmetry components, while the isoscalar combination

$$F_S=\frac{2F^{^3\mathrm{He}}+F^{^3\mathrm{H}}}{3G_E^S} \tag{1c}$$

is the Fourier transform of the matter density. Equation (1c) follows directly when one uses only the totally symmetric $\boxed{4}$ S state. The Coulomb energy *difference* of ^{3}He and ^{3}H is then obtained from the expression[10]

$$\Delta E_C=\frac{2\alpha}{\sqrt{3}\pi}\int_0^\infty dq\,G_E^S(\tfrac{1}{3}q^2)G_E^V(\tfrac{1}{3}q^2)[F_S(q^2)+F_V(q^2)]\,. \tag{2}$$

The factors of 3 result from the identity $C(q^2)=F(3q^2)$ discussed above and a change of variable. It is useful to examine this equation more closely. The integrand may be determined in terms of $F^{^3\mathrm{He}}$ and $F^{^3\mathrm{H}}$. For simplicity we neglect the small neutron form factor, so that $G_E^V=G_E^S=G_E$. In this limit we have

$$F_S+F_V=\frac{4F^{^3\mathrm{He}}-F^{^3\mathrm{H}}}{3G_E}\,, \tag{3}$$

which demonstrates that ΔE_C is four times as sensitive to the ^{3}He form factor data as the ^{3}H data. This is a fortuitous circumstance, since abundant data at low- and high-momentum transfer q^2 exist only for ^{3}He.

If we neglect the mixed-symmetry components, whose effect on the Coulomb energy will be shown to be small, the total Coulomb energies of ^{3}He and ^{3}H are given, respectively, by

$$E_C^{^3\mathrm{He}}=\frac{2\alpha}{\sqrt{3}\pi}\int_0^\infty dq[(G_E^p)^2+2G_E^pG_E^n]F_S\,, \tag{4a}$$

$$E_C^{^3\mathrm{H}}=\frac{2\alpha}{\sqrt{3}\pi}\int_0^\infty dq[2G_E^pG_E^n+(G_E^n)^2]F_S\,, \tag{4b}$$

which clearly reproduces $\Delta E_C=E_C^{^3\mathrm{He}}-E_C^{^3\mathrm{H}}$ in Eq. (2) with $F_V=0$. A similar treatment may be made of the Coulomb energy of the isoscalar α particle, for which $F_V=0$. In this case, neglect of some of the mixed-symmetry components lead to

$$E_C^{^4\mathrm{He}}=\frac{\sqrt{6}\,\alpha}{2\pi}\int_0^\infty dq\frac{F^{^4\mathrm{He}}(q^2)}{G_E^S(q^2)} \times\left\{\left[G_E^p\left(\frac{3q^2}{8}\right)\right]^2+(G_E^n)^2+4G_E^pG_E^n\right\}, \tag{5}$$

where all of the nucleon form factor terms in Eq. (5) have the same argument $(3q^2/8)$. To our knowledge, an expression similar to Eq. (5) has been written down only once before[18]; the square-bracketed quantity was not correct in that instance.

The method we will use to evaluate the integrals numerically in Eqs. (2), (4), and (5) is simple and permits us to estimate the statistical error in E_C due to the (random) error in the data. We assume a set of form factor data which are ordered according to increasing q^2 and with no data at the same q^2. Duplicate data can be combined into a single datum. The point at $q^2=0$ is identically one. Neighboring data points are assumed to be connected by straight lines or parabolas; this defines our interpolation for q^2 between data points. Performing an integral over the form factor leads to trapezoidal and Simpsons' type rules, respectively, with the integral expressed as a linear combination of the data. Treating the individual data as random variables, the mean and variance of the integral can be obtained in the usual way. Such a scheme has the advantage that no analytic form for the form factor need be assumed, and no direct fits to the data need be made. It has the disadvantage that the integrand is not as smooth as in a fit by an analytic form. Noisy data can distort the result somewhat, and this should be signaled by a large statistical error estimate. Using both the trapezoidal rule and the Simpsons' rule provides a good indicator. Although the trapezoidal rule has a larger truncation error (i.e., it is a lower-order quadrature rule), it is less noisy, since the noise amplification factor is proportional to the square of the weight of each individual point and is minimized for equal weights.[19] A very large error in the Simpsons' result compared to the trapezoidal result is the signal for the presence of significant noise in the data. Where sufficient data exist, we will use the trapezoidal rule.

III. NUMERICAL RESULTS FOR ^{3}He-^{3}H AND ^{4}He

Only a single set of data[20] exists for ^{3}H. These 11 data span a q^2 range from 1.0 to 8.0 fm^{-2}. The ^{3}He data are extensive and we have used four recent sets,[20-24] including three not available when the results of Ref. 10 were calculated and one which followed the calculations of Ref. 5. These data cover a range from very small q^2 to very large q^2. For ^{4}He we have used three sets of data,[24-27] which also range from very small to very large q^2. In conjunction with these nuclear data we have used the proton form factor fit (5.3) and the proton and neutron form factor fit (8.2) of Höhler *et al.*[28] These fits do not differ significantly from each other in the range $q^2 \leq 10$ fm^{-2} or from other recent fits.[29] We have used the Höhler *et al.* form factor because of their attention to a number of experimental details. It was found that fit (5.3) together with the phenomenological neutron form factor of Ref. 30 did not produce results which differed significantly from those generated using fit (8.2). In what follows we will use only fit (8.2), which have proton and neutron rms radii of 0.84 and -0.34 fm, respectively.

In calculating the ^{3}He-^{3}H Coulomb energy, Eq. (2) was evaluated separately for the ^{3}He and ^{3}H contributions. At least 98% of the values of the integrals resulted from the integration range $0 \leq q^2 \leq 6$ fm^{-2}. For ^{3}H it was found that the trapezoidal rule and Simpsons' rule procedures gave nearly the same results with comparable statistical errors, and the Simpsons' rule value was adopted. In the ^{3}He calculation, the Natl. Bur. Stand. form factor data set[23] were renormalized by increasing $F(q^2)$ by $\frac{1}{2}$% and the standard deviations by 1.66 in accordance with their fits. The first data point in the set then corresponds to a form factor greater than 1.0, and it was deleted; including that point in the analysis produces significant noise in the Simpsons' result. Although both quadratures were consistent with each other in the statistical sense, there was more noise in the Simpsons' rule case; because there were many closely spaced data, it was felt that the truncation error was not a serious problem. (This latter statement applies to ^{4}He as well.) All results quoted below for ^{3}He and ^{4}He were calculated using the trapezoidal rule. We find for the ^{3}He-^{3}H Coulomb energy difference:

$$\Delta E_C = (886.0 \pm 1.7) - (246.9 \pm 1.1) = 639 \pm 2 \text{ keV},$$

which is consistent with the 638± 2 keV found in Ref. 10, the 638± 17 keV found[31] in Ref. 5, and slightly larger than the result of Ref. 32. The finite-size effect can be calculated by setting $G_E^V = G_E^S = 1$ in Eq. (2) [but not in Eq. (1)]; it decreases ΔE_C by 43 keV. The isoscalar result obtained by setting F_V to zero is

$$\Delta E_C^{\text{iso}}\ (G_E^n = 0) = (432.70 \pm 0.80) + (234.14 \pm 1.02)$$
$$= 667 \pm 1.3 \text{ keV},$$

and we see that the mixed-symmetry effect (from F_V) is -28 keV. The neutron-proton Coulomb interaction may be estimated using Eq. (4), and one finds 8 keV for *both* ^{3}He and ^{3}H. The $(G_E^n)^2$ term in ΔE_C accounts for less than 0.1 keV.

Because of the factor of 4 weight of the ^{3}He data, the lack of high (and low) q data for ^{3}H is probably not very important. More important is the limit of validity of the hyperspherical formulas for evaluating Coulomb energies. The calculations of Ref. 5 suggest that the error in the formula for ^{3}He-^{3}H may be 1–4%. We will *assume* a 3% error. Thus for ΔE_C, we have an estimate of 639± 19 keV, compatible with that of Ref. 5.

For ^{4}He the results were calculated using Eq. (5), both with and without the neutron contributions. We find

$$E_C^{^4\text{He}} = 757 \pm 1.4 \text{ keV}$$

and

$$E_C^{^4\mathrm{He}}(G_E^n = 0) = 736 \pm 1.3 \text{ keV},$$

so that neutrons contribute approximately +21 keV. The finite-size effect of the proton and neutrons ($G_E^p \equiv 1$ and $G_E^n \equiv 0$) together is − 50 keV. We will *assume* a 3% error in the hyperspherical formula, so that we have an estimate of $E_C^{^4\mathrm{He}} = 757 \pm 23$ keV.

The Coulomb energy of the α particle is thus similar in magnitude to the observed binding energy difference in the trinucleon isodoublet:

$$E_C^{^4\mathrm{He}} \approx B(^3\mathrm{H}) - B(^3\mathrm{He}),$$

which implies that the additional Coulomb energy of ^{4}He compared to ΔE_C is of the same magnitude as that due to non-Coulomb charge asymmetry effects in the ^{3}He-^{3}H binding energy difference. It is also clear that calculated binding energies for the α particle which fail to include Coulomb effects should be of the order of 29 MeV; for a point proton and with $G_E^n = 0$ the Coulomb energy should be approximately 0.81 MeV.

The question of compression effects and the increase in $E_C^{^4\mathrm{He}}$ compared to ΔE_C is not so trivial. Because ^{4}He is an isoscalar object, we should compare with the isoscalar Coulomb energy difference for ^{3}He-^{3}H. Also, because the n-p pairs enter the three-body and four-body calculations in a slightly different manner, we shall consider the results for $G_E^n = 0$. Thus the relevant comparison is $E_C^{^4\mathrm{He}}$ ($G_E^n = 0$) = 736 keV and $\Delta E_C^{\mathrm{iso}}$ ($G_E^n = 0$) = $E_C^{^3\mathrm{He}}$ ($G_n^E = 0$) = 667 keV. The corresponding isoscalar radii (corrected for the finite size of the nucleons, $\langle r_{\mathrm{iso}}{}^2\rangle^{1/2} = 0.77$ fm) may be obtained from Refs. 20 and 24; we find $\langle r^2(^4\mathrm{He})\rangle^{1/2} = 1.485$ fm and $\langle r_{\mathrm{iso}}{}^2(^3\mathrm{He}\text{-}^3\mathrm{H})\rangle^{1/2} = 1.63$ fm. It is now simple to ascertain that the relevant Coulomb energy due to the repulsion of the protons scales approximately inversely as the radii; the ratio of the Coulomb energies is 1.103 and the ratio of isoscalar radii is given by 1.098. This equality is just what one would naively anticipate.

We have ignored a number of relativistic effects in our analysis, which presumably make a very small contribution to both the Coulomb form factors (at small momentum transfers) and the overall electromagnetic energy shifts. As discussed in detail in Ref. 35, the small spin-orbit and Darwin-Foldy contributions to the form factor (see Sec. 2 of Ref. 35) generate the spin-orbit and Darwin-Foldy contributions [of order $(v/c)^2$] to the Breit interaction and consequently to ΔE. Thus the same physical effect occurs in *both* the form factor and ΔE; if we ignore the fact that the hyperspherical formula was derived for the static Coulomb potential and the nonrelativistic (plane wave) form factor, a direct evaluation of Eqs. (4) and (5) in terms of the experimental form factor should include much of the contribution to ΔE from these small effects. It is possible to demonstrate that this is true for the Darwin-Foldy contribution; an exact treatment would give a result slightly larger than what results from the direct and naive use of Eqs. (4) and (5). We expect that a similar result holds for the spin-orbit term, whose contribution to ΔE_C is ordinarily not evaluated.

A similar expectation should exist for the meson exchange contributions to the form factor *and* to ΔE_C, which are discussed in Sec. 6 of Ref. 35. Physical processes which lead to a smaller charge radius should increase the electromagnetic energy as well and *vice versa*. Exchange contributions to the charge operator unfortunately suffer from an additional problem[36]; a consistent (semi-) relativistic calculation of the wave function must be made at the same time in order to obtain unambiguous results for the form factor. The same problem exists for the Coulomb energy. Fortunately, these mesonic effects in the form factor are *relatively* large only for large momentum transfers, and thus they are not expected to make a very large contribution to ΔE_C, which is dominated by low-momentum contributions to F. In summary, we expect our naive use of the uncorrected Eqs. (4) and (5) includes *part* of the small mesonic and other relativistic corrections to ΔE_C.

IV. ΔB_Λ^C PROBLEM

Now let us turn to the question of Coulomb effects in the binding energies of the $^4_\Lambda$He-$^4_\Lambda$H isodoublet. Dalitz and von Hippel[12] first looked at the effect of Coulomb repulsion between the two protons in the Λ-separation energy difference ΔB_Λ. Based upon a simple ^{3}He core compression estimate of 10% within $^4_\Lambda$He, an estimate which ignored the effects of short-range repulsion, they deduced that $\Delta B_\Lambda^C \cong -80$ keV. However, they also noted that short-range repulsion would significantly reduce that estimate.

Gibson, Goldberg, and Weiss[15] estimated that ΔB_Λ^C was of the order of − 20 keV. This was based upon binding energies calculated in the Hartree-Fock approximation using two-term Gaussian potentials which did have a soft, short-range repulsion. The corresponding core compression of the ^{3}He within the $^4_\Lambda$He was determined to be only some 0.04–0.05 fm. These results do not disagree with those of Ref. 12 when one considers their neglect of short-range repulsion in the two-body force. This estimate for ΔB_Λ^C is also consistent with that which one would make based upon the results of the previous section. The N-N interaction is much stronger than the Λ-N interaction: $B_\Lambda({}^4_\Lambda\mathrm{He}) \cong 2.3$ MeV compared to $B_n({}^4\mathrm{He}) \cong 20$ MeV. Thus the compression of the ^{3}He core in the nuclear case (^{4}He) should certainly be much greater than in the corresponding hypernuclear case (${}^4_\Lambda$He). One would therefore infer that $|\Delta B_\Lambda^C|$ should be some 10–20% of the difference

$$E_C^{^4\mathrm{He}}(G_E^n=0) - E_C^{^3\mathrm{He}}(G_E^n=0)$$

or some 10–20 keV.

These conclusions clearly conflict with the oft-quoted[33,34] variational results of Refs. 13 and 14. There it was found that $\Delta B_\Lambda^C \cong -200$ keV, which is an order of magnitude larger. One can only speculate as to the origin of this discrepancy, but it is interesting to note that a Coulomb energy enhancement of -200 keV would require a change in the radius of the ^{3}He core much larger than that seen in going from ^{3}He to ^{4}He, a situation which seems very unlikely.

V. CONCLUSIONS

We estimate that the true Coulomb energy difference in the ^{3}He-^{3}H isodoublet is $\Delta E_C = 639$ keV with an "uncertainty" of some 3%. The isoscalar value, which ignores effects of mixed-symmetry components of the three-body wave function (and with $G_E^n = 0$), is 667 keV; if one treats the protons as point charges, this is increased to 721 keV. In the ^{4}He system, our estimate of the Coulomb energy is $E_C^{^4\mathrm{He}} = 757$ keV, again with an "uncertainty" of 3%. Neglecting neutron effects decreases this to 736 keV; if point protons are also assumed the Coulomb energy is 806 keV. We infer from these results and the corresponding radii that the Coulomb contribution to $\Delta B_\Lambda = B_\Lambda({}^4_\Lambda\mathrm{He}) - B_\Lambda({}^4_\Lambda\mathrm{H})$ is quite likely to be of the order of -20 keV and therefore small compared to the observed $\Delta B_\Lambda \approx 340$ keV.

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