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Dispersion relation approach to three-body systems

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Partial-wave dispersion relations are presented which describe both $2 \to 2$ bound-state amplitudes, $2 \to 3$ breakup amplitudes, and $3 \to 3$ three-particle scattering amplitudes. The dispersion relations are reduced to one-dimensional integral equations by absorbing the momentum dependence of the $2 \to 3$ and $3 \to 3$ amplitudes in the three-body phase-space factors. The momentum dependence is obtained by solving final-state unitarity equations. For S-wave nucleon-deuteron scattering the nature of this off-shell dependence is described in detail. Dynamical singularities of the amplitudes which occur in the dispersion relations are discussed, and the modifications of the dispersion relations due to the presence of anomalous thresholds are written out explicitly. The physical importance of three-body unitarity for the bound-state properties of three-body systems is reviewed and possible applications of the present theory to few-nucleon and pionic systems are briefly discussed.

[NUCLEAR REACTIONS Three-body equations. Partial-wave dispersion relations. Three-body unitarity. Final-state unitarity and zero-range approximations. Nucleon-deuteron scattering and breakup.]

I. INTRODUCTION

Since 1960, when the N/D method was first used in the realm of elementary particle physics,[1,2] the method has been widely applied in the field of few-body problems.[3-10] The interest for this approach was based both on the (mathematical) simplicity and physical transparency which one thought inherent to the approach, and on the need for a microscopic theory which could easily be extended from the three- to the N-body case ($N \geq 4$), without the formal and interpretational difficulties occurring in the extensions of the Faddeev formulations.

Since the potential concept plays only a minor role in the dispersion theoretic approach, one could not expect that this method would be very useful in determining the details of the two-particle interaction in comparison to e.g., the Faddeev approach in the three-nucleon system. On the contrary, one of the motivations for using the dispersion theory lay in the expectation that the scattering properties of few-body systems would be rather insensitive to the details of the two-nucleon interaction, an expectation which since then has been borne out by Faddeev calculations for the three-nucleon system. In the language of dispersion relations this insensitivity would be reflected in the importance of nearby rescattering singularities, and the relative unimportance of far-away potential singularities. The dispersion method would also provide a framework for including pionic degrees of freedom, which in a potential approach such as the Faddeev equation cannot be treated properly. In the three-nucleon system the singularities of one pionic diagram lie very close to the physical threshold, so that this diagram is expected to be quite important for determining the low energy properties of the system, and therefore, even puts some doubt on the physical relevance of the usual potential approaches to few-body systems.[11] It is this flexibility in the choice of diagrams and the simplicity of the calculations (the kernel of the integral equations is regular as long as one does not include breakup explicitly) which provides enough incentive to go along with the development of the method. The two main problems in this development—especially for the three-body system—are the study and importance of higher order diagrams, and the effects of three-particle unitarity. The most natural framework for studying these problems and the correctness of approximations made in their treatment is the one-term separable model of the three-nucleon system, usually known as the Amado model. Using the exact phase-space factor accounting for two- and three-particle unitarity, Stelbovics and Dodd[8] calculated S-wave scattering phases and bound-state properties within this model using different diagrams to represent the left-hand cut. This analysis showed that the quartet phases (which mainly determine the experimental cross section) were quite insensitive to the order of input, whereas the doublet phases were more sensitive, and the bound-state properties were quite sensitive to the order of approximation. Other analyses using different prescriptions for the phase-space factors[5, 10] lead to similar conclusions. Although this indicates that the criterion of the nearest singularities has a limited range of validity, it does not make the N/D approach obsolete since in the aforementioned studies the important pion-exchange diagram was not taken into consideration, and the inclusion of such a diagram may well render the higher order potential diagrams less important (we hope to investigate this possibility in a forthcoming study). Also, one can try to find suitable approximation schemes for higher order terms,[10] although the necessity of calculating left-hand projections in some of these schemes represents a serious drawback. Finally one can use some phenomenological models for the higher order input,[5] an approach which most likely will be necessary in $N \geq 4$ systems.

The second problem, the treatment of three-particle unitarity, is the main topic of this paper. Evidence for the importance of three-particle unitarity was found in an N/D calculation of the three-nucleon system,[10] both in the case of scattering and bound-state properties. Further indications for its importance come from qualitative properties of the three-body system in some limiting cases. When the range of the two-nucleon force goes to zero, the Thomas theorem[12] states that the binding energy of the three-particle system goes to infinity. However, if one only takes into account two-particle unitarity, this theorem does not apply. Another property of the three-body system known as the Efimov effect[13] states that the number of bound states becomes infinite, and accumulate near $E = 0$ if the two-body scattering length goes to infinity. This effect may also be absent, if one represents the three-body system by an inert two-body bound state plus a third particle.

In a treatment of three-particle unitarity neither the approach of Stelbovics and Dodd,[8] where complete Faddeev calculations have to be performed for a series of energies in order to determine the phase-space factors, nor the Frye-Warnock equations used in Ref. 5, where the phase-space factor is determined from experimental absorption coefficients, is a serious candidate for giving the N/D approach an independent status. Rather, we need a method which also applies if there are no full scattering calculations or detailed experimental phase shifts available. Such was the approach taken in the quartet calculation of Ref. 5, and in a subsequent paper dealing with the many-channel case.[10] One of the shortcomings of Ref. 10 was that three-particle channels were treated as two-body bound-state channels by increasing the binding in the two-particle subsystems. Furthermore the off-shell behavior of the amplitudes was neglected in the calculation of effective phase-space factors, an approximation which—though leading to good agreement with the exact results—did not (yet) have a microscopic foundation.

In the present paper we will address both these problems and will try to carry the program of the implementation of three-body unitarity one step further, while still keeping the equations in a tractable form. The first step towards an exact treatment of three-body unitarity, especially within the framework of the N/D method, was made by Blankenbecler,[14] and worked out in more detail shortly afterwards.[15, 16] It was understood that three kinds of amplitudes enter the unitarity equations [UE], describing respectively $2\to 2$, $2\to 3$, and $3\to 3$ processes. Partial-wave amplitudes were defined which depend on the total energy, and in the case of $2\to 3$ and $3\to 3$ amplitudes also on the subenergies σ of the two-particle systems. New N/D equations could be formulated which contain additional integrations over the subenergies. In order to reduce these new equations to standard one-dimensional integral equations, one can use the isobar ansatz, so that the amplitudes taken at certain resonance values of the subenergies can be factorized out of the integrals. If there are no two-body resonances one can still factorize out the amplitudes at some effective value for σ, or at several such points. In this respect it is important that Rubin, Sugar, and Tiktopoulos[17] have proved that the $2\to 3$ and $3\to 3$ amplitudes are analytic in the energy plane if one keeps σ and σ' proportional to the full energy E. The remaining dependence on σ and σ' under the integrals can be determined using discontinuity equations in σ and σ'.

Unfortunately the papers[14-16] missed some terms in the discontinuity equations for E and σ. One of these, the so-called exchange term, represents the exchange of a particle in the intermediate three free particles state. It was correctly given by Freedman *et al.* in Ref. 18. Subsequently, various derivations of all terms in the three-body UE were presented both starting from the Faddeev equations,[19-21] and from general S-matrix unitarity.[22] The three-body UE possess some special features due to the occurrence of δ-function contributions. In order to get rid of these contributions one introduces connected $3\to 3$ amplitudes, thereby introducing terms in the UE which are linear or quadratic in the two-body t matrix. By choosing the imaginary parts of the external subenergies in the UE as $\mathrm{Im}\sigma = -\mathrm{Im}\sigma' = -i\epsilon$ ($\epsilon > 0$) one obtains quasi two-body amplitudes $C(E,\sigma,\sigma')$ or $A(E,\sigma,\sigma')$, which satisfy UE which do not contain terms linear in t (Refs. 22 and 23) and therefore obey the same type of equation as the bound-state amplitudes do. The inhomogeneous term, which is quadratic in $t(\sigma)$ remains, however. After partial-wave projection this term features a cut which lies on the unitarity cut for $\sigma, \sigma' > 0$. Since this term can be considered as the first-order input of the $3 \to 3$ amplitude, its cut can be considered as a dynamical cut. The superposition of unitarity and dynamical cuts can easily be built into the N/D formalism, although the resulting equations are of singular nature. It has been shown[24, 25] that the equations can be regularized by using an exact analytical solution. For $\sigma = \sigma' = 0$ the cut of the first-order term turns into a pole at $E = 0$. In this case the equations can be solved without difficulty since the phase-space factor behaves at least like E^2 at the breakup threshold. It is thus natural to define the quasi two-body amplitudes at zero subenergies, especially if no resonances are present. Consequently this is the choice made in our investigation of the three-nucleon system.

Information about the σ and σ' dependence of the amplitudes within the UE can be obtained by using the discontinuity equations of these amplitudes in σ or σ', which we call final-state interaction (FSI) equations. The FSI equations were discussed by many authors, and their correspondence with Faddeev type (off-shell dynamical) equations was specifically noted in Refs. 26–29. This correspondence is not an equivalence, since the FSI equations do not contain an inhomogeneous term with dynamical singularities in σ', in contrast to the Faddeev equations. The lack of a homogeneous term implies that there exist an infinite number of solutions of the FSI equations. This arbitrariness can be removed by using the Skornyakov-Ter Martirosian[30] equations, whose solutions satisfy the FSI equations in the zero-range limit and in the limit of infinite scattering length. These equations provide a unique solution in the quartet case for every angular momentum,[31] and in the doublet case for all $l \neq 0$. Therefore the present solution generalizes the one in Ref. 21 to nonzero angular momentum. In the S-wave doublet case the equations have a homogeneous solution, so that there is no unique solution. Beloozerov resolves this ambiguity by using further constraints on the solution; however, since we think that these constraints are somewhat arbitrary, and in addition lead to a solution with certain undesirable features, we employ the homogeneous solution as a sole indicator of the subenergy dependence of the amplitudes. This solution of the S-wave FSI equations is the same as used in Ref. 21.

Using this prescription for the dependence on the subenergies, we can now determine both the direct and exchange part of the three-body phase-space factor, which in turn can be used in the N/D equations.

An important problem arises in the formulation of these N/D equations in the presence of three-body channels because of the occurrence of anomalous thresholds.[15] Recently this problem has been thoroughly reconsidered,[32] and we use the results from that paper to formulate the modified N/D equations for the present case.

The outline for this paper is as follows. In Sec. II we review the full unitarity equations for the $2 \to 2$, $2 \to 3$, and $3 \to 3$ amplitudes. A separable model for the two-particle interaction is used to factorize out the evident $t(\sigma)$ and $t(\sigma')$ dependence and the threshold behavior factor, leaving us with a quasi two-body amplitude $A_l(E)$. The discontinuity equations for this amplitude are investigated in Sec. III with the main emphasis on the question: Which values of σ' contribute to the unitarity integrals? This investigation enables us to define suitable "on-shell" amplitudes $A(E)$, which are taken outside the integrals. The remaining σ and σ' dependence in the three-body phase-space factors is parametrized by so-called off-shell functions. In Sec. IV the FSI equations and the corresponding Skornyakov-Ter Martirosian equations are studied. The solutions of the latter are given for the quartet and doublet case ($l \neq 0$). The properties of the solutions are discussed, and numerical calculations of the off-shell amplitudes and corresponding off-shell functions are given.

The analytic properties of the first- and second-order dynamical inputs are discussed in Sec. V. Our choice $\sigma = \sigma' = 0$ leads to a very simple analytic structure of the input: The first-order $3 \to 3$ input has a pole at $E = 0$. In Sec. VI we present the modified N/D equations. Section VII contains a discussion of the physical importance of three-particle unitarity. The emphasis is on bound-state properties, and the Thomas theorem and the Efimov effect are briefly discussed. Finally we comment on the general applicability of the formalism to few-nucleon and pionic systems.

II. THREE-BODY EQUATIONS AND UNITARITY

The three-body transition operator, containing all the information about bound-state scattering, breakup, and $3 \to 3$ processes, can be separated as follows[33]:

$$T = \sum_{\gamma} T_{\gamma} + \sum_{\alpha\beta} M_{\alpha\beta} . \tag{2.1}$$

Here T_γ are the two-body t matrices in the three-body space, and $M_{\alpha\beta}$ are connected amplitudes for the transition with the pair α interacting first and the pair β interacting last. The contribution of the bound-state poles to T can be explicitly written as follows[33, 21]:

$$\langle \vec{q}\vec{p}|M_{\alpha\beta}|\vec{q}'\vec{p}'\rangle = \frac{g_\alpha(\vec{q})\langle\vec{p}|{}^R M^R_{\alpha\beta}|\vec{p}'\rangle g_\beta(q')^*}{(E-p^2/2n_\alpha - E_\alpha)(E-p'^2/2n_\beta - E_\beta)} + \frac{g_\alpha(\vec{q})\langle\vec{p}|{}^R M_{\alpha\beta}|\vec{q}'\vec{p}'\rangle}{E-p^2/2n_\alpha - E_\alpha} + \frac{\langle\vec{q}\vec{p}|M^R_{\alpha\beta}|\vec{p}'\rangle g_\beta(\vec{q}')^*}{E-p'^2/2n_\beta - E_\beta} + \langle\vec{q}\vec{p}|\tilde{M}_{\alpha\beta}|\vec{q}'\vec{p}'\rangle . \quad (2.2)$$

Here we have introduced the usual notations: $\vec{p}$ and $\vec{q}$ are the channel and pair momentum, n_α is the channel reduced mass, and $g_\alpha(\vec{q})/(q^2/2m_\alpha - E_\alpha)$ is the bound-state wave function with energy E_α. The amplitude $\tilde{M}$ has no pole contributions.

We can now consider ${}^R M_{\alpha\gamma}$ as a connected transition operator whose matrix elements give the correct physical $2\to 3$ amplitudes. The $3\to 2$ transition operator is defined analogously, whereas the connected bound-state amplitude is given by the operator ${}^R M^R_{\alpha\beta}$. For completeness we note that the operators $M_{\alpha\beta}$ satisfy the equation[33]

$$M_{\alpha\beta} = (1-\delta_{\alpha\beta})T_\alpha G_0 T_\beta + \sum_{\gamma\neq\alpha} T_\alpha G_0 M_{\gamma\beta} . \quad (2.3)$$

Similar equations hold for the reduced amplitudes ${}^R M_{\alpha\beta}$, $M^R_{\alpha\beta}$, and ${}^R M^R_{\alpha\beta}$ if the appropriate T matrices are replaced by the form factors $g_\alpha(\vec{q})$.

The unitarity of the S matrix, $SS^+ = 1$, immediately implies corresponding expressions for the discontinuity of T, which is related to S by $S_{fi} = 1 - 2\pi i\,\delta(E_f - E_i)T$. We need, however, more detailed equations for the connected parts of the amplitudes ${}^R M^R$, ${}^R M$, and M; to this end we use Eqs. (2.1), (2.2), and the on-shell conditions $E = q_\alpha^2/2m_\alpha + p_\alpha^2/2n_\alpha$ for the reduced amplitudes. Our derivation here is similar to that of Ref. 21 and uses Eq. (7.31) of Ref. 33. We obtain the following equations:

$$\mathrm{disc}\,{}^R M^R_{\alpha\beta}(E) = -2\pi i \sum_\gamma {}^R M^R_{\alpha\gamma}(E+i\epsilon)\delta(E-H_\gamma)\,{}^R M^R_{\gamma\beta}(E-i\epsilon) - 2\pi i \sum_\gamma {}^R M_{\alpha\gamma}(E+i\epsilon)[\delta^d(E-H_0)+\delta^e(E-H_0)]M^R_{\mu\beta}(E-i\epsilon), \quad (2.4)$$

$$\mathrm{disc}\,M^R_{\alpha\beta}(E) = -2\pi i \sum_\gamma M^R_{\alpha\gamma}(E+i\epsilon)\delta(E-H_\gamma)\,{}^R M^R_{\gamma\beta}(E-i\epsilon) - 2\pi i \sum_{\mu\gamma} [M_{\alpha\gamma}(E+i\epsilon)+\delta_{\alpha\gamma}T_\alpha(E+i\epsilon)][\delta^d(E-H_0)+\delta^e(E-H_0)]M^R_{\mu\beta}(E-i\epsilon) , \quad (2.5)$$

$$\begin{aligned}\mathrm{disc}\,M_{\alpha\beta}(E) = &-2\pi i \sum_\gamma M^R_{\alpha\gamma}(E+i\epsilon)\delta(E-H_\gamma)\,{}^R M_{\gamma\beta}(E-i\epsilon) - 2\pi i(1-\delta_{\alpha\beta})T_\alpha(E+i\epsilon)T_\beta(E-i\epsilon)\delta^e(E-H_0)\\ &-2\pi i\sum_\gamma M_{\alpha\gamma}(E+i\epsilon)[\delta^d(E-H_0)+\delta^e(E-H_0)]\,T_\beta(E-i\epsilon)\\ &-2\pi i\sum_\gamma T_\alpha(E+i\epsilon)[\delta^d(E-H_0)+\delta^e(E-H_0)]M_{\gamma\beta}(E-i\epsilon)\\ &-2\pi i\sum_{\gamma\mu} M_{\alpha\gamma'}(E+i\epsilon)[\delta^d(E-H_0)+\delta^e(E-H_0)]M_{\mu\beta}(E-i\epsilon) .\end{aligned} \quad (2.6)$$

In Eqs. (2.4)–(2.6) we separated $\delta(E-H_0)$ in a direct $[\delta^d(E-H_0)]$ and an exchange part $[\delta^e(E-H_0)]$. The different terms in (2.4)–(2.6) allow a simple diagrammatic representation, as shown in Fig. 1.

The physical cross sections can now be expressed in terms of the amplitudes $M_{\alpha\beta}$. In the three-nucleon system the equations (2.3)–(2.6) can be simplified considerably because of the identity of the particles. The relevant physical amplitude in this case is a sum of direct and exchange terms

$$M = \tfrac{1}{3}\sum_{\alpha\beta} M_{\alpha\beta} = M_d + 2M_e . \quad (2.7)$$

Equation (2.3) is now replaced by $(T \equiv T_\gamma)$

$$M = 2TG_0^e T + 2TG_0^e M , \quad (2.8)$$

where we stressed the exchange nature of the intermediate Green's functions. The unitarity equations for these symmetrized amplitudes have the same structure as before; one simply omits the partition indices and summations and multiplies the exchange δ function with a factor 2. In the nucleon-deuteron case one obtains the following expression for the elastic cross section:

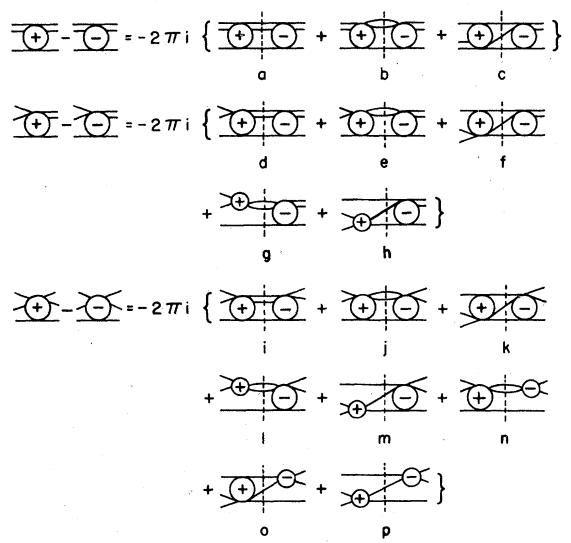

FIG. 1. Unitarity equations for the amplitudes ${}^R M^R$, M^R, and M. Parallel lines represent bound states and dashed lines indicate on-shell δ functions.

$$\frac{d\sigma}{d\Omega} = \left(-\frac{8\pi^2}{3}\right)^2 \tfrac{1}{3}\{2\,|{}^R M^{R(3/2)}|^2 + |{}^R M^{R(1/2)}|^2\}\,, \tag{2.9}$$

where the ${}^R M^R$ amplitudes are now defined in a spin-isospin representation. The relation of the various breakup cross sections to the amplitudes ${}^R M$ is more complicated because of the spin algebra involved. We refer the reader to Ref. 34, in which the same normalization for the amplitude is used and where expressions for the breakup cross sections are given.

In the following we prefer to work with quasi two-body scattering amplitudes instead of the $2\to3$ and $3\to3$ amplitudes ${}^R M$ and M. This can be done very easily in the separable model. Assuming that the two-particle system is always in an S state,[35] we have the following expression for the two-body t matrix:

$$\langle \vec p_\alpha \vec q_\alpha | T | \vec p_\alpha' \vec q_\alpha' \rangle = \delta(\vec p_\alpha' - \vec p_\alpha)\, t(E - p_\alpha{}^2; \vec q_\alpha', \vec q_\alpha')\,,$$

$$t_\alpha(z; \vec q_\alpha, q_\alpha') = g_\alpha(q_\alpha)\tau_\alpha(z) g_\alpha(q_\alpha')\,, \tag{2.10}$$

so that the on-shell $3\longrightarrow3$ amplitude can be written as

$$\langle \vec q \vec p | M_{\alpha\beta} | \vec q' \vec p' \rangle = g_\alpha(q)\tau_\alpha(q^2) A_{\alpha\beta}(E; \vec p, \vec p')\tau_\beta(q'^2) g_\beta(q')\,. \tag{2.11}$$

Similar expressions hold for ${}^R M_{\alpha\beta}$ and $M^R_{\alpha\beta}$, whereas ${}^R M^R_{\alpha\beta} \equiv A_{\alpha\beta}$. We can now insert (2.11) in the UE (2.4)–(2.6), in order to get UE for $A_{\alpha\beta}(E;\sigma,\sigma')$, where we changed to subenergy arguments in view of the coming discussion. First we note that the terms which are linear in T and are multiplied with $\delta^d(E-H_0)$ disappear because they also occur on the left. Next, by using amplitudes $A_{\alpha\beta}(E;\sigma+i\epsilon,\sigma'-i\epsilon)$ with fixed imaginary parts $i\epsilon$, we obtain UE for A which are free of terms which are linear in T, in accordance with Eq. (24) in Ref. 22. In order to obtain a closed set of equations in which only one type of amplitude appears we then have to perform an analytic continuation[22] in the arguments σ and σ' to obtain a set of equations in terms of $A_{\alpha\beta}(E;\sigma-i\epsilon,\sigma'+i\epsilon)$. The UE for these amplitudes still contain terms which are quadratic in T; however, these can be regarded as a dynamical input so that they will no longer be considered as part of the UE. As a result the amplitudes A satisfy the following UE:

$$\begin{aligned}\mathrm{disc}_E A_{\alpha\beta}(E;\sigma_-,\sigma_+') = -2\pi i\Big[&\sum_\gamma A_{\alpha\gamma}(E_+;\sigma_-,\sigma_+'')\delta(E-H_\gamma)A_{\gamma\beta}(E_-;\sigma_-'',\sigma_+') \\ &+\sum_\gamma \int A_{\alpha\gamma}(E_+;\sigma_-,\sigma_+'')\tau_\gamma(\sigma_+'')g_\gamma(\sigma_+'')\delta^d(E-H_0)g_\gamma(\sigma_-'')\tau_\gamma(\sigma_-'')A_{\gamma\beta}(E_-;'',\sigma_+') \\ &+\sum_{\gamma\neq\mu}\int\int A_{\alpha\gamma}(E_+;\sigma_-,\sigma_+'')\tau_\gamma(\sigma_+'')g_\gamma(\sigma_+'')\delta^e(E-H_0)g_\mu(\sigma_-''')\tau_\mu(\sigma_-''')A_{\mu\beta}(E_-;\sigma_-''',\sigma_+')\Big]\,.\end{aligned} \tag{2.12}$$

The integral signs represent integrations over the intermediate momenta. In the following we will use the momentum representation and denote the amplitudes by $A_{\alpha\beta}(E;\vec p,\vec p')$, but we have to keep in mind that they were defined with the aforementioned conventions for the subenergies.

III. UNITARITY EQUATIONS IN TERMS OF ONE VARIABLE AMPLITUDES

In this section we rewrite the unitarity equations in a form which enables us to apply N/D methods. This means that we have to eliminate in some way the momentum dependence of the amplitudes in the UE (2.12). For particle-resonance amplitudes one expects the region $\sigma \approx \sigma_r$ (= resonance energy) to be important so that one can use an isobar ansatz for the amplitude at this energy. However, in the three-nucleon system only bound states and antibound virtual states exist, and for this system, like many others, the choice for a "resonant" energy σ_r is not so obvious. In the present section we investigate which region of the σ space is most important in the UE, since this would suggest the use of an isobar ansatz in that region of the space. The resulting UE would then contain quasi two-body amplitudes A at these subenergies. In principle one can employ a set of resonant energies σ_r which would lead to equations similar to those considered in Refs. 15 and 16; however, practicality forces us to restrict the effective σ_r values to one or two. The remaining σ dependence under the integrals in the UE can be specified by the FSI equations and will be discussed in the next section.

We start by expanding the amplitudes A in a partial-wave series:

$$A_{\alpha\beta}(E;\vec{p},\vec{p}\,') = 4\pi \sum_{l m} p^l Y^*_{l m}(\hat{p}) A^l_{\alpha\beta}(E;p,p') \times p'^l Y_{l m}(\hat{p}') \Big/ (E+E_0)^l , \qquad (3.1)$$

where the cutoff factor $(E+E_0)^{-l}$ has been introduced, to guarantee convergence in the dispersion relations for $l \neq 0$. The resulting lth-order pole at $E=-E_0$, which will occur if we compute the amplitudes $A^l_{\alpha\beta}$ approximately, is not expected to affect the behavior of the amplitudes in the physical region very much, as long as $-E_0$ lies far to the left. From now on we will restrict ourselves to the three-nucleon system and use symmetrized amplitudes. Therefore we can omit the partition indices, and instead will employ channel indices i, j, or k which characterize the spin state of the two-particle subsystems.

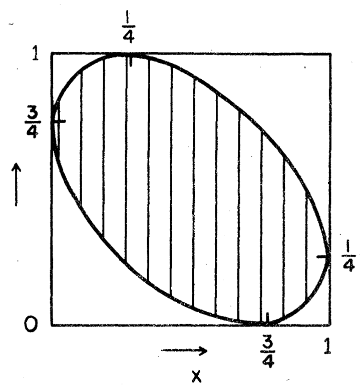

FIG. 2. Region of integration in the two-dimensional momentum space for the exchange contribution to three-particle phase-space factor ($x=\sigma/E$, $y=\sigma'/E$).

Using the partial-wave expansion in Eq. (3.1) one now gets the following expression for the direct contribution to three-particle unitarity (cf. Refs. 5, 9, and 10; we omit the angular momentum label):

$$\mathrm{disc}_d A_{ij}(E;p,p') = -16\pi^3 i \, \frac{4}{3\sqrt{3}} E^2 \left(\frac{4}{3}\,\frac{E}{E+E_0}\right)^l \sum_k \int_0^1 dx\, x^{1/2}(1-x)^{l+1/2} |\tau_k(Ex) g_k(Ex)|^2 \times A_{ik}(E;p,\{\tfrac{4}{3}E(1-x)\}^{1/2}) A^*_{kj}(E;\{\tfrac{4}{3}E(1-x)\}^{1/2},p') . \qquad (3.2)$$

In this expression x replaces the two-particle energy ($\sigma = Ex$). The exchange contribution can be written in several ways. The most symmetric expression is (cf. Ref. 22)

$$\mathrm{disc}_e A_{ij}(E;p,p') = -16\pi^3 i \, \tfrac{8}{9} E^2 \left(\frac{4}{3}\,\frac{E}{E+E_0}\right) \sum_{k,m} \lambda_{km} \int\!\!\int_A dx\, dy (1-x)^{l/2} (1-y)^{l/2} P_l \left(\frac{-\frac{5}{4}+x+y}{(1-x)^{1/2}(1-y)^{1/2}}\right) \times \tau_k(Ex) g_k(Ex) \tau^*_m(Ey) g^*_m(Ey) A_{ik}(E;p,\{\tfrac{4}{3}E(1-x)\}^{1/2}) \times A^*_{mj}(E;\{\tfrac{4}{3}E(1-y)\}^{1/2},p') , \qquad (3.3)$$

where the integration is over the two-particle energies $x(=\sigma/E)$ and $y(=\sigma'/E)$. The region of integration is the ellipse shown in Fig. 2. It is defined by the requirement that the modulus of the argument of the Legendre polynomial is less than one. Since the region is symmetric, the integral (3.3) will be real for $p=p'$, and $i=j$. The λ_{km} are spin-isospin factors which equal unity for spinless bosons.

Another representation for the exchange contribution, which is more similar to (3.2) and very convenient for use in the low energy region and in actual calculations, was derived in Ref. 10:

$$\mathrm{disc}_e A_{ij}(E;p,p') = -16\pi^3 i\,\frac{8}{3\sqrt{3}}E^2\left(\frac{4}{3}\frac{E}{E+E_0}\right)^l(-2)^{-l}$$
$$\times\sum_{k,m}\lambda_{km}\int_0^1 dx\,x^{1/2}(1-x)^{(l+1)/2}\sum_{h=0}^{l}\binom{l}{h}(3x)^{h/2}(1-x)^{(l-h)/2}F_h^{ikmj}(x)\,, \tag{3.4}$$

where

$$F_h^{ikmj}(x) = \tfrac{1}{2}\int_{-1}^{1} d\cos\theta\,P_h(\cos\theta)\tau_k(Ex)g_k(Ex)\tau_m^*(Ey)g_m^*(Ey)A_{ik}(E;p,\{\tfrac{4}{3}E(1-x)\}^{1/2})A_{mj}^*(E;\{\tfrac{4}{3}E(1-y)\}^{1/2},p') \tag{3.5}$$

and

$$y = \tfrac{3}{4} - \tfrac{1}{2}x - [\tfrac{3}{4}x(1-x)]^{1/2}\cos\theta \tag{3.6}$$

relates the two-particle energy of one pair to the two-particle energy of the other pair subject to the on-shell condition. The summation over h can be performed and the equivalence with (3.3) is then easily shown. Equation (3.4) allows a direct calculation of the zero energy behavior of the exchange contribution in the one-channel case:

$$\mathrm{disc}_e A_{ii}(E;p,p) = -16\pi^3 i\,\frac{8}{3\sqrt{3}}E^2\left(\frac{4}{3}\frac{E}{E+E_0}\right)^l(-2)^{-l}\lambda_{ii}\,\frac{\Gamma(\frac{3}{2})\Gamma(l+\frac{3}{2})}{\Gamma(l+3)}\,|\tau(0)g(0)A_{ii}(0;p,0)|^2 + O(E^{3+l})\,. \tag{3.7}$$

In the same limit the direct term is

$$\mathrm{disc}_d A_{ii}(E;p,p) = -16\pi^3 i\,\frac{4}{3\sqrt{3}}E^2\left(\frac{4}{3}\frac{E}{E+E_0}\right)^l\,\frac{\Gamma(\frac{3}{2})\Gamma(l+\frac{3}{2})}{\Gamma(l+3)}\,|\tau(0)g(0)A_{ii}(0;p,0)|^2 + O(E^{3+l})\,, \tag{3.8}$$

which shows that the ratio of direct and exchange term for zero energy is $\frac{1}{2}(-2)^l/\lambda_{ii}$. The spin factor λ_{ii} equals $-\frac{1}{2}$ in the quartet case of nucleon-deuteron scattering showing that direct and exchange contributions are exactly opposite if $l=0$. For higher energies we expect the exchange term to be less important since the kinematical constraints, as illustrated in Fig. 2, prevent the t matrices of different pairs from becoming maximal simultaneously.

In the following we want to determine the part of the integration region which contributes most to the direct and exchange integrals, for different values of l and E. For very low energies this analysis is quite simple, at least for the direct term. Computing the expectation value of x in (3.2) for low energies by neglecting the x dependence of τ, g, and A one finds

$$\langle x\rangle = 3/(2l+6)\,,\quad E \ll \alpha^2 \tag{3.9}$$

where α is the usual bound-state parameter (α_d^2 = binding energy of the deuteron). It is clear from (3.9) that the low (two-particle) energy region dominates, especially for high l values. If the energy becomes slightly larger, and the energy dependence of the two-particle propagator is also taken into account one obtains

$$\langle x\rangle = (2l+4)^{-1}\,,\quad \alpha^2 \ll E \ll \beta^2 \tag{3.10}$$

where β is the range parameter in the separable model. For very high energies one obtains in the S-wave case if one neglects the x dependence of A

$$\langle x\rangle = \left(\frac{\alpha+\beta}{E}\right)^{1/2}\,. \tag{3.11}$$

For the exchange term the situation is more complicated. For $l=0$ and small E we have the same situation as before [use (3.4)]; however, for $l>0$ the expansion of (3.5) in E does not lead to a simple expansion in x, from which we can infer the value of $\langle x\rangle$. However, by neglecting the x dependence in τ, g, and A, and by deducing the expectation values of $\langle x\rangle = \langle y\rangle$ from the remaining integrals we end up with the estimates

$$\begin{aligned}&\langle x\rangle = \langle y\rangle = 0.5,\quad l=0,1,\quad E\ll\alpha^2\,,\\ &\langle x\rangle = \langle y\rangle = 0.432,\quad l=2,\quad E\ll\alpha^2\,,\end{aligned} \tag{3.12}$$

whereas for higher energies the regions

$$\begin{aligned}&\langle x\rangle = 0.75,\quad \langle y\rangle = 0,\\ &\qquad\qquad E\gg\alpha^2,\ l \text{ arbitrary},\\ &\langle x\rangle = 0,\quad \langle y\rangle = 0.75\end{aligned} \tag{3.13}$$

tend to be more important.

We could now define amplitudes for all the corresponding momenta. For the set (3.13) this would mean amplitudes $A(E;p,(\frac{4}{3}E)^{1/2})$ and $A(E;p,(E/3)^{1/2})$, for the set (3.9) $A(E;p,[4E/(2l+6)]^{1/2})$. The momenta p could then be chosen accordingly to define a closed set of equations in these amplitudes. Since this procedure would lead to a large set of coupled integral equations, especially if there are more channels, it is more practical to define just one on-shell amplitude and improve—if possible—the calculation of the off-shell factors through final-state unitarity. Apart from this, the choice $A(E;p,(E/3)^{1/2})$ forbids itself since this amplitude has a logarithmic singularity for $p=(4E/3)^{1/2}$.

The best choice seems to be $A(E;p,(\frac{4}{3}E)^{1/2})$, since this amplitude is important for the direct contribution at higher l values or higher energies, and it is important in the exchange contribution for high energies [although it is multiplied with the amplitude $A(E;p,(\frac{1}{3}E)^{1/2})$ in the latter case]. It also has a simple physical meaning as it corresponds to two particles going out with zero relative energy (since $\sigma=0$), a final-state mechanism which can be observed experimentally. A further reduction of the set of equations occurs if one chooses the on-shell momentum in the triplet three-body channel to be the bound-state momentum $p=[\frac{4}{3}(E+\alpha_d^2)]^{1/2}$, so that all $3\to3$ amplitudes involving a triplet in or out state can be related to the $2\to2$ bound state or $2\to3$ breakup amplitudes.

The unitarity equations have the following form in terms of the amplitudes $A_{ij}(E)$:

$$\mathrm{disc}A_{ij}(E)=-2\pi i\sum_{k,m}A_{ik}(E)\rho_{km}(E)A^*_{mj}(E)\,, \tag{3.14}$$

where we have used off-shell functions to express the "half-shell" amplitudes in terms of the "on-shell" amplitudes $A_{ik}(E)$:

$$A_{im}(E;p_i^{\mathrm{on}},p_m)=\sum_k A_{ik}(E;p_i^{\mathrm{on}},p_k^{\mathrm{on}})\chi_{km}(E,p_m). \tag{3.15}$$

Other definitions of the off-shell function are also possible, for example by including the q dependence of the propagator $\tau_k(q^2)$ and the vertex function $g_k(q^2)$ in the off-shell functions. Introducing the notation 3 for the set of all three-particle channels we obtain

$$\begin{aligned}\rho_{ij}=8\pi^2\Big\{&\delta_{ij}\theta(E+\alpha_i^2)\frac{1}{3\pi}[\tfrac{4}{3}(E+\alpha_i^2)]^{l+1/2}+\theta(E)\frac{\sqrt{3}}{4}(\tfrac{4}{3}E)^{2+l}\\&\times\sum_{m\in3}\int_0^1 dx\,x^{1/2}(1-x)^{l+1/2}\,|\tau_m(Ex)g_m(Ex)|^2\chi_{im}(E,\{\tfrac{4}{3}E(1-x)\}^{1/2})\chi^*_{jm}(E,\{\tfrac{4}{3}E(1-x)\}^{1/2})\\&+\theta(E)\tfrac{1}{2}(\tfrac{4}{3}E)^{2+l}\sum_{m,n,\in3}\lambda_{mn}\int_A\int dx\,dy\,(1-x)^{l/2}(1-y)^{l/2}P_l\left[\frac{-\frac{5}{4}+x+y}{(1-x)^{1/2}(1-y)^{1/2}}\right]\\&\times\tau_m(Ex)g_m(Ex)\tau^*_n(Ey)g^*_n(Ey)\chi_{im}(E,\{\tfrac{4}{3}(1-x)\}^{1/2})\chi^*_{jn}(E,\{\tfrac{4}{3}E(1-y)\}^{1/2})\Big\}/(E+E_0)^l\,.\end{aligned} \tag{3.16}$$

The unitarity equations (3.14) can be written in the following form:

$$\mathrm{disc}\underline{A}=-2\pi i\underline{A}\,\underline{\rho}\,\underline{A}^*\,, \tag{3.17}$$

and hence

$$\mathrm{disc}\underline{A}^{-1}=2\pi i\underline{\rho}\,. \tag{3.18}$$

The usual N/D decomposition

$$\underline{A}=\underline{N}\,\underline{D}^{-1} \tag{3.19}$$

separates $\underline{A}$ into functions having only dynamical singularities ($\underline{N}$) and functions having only unitarity singularities ($\underline{D}$). Equation (3.18) can be written as

$$\mathrm{disc}\underline{D}(E)=2\pi i\underline{\rho}\,\underline{N}\,. \tag{3.20}$$

This equation can be used to write a dispersion integral for $\underline{D}$, which in turn can be used in deriving a closed set of integral equations for N.

These equations and the modifications therein due to anomalous thresholds will be discussed in Sec. VI.

IV. FINAL-STATE INTERACTION (FSI) EQUATIONS

In Sec. II we derived discontinuity equations for the amplitudes in the variable E. In order to obtain such equations for the $2\to3$ and $3\to3$ amplitudes, we had to eliminate terms which were linear in the two-body t matrix. This elimination was accomplished by equating such terms with discontinuities of the amplitudes in the subenergies σ and σ'. In the present section we will investigate these latter discontinuity equations in σ and σ' (FSI equations), and use them as a source of information on the σ and σ' dependence of the amplitudes. As shown in the last section such information is required to define the three-body phase-space factors.

As is usual in treatment of the FSI (Refs 15 and 22), one introduces amplitudes C which are related to the full on-shell amplitude M by

$$\langle \vec{p}_\alpha \vec{q}_\alpha | M_{\alpha\beta} | \vec{p}_\beta' \vec{q}_\beta' \rangle = \frac{1}{p_\alpha p_\beta'} \sum_l (2l+1) P_l(\hat{p}_\alpha \hat{p}_\beta') t_\alpha(q_\alpha) \times C^l_{\alpha\beta}(E; q_\alpha, q_\beta') t_\beta(q_\beta') . \tag{4.1}$$

where the usual S-wave assumption for the two-particle state is made. The relation of the new amplitude C to A is

$$A^l_{\alpha\beta}(E; p_\alpha, p_\beta') = (p_\alpha p_\beta')^{-l-1}(E+E_0)^l g_\alpha(q_\alpha) \times C^l_{\alpha\beta}(E; q_\alpha, q_\beta') g_\beta(q_\beta') , \tag{4.2}$$

where the separable approximation—necessary for defining A—has been used. We will investigate the momentum dependence of this function C and relate this back to the off-shell dependence of A and the off-shell functions χ via (4.2) and (3.15). The discontinuity of $M_{\alpha\beta}$ in the variable $\sigma_\beta' = q_\beta'^2$ is equal to the exchange contribution in the third term of Eq. (2.6). Using the expansion (4.1) and introducing symmetrized amplitudes with channel labels we obtain

$$\mathrm{disc}_{\sigma'} C^l_{ij}(E+i\epsilon; \sigma - i\epsilon, \sigma') = -\frac{16\pi^2}{3}\, i \sum_k \lambda_{kj} \int_{\sigma_-}^{\sigma_+} \frac{d\sigma''}{p''} C_{ik}(E+i\epsilon; \sigma - i\epsilon, \sigma'' + i\epsilon)\, t_k(\sigma'' + i\epsilon) P_l\left(\frac{E - p'^2 - p''^2}{p' p''}\right). \tag{4.3}$$

As usual the exchange contribution acquires a factor 2 (absorbed in the constant $-16\pi^2/3$), since there are two identical terms. The end points of integration in (4.3) are determined by the condition that the modulus of the argument of the Legendre function equals unity [i.e., $\sigma_\pm = \frac{3}{4}E - \frac{1}{2}\sigma' \pm (\frac{3}{4}\sigma'(E-\sigma'))^{1/2}$]. The discontinuity in (4.3) can also be obtained by applying Cutcosky's rules.[22]

Our next step is to simplify Eq. (4.3) by the pole approximation:

$$t_k(q) = -\frac{1}{2\pi^2} \frac{1}{q \cot\delta_k - iq} \approx \frac{1}{2\pi^2 iq} , \tag{4.4}$$

which is valid if $q \gg \alpha$ and $q \ll 2\beta/3$ (as before, α and β are the usual two-nucleon separable potential parameters, $\beta \approx 3/r_e$, where r_e is the effective range parameter). This approximation seems reasonable for the two nucleon-system, where $\beta \approx 6|\alpha|$; however, for larger systems similar range parameters β tend to become smaller so that (4.4) may be less reliable. One can now show that in the one-channel case the solution of the Skornyakov-Ter Martirosian equations[30]

$$C^l(E; q, q') = \kappa \lambda_{11} Q_l\left(\frac{E - p^2 - p'^2}{pp'}\right) + \frac{4i}{\pi} \lambda_{11} \int_0^\infty \frac{dp''}{q''} C_l(E; q, q'') \times Q_l\left(\frac{E - p''^2 - p'^2}{p'' p'}\right) \tag{4.5}$$

satisfies Eq. (4.3) under the approximation (4.4). In addition it gives the correct superposition of the solutions of Eq. (4.3) by specifying the inhomogeneous term. The normalization κ is unimportant for our purposes. The equations (4.5) have been formally solved by Beloozerov.[31] Generalizing his solution to the many-channel case one can write for negative E and positive momenta

$$C^l(E; q, q') = \kappa (\sinh\phi)^{1/2} (\sinh\chi)^{1/2} \int_0^\infty d\tau\, \tau \sinh(\tau\pi) |\Gamma(l+1+i\tau)|^2 \times P^{-l-1/2}_{i\tau-1/2}(\cosh\phi) P^{-l-1/2}_{i\tau-1/2}(\cosh\chi) \left(\underline{1} - \frac{8}{\sqrt{3}}(-1)^l L_l(\tau) \underline{\lambda}\right)^{-1} \underline{\lambda} (-1)^l L_l(\tau), \tag{4.6}$$

where $\sinh\phi = p/(-\frac{4}{3}E)^{1/2}$ and $\sinh\chi = p'/(-\frac{4}{3}E)^{1/2}$. As usual p and q are related by the on-shell condition $q^2 + \frac{3}{4}p^2 = E$. The function $L_l(\tau)$ is defined by

$$L_l(\tau) = (\sin\theta)^{1/2} |\Gamma(l+1+i\pi)|^2 P^{-l-1/2}_{i\tau-1/2}(0) P^{-l-1/2}_{i\tau-1/2}(\cos\theta) , \tag{4.7}$$

where $\theta = \pi/6$ is introduced for convenience. The special Legendre functions are defined in the Appendix. Unfortunately, in the doublet S-wave case the integrand has a pole at the integration interval, which makes the solution undefined since the pole corresponds to a solution of the homogeneous equation. We will return to this problem later on in this section. In order to extend Eq. (4.6) to positive energies one first rewrites the integrand in terms of positive and negative frequency parts:

$$sh(\pi\tau)\left|\Gamma(l+1+i\tau)\right|^2 P^{-l-1/2}_{i\tau-1/2}(\cosh\phi)=(-1)^l[Q^{l+1/2}_{i\tau-1/2}(\cosh\phi)-Q^{l+1/2}_{-i\tau-1/2}(\cosh\phi)], \tag{4.8}$$

where $Q^{l+1/2}_{i\tau-1/2}$ essentially behaves like $e^{-i\phi\tau}$ and therefore goes to zero if $\mathrm{Im}\tau\to-\infty$. Using Cauchy's theorem we can now express the integral (4.6) in four sums over the residues of the integrand at the poles on the positive or negative imaginary axis. After collecting similar terms and using symmetry properties in τ one obtains in the one-channel case

$$C^l(E;q,q')=2\pi\kappa\sum_{r_j>0} r_j\tilde{Q}^{l+1/2}_{r_j-1/2}(\cosh\phi)\tilde{P}^{-l-1/2}_{r_j-1/2}(\cosh\chi)\left[-\frac{8}{\sqrt{3}}(-1)^l\frac{\partial\ln L_l}{\partial\tau}\bigg|_{\tau=ir_j}\right]^{-1}, \quad p\geq p' \tag{4.9}$$

where we introduce reduced Legendre functions

$$\tilde{Q}^{-l-1/2}_{i\tau-1/2}(\cosh\phi)=Q^{-l-1/2}_{i\tau-1/2}(\cosh\phi)(\sinh\phi)^{1/2}. \tag{4.10}$$

For $p'>p$ the roles of $\tilde{Q}$ and $\tilde{P}$ are interchanged. This does not mean that $C(E;q,q')$ has a singularity in p' for $p'=p$; it simply means that as an analytic function of q', $C(E;q,q')$ has different pole representations in the region $p\geq p'$ and $p'>p$. We can now continue $C(E;q,q')$ to positive energies, keeping q^2 and q'^2 negative in order to avoid the singularities at $q=0$ or $q'=0$. We have for $\mathrm{Im}E>0$

$$\phi=\frac{i\pi}{2}+\ln\{p(\tfrac{4}{3}E)^{-1/2}+[p^2/(\tfrac{4}{3}E)-1]^{1/2}\}, \quad E\leq\tfrac{3}{4}p^2. \tag{4.11}$$

Next we continue to $E>\frac{3}{4}p^2$, hence to positive values of the subenergy σ (Ref. 36)

$$\phi=i\sin^{-1}[p(\tfrac{4}{3}E)^{-1/2}], \quad \mathrm{Im}\sigma>0 \tag{4.12a}$$

$$\phi=i\pi-i\sin^{-1}[p(\tfrac{4}{3}E)^{1/2}], \quad \mathrm{Im}\sigma<0, \quad E\geq\tfrac{3}{4}p^2. \tag{4.12b}$$

The two different expressions correspond to different branches of the second square root function in (4.11). Since the "off-shell" subenergy σ'' in (4.3) has a positive imaginary part, the off-shell dependence will be described through relation (4.12a). The negative imaginary part in the "on-shell" subenergy σ in (4.3) is not operative, as the on-shell value is either zero ($q^2=0$) or negative ($q^2=-\alpha^2$) so that the on-shell dependence is described through (4.11). For $\sigma>0$ we can continue the expression (4.9) straightforwardly into the whole region $0<\sigma'<E$ using (4.12a); however, if $\sigma=0$ the function C^l has a logarithmic singularity for $p'=\frac{1}{2}p$, which manifests itself in a divergence of the sum in (4.9). For $p'<\frac{1}{2}p$, the right-hand side of (4.9) represents the first branch of the multivalued function C^l.

If $\sigma>0$ the logarithmic singularity originally located at $p'=\frac{1}{2}p$ opens up into two singularities. These logarithmic branch points locate the region where the $3\to3$ one-particle exchange process is physical, i.e., they represent physical thresholds. They are usually considered as singularities in the full energy E, and will be discussed briefly in the next section. With the present choice of our on-shell momenta we succeeded in an almost complete separation of these logarithmic singularities from the square root singularities in the subenergies. For $p'\to0$, $\tilde{P}^{-l-1/2}_{r_j-1/2}(\cosh\chi)\sim p'^{l+1}$, which appears to guarantee the required threshold behavior of the function C^l for $p'\to0$. However, as we will see shortly one is not always allowed to take the limit for $p'\to0$ under the summation sign, so that the proof of this analytic property is not completely trivial.

Before studying the off-shell behavior of C^l quantitatively let us make two remarks. First, the functions $\tilde{P}^{-l-1/2}_{r_j-1/2}(\cosh\chi)$ are solutions of Eq. (4.3) for all j; the appropriate linear combination of these in (4.9) is determined by the inhomogeneous term in Eq. (4.5). Second, the unitarity equations (2.11) contain two amplitudes: $A_{ij}(E+i\epsilon;\sigma-i\epsilon,\sigma'+i\epsilon)$ and $A_{ji}(E-i\epsilon;\sigma'-i\epsilon,\sigma+i\epsilon)$. Since these amplitudes are each other's complex conjugates, we find the off-shell function for the second amplitude simply by complex conjugation of the off-shell function of the first amplitude, in accordance with Eq. (3.16).

In the quartet case ($\lambda=-\frac{1}{2}$) we have to solve the following equation:

$$L_l(ir_j)=-(-1)^l\sqrt{3}/4. \tag{4.13}$$

Since the influence of three-particle unitarity is expected to be largest in the S-wave case we concentrate on this case, so that

$$\frac{1}{r_j}\frac{\sin(\pi r_j/6)}{\cos(\pi r_j/2)}=-\frac{\sqrt{3}}{4}, \tag{4.14}$$

which has roots $r_j=2, 2.16622, 5.12735, \ldots$ for $j=0,1,2,\ldots$. For large j one can show that

$$r_j=2j+1+\frac{a_j}{\pi\sqrt{3}\,j}+O(j^{-2}), \tag{4.15}$$

where $a_j=2$ for $j\neq3h+1$, and $a_j=-4$ for $j=3h+1$. In our actual calculations of r_j we have used an expression up to order j^{-4}. We warn the potential user of these formulas that there are some tricky convergence problems in determining the amplitude near $p'=0$.

In Fig. 3 we show the results of a calculation of the off-shell amplitude for $q=0$ $[p=(\frac{4}{3}E)^{1/2}]$ and

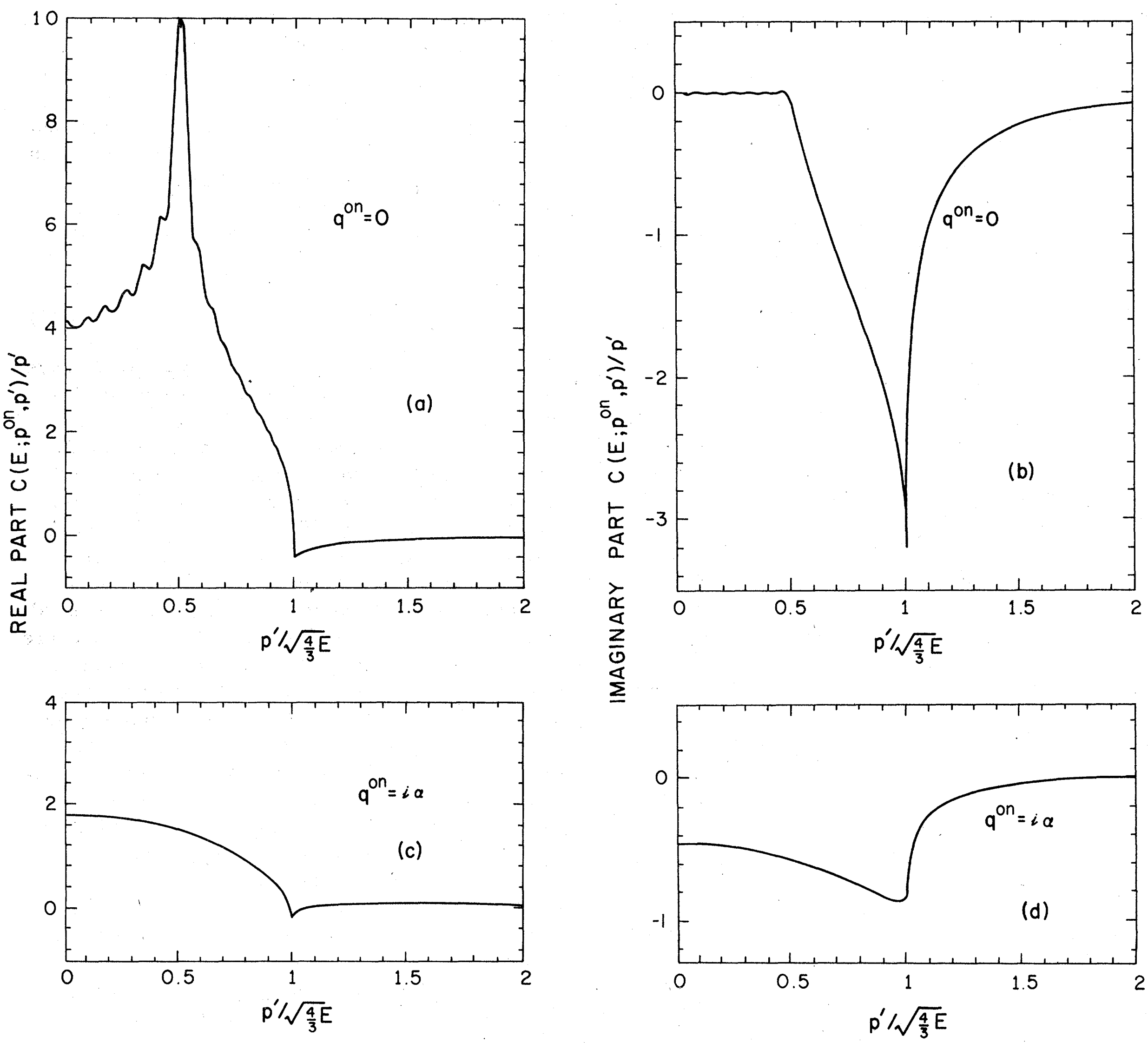

FIG. 3. Behavior of the S-wave quartet half-shell amplitudes for $q^{\text{on}}=0$ and $q^{\text{on}}=i\alpha$. The number of poles included in this calculation is 40. The amplitudes were calculated for a lab energy of 14.4 MeV.

$q=i\alpha$, where α^2 represents the deuteron binding energy. Only the p' dependence in the region $0\leq p'\leq(4E/3)^{1/2}$ enters the unitarity equations, but we have shown a larger region to illustrate the behavior of the amplitude near the square root singularity at $p'=(4E/3)^{1/2}$. In Fig. 4 we show the off-shell function calculated from this amplitude for $q=i\alpha$ and $q=0$. The energy (E) dependence of the off-shell function only enters via the on-shell value of $p^{\text{on}}=[\frac{4}{3}(E+\alpha^2)]^{1/2}$, i.e., it only determines the (complex) normalization of the off-shell function. The off-shell functions have been determined by replacing the form factors $g(p)$ in Eq. (4.2) by constants, in accordance with the zero-range approximation (4.4). The use of Yamaguchi form factors would slightly decrease the magnitude of χ, especially for $p'=0$ and large E. If $E\to\infty$, $p^{\text{on}}/(\frac{4}{3}E)^{1/2}\to 1$, and we get the off-shell function shown in Fig. 4(a). In Sec. III we found that for high energies the main contribution to the exchange part of the phase-space integral (3.7b) comes from the product $A_{ij}(E;p,(4E/3)^{1/2})A^*_{ij}(E;p,(E/3)^{1/2})$, assuming that the amplitude is not strongly varying over the region of integration. Now we see that the corresponding amplitude A in Fig. 3(a) is enhanced near the momentum $p'=(E/3)^{1/2}$ due to the logarithmic singularity, so that the dominance of this region is even more pronounced. It therefore seems to be a very good approximation to replace the exchange integral

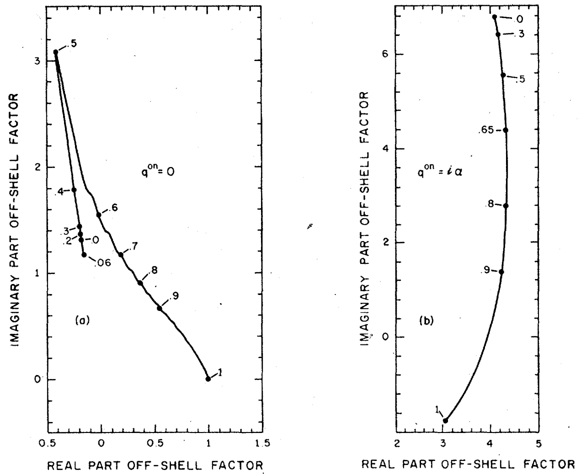

FIG. 4. Quartet off-shell functions corresponding to the half-shell amplitudes in Fig. 3. The numbers along the curve represent the value of p' in units of $(4E/3)^{1/2}$.

for higher energies by a sum of two terms near these momenta and to neglect the remainder of the integral. The logarithmic singularity does not invalidate this procedure since we can easily integrate it. We can then replace the full unknown off-shell behavior by one unknown (energy-dependent) coefficient of the logarithmic singularity which again can be calculated in terms of a zero-range model. Alternatively, we could introduce this coefficient as an unknown in the N/D equations, although this requires modifications in these equations because of the logarithmic singularities in the physical region.

The present analysis favors an off-shell function whose magnitude varies between 4 and 7 [Fig. 4(b)]. This is much larger than the unit off-shell function which gave good results in Ref. 10. In order to check whether this magnitude is typical for a larger class of interactions, we also analyzed amplitudes obtained by Bruinsma in a breakup calculation of the three-nucleon system at 14.1 MeV, in which he used a one term separable potential with charge dependent parameters. The two calculations agree in the sense that the amplitudes are fairly constant in the region $0<p'<(E/3)^{1/2}$, but for $(E/3)^{1/2}<p'<(4E/3)^{1/2}$ our amplitudes *decrease* slowly, whereas in Bruinsma's calculation they *increase* slowly. The main difference occurs, however, if we consider the off-shell function, since in Bruinsma's calculation the amplitude increases sharply if $p>(4E/3)^{1/2}$. As a result off-shell functions computed from his amplitudes never exceed 0.25, and therefore do not agree at all with our results. Other quartet Faddeev calculations have to be analyzed in order to determine whether this difference is due to the inadequacy of the zero-range approximation, or to the sensitivity of the off-shell behavior of the scattering amplitudes to the assumption of charge dependence. We return to this question in the doublet case. In any case it seems necessary to introduce a second "on-shell" momentum at $q=0$, in order to avoid the uncertain extrapolation to the on-shell momentum $q=i\alpha$.

In the doublet case the integrand has poles for τ_j satisfying Eq. (4.13) and the following equation:

$$L_l(\tau_j)=(-1)^l\sqrt{3}/8\,. \tag{4.16}$$

This latter condition is the same as we would have had in the spinless case. In the S-wave case the purely imaginary solutions $\tau_j=ir_j$ of (4.16) are $r_j=4$, 4.46529, 6.81836, ... For large j the behavior of r_j is given by Eq. (4.15), with $a_j=8$ for $j=3h+1$, and $a_j=-4$ for $j\neq 3h+1$. There is one real solution to (4.16), namely $\tau_0=1.006238$. This solution is the source of the ambiguity in the S-wave zero-range problem, since it corresponds to a solution of the homogeneous integral equation. Beloozerov[31] uses Danilov's[37] procedure to resolve this ambiguity. The procedure essentially determines the coefficient of the homogeneous solution by requiring that the asymptotic behavior of the total solution for $p'\to\infty$ is the same as that of the solution of a set of finite range equations in which terms linear in the range parameters are kept and in which the triton binding energy acts as a parameter determined by experiment. This procedure seems to be rather *ad hoc*, and therefore it is of interest to investigate recent attempts to formulate unambiguous zero-range theories.[38]

Apart from the *ad hoc* nature of the solution of this ambiguity, the doublet solution of Beloozerov has some severe defects. First, the solution does not have a pole at the triton binding energy despite the fact that the triton binding energy has been used as an input parameter. Second, the solution of Beloozerov appears to be singular in $p=p'$ for positive energy. The presence of this singularity might indicate that the sum, representing the amplitude, has been extended to a region where it is no longer valid. Since the present state of the art, therefore, does not allow us to obtain a satisfactory, unambiguous full solution of the Skornyakov-Ter Martirosian equations, we have used only the homogeneous solution of the equations. We expect this choice to be quite reasonable, since on the one hand the homogeneous solution seems to dominate the momentum dependence of the amplitude in the case considered by Beloozerov, whereas on the other hand the solution satisfies the FSI equations, and has the required analytical properties for $p' \to 0$ and $p' \to (\frac{1}{3}E)^{1/2}$. The absence of the logarithmic singularities in this amplitude does not seem to be too serious, since even for Beloozerov's solution the singularity hardly shows up; in addition the singularity will be further washed out in the off-shell functions which also depend on the other amplitudes which are free of these singularities. The resulting off-shell matrix is diagonal, and the elements are given by

$$\chi_{ij}(E;p_j)=\delta_{ij}\,\frac{p_j^{\rm on}\sin(\tau_0\phi_j)g_j(E-\frac{3}{4}p_j^{2})}{p_j\sin(\tau_0\phi_j^{\rm on})g_j[E-\frac{3}{4}(p_j^{\rm on})^2]}\,, \tag{4.17}$$

where ϕ_j is given by Eq. (4.12a), and $\phi_j^{\rm on}$ is related to $p_j^{\rm on}$ by Eq. (4.11). In Fig. 5 we show the form of this off-shell function for $j=1$ and $j=2$. We do not show the amplitude itself since we do not known the (complex) normalization of the homogeneous solution, and a plot of the real and imaginary part of the amplitude requires the knowledge of the phase of this coefficient. If one accepts the diagonal nature of the off-shell function as a good approximation then one can calculate this function from a single off-shell amplitude since $A_{ij}(E;p_j) = A_{ij}(E;p_j^{\rm on})\chi_{jj}(E;p_j)$. In that case we can compare our calculation of the triplet off-shell function χ_{11} with the calculations of Bruinsma and Brady and Sloan.[39] The off-shell function derived from Bruinsma's amplitudes has approximately a constant phase as ours does [Fig. 5(a)], however its magnitude is smaller by roughly a factor 6. Using the amplitudes of Brady and Sloan we find the opposite, the phase varies between $-45°$ and $45°$, and the magnitude is about a factor 4 larger than ours, i.e., a factor 24 larger than Bruinsma's.

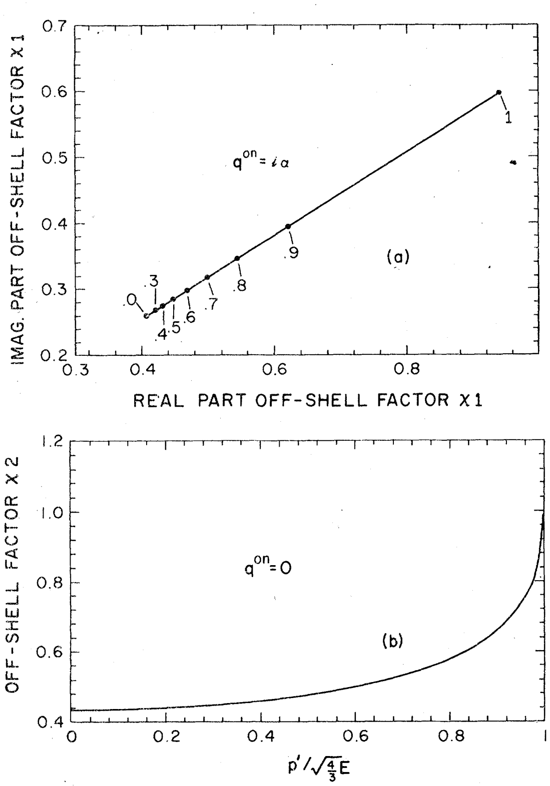

FIG. 5. S-wave doublet off-shell factors at a lab energy of 14.4 MeV. The meaning of the numbers in Fig. 5(a) is the same as in Fig. 4. The second off-shell factor χ_2 is real [Fig. 5(b)].

The major uncertainty in the off-shell function seems to originate from the extrapolation from physical momenta q to the on-shell momentum $q = i\alpha$. This uncertainty would be eliminated if a second triplet "on-shell" amplitude is introduced at $q=0$ this amplitude and the singlet amplitude will have the same off-shell function for the zero-range model (4.17).

The singlet amplitudes cannot easily be compared with Bruinsma's calculation as he uses the on-shell value $q=i\alpha$. If one ignores this difference, one finds in both cases that the single off-shell function drops sharply near $p'=(4E/3)^{1/2}$. However, the behavior for $p'\to 0$ is quite different as Bruinsma's off-shell function levels off to a value of -0.4 in the np-singlet case, and -1 in the pp-singlet case, whereas we found 0.4 [Fig. 5(b)].

Even if one introduces the "on-shell" amplitude at $q=0$, the present analysis shows strong differences in off-shell function originating from different models. Therefore, it may even be necessary (and further calculations have to show this) to introduce another "on-shell" amplitude, and the high energy analysis of the previous section has shown that $p=(E/3)^{1/2}$ is the most obvious choice.

A more radical procedure would be to parametrize the three-body phase-space factor directly using its asymptotic behavior near $E=0$ ($\sim E^2$) and the mild behavior for larger energies (cf. Ref. 10). The parameters in such a model could both come from microscopic sources as in the previous sections and from experimental information.

V. DYNAMICAL SINGULARITIES OF THE AMPLITUDES

The left-hand or dynamical singularities of the bound-state three-body amplitudes have been discussed before,[10] and therefore we will concentrate on the $2\to 3$ and $3\to 3$ amplitudes in the present section. The lowest order exchange graph is pictured in Fig. 6. It has the general form

$$\begin{aligned}A_{ij}^{(1)}(E;\cos\theta) \sim 2\lambda_{ij}\, & g_i(\vec{p}'+\tfrac{1}{2}\vec{p})\\ &\times(E-p^2-p'^2-pp'\cos\theta)^{-1}\\ &\times g_j(\vec{p}+\tfrac{1}{2}\vec{p}')\,,\end{aligned} \tag{5.1}$$

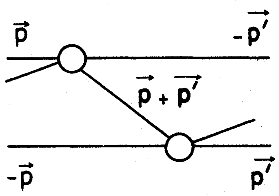

FIG. 6. One-nucleon exchange graph.

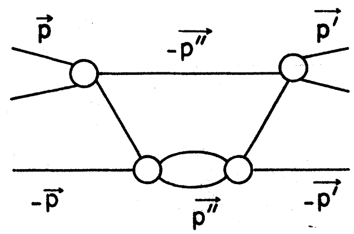

FIG. 7. Repeated one-nucleon exchange graph.

where λ_{ij} is the spin-isospin recoupling coefficient, and $\cos\theta = \hat{p}\cdot\hat{p}'$. The factor 2 is due to the presence of two such diagrams. For the $3\rightarrow 3$ amplitude the "on-shell" momenta are given by $p = p' = (4E/3)^{1/2}$, so that the E dependence in the propagator in $A_{ij}^{(1)}$ factorizes, and the amplitude has a pole for $E=0$. The pole can be considered as a contraction of the left-hand cut of the bound-state amplitude in the limit $\alpha\rightarrow 0$, or, alternatively, as the contraction of the positive energy cut of the $3\rightarrow 3$ amplitude between $\frac{4}{3}\sigma$ and 4σ, when $\sigma\rightarrow 0$. Since this pole term is much easier to handle than the logarithmic branch points, our choice for the "on-shell" momenta reduces the complexity of the calculation of the input considerably. The E^{-1} singularity of the amplitude does not endanger our N/D description since it is compensated by the E^2 behavior of the phase-space factor near $E=0$.

The second-order amplitude has the following structure (see Fig. 7):

$$A_{ij}^{(2)}(E,\cos\theta) \sim \sum_k \int d\vec{p}''\, g_i(\vec{p}''+\tfrac{1}{2}\vec{p})\, \frac{\lambda_{ik}}{E-p''^2-p^2-\vec{p}\vec{p}''}\, g_k(\vec{p}+\tfrac{1}{2}\vec{p}'')S_k(E-\tfrac{3}{4}p''^2)g_k(\vec{p}'+\tfrac{1}{2}\vec{p}'')$$
$$\times\frac{\lambda_{kj}}{E-p''^2-p'^2-\vec{p}''\vec{p}'}\, g_j(\vec{p}''+\tfrac{1}{2}\vec{p}'), \qquad (5.2)$$

where S_k is the dressed propagator of the intermediate three-particle state. The Amado model allows a simple analytical expression for this propagator. For $A>3$ systems, this propagator becomes a complicated convolution of several propagators (cf. Fonseca[40] for a calculation in the $A=4$ system). Neglecting for the moment the form factors (i.e., by letting $\beta\rightarrow\infty$) and by putting the momenta on-shell one obtains for the lth partial-wave $3\rightarrow 3$ amplitude

$$A_{ij}^{l(2)}(E) \sim \frac{(E+E_0)^l}{E^{l+1/2}} \sum_k \int_0^\infty dx\left[Q_l\left(\frac{-1-4x^2}{4x}\right)^2\right] \times S_k(E(1-x^2))\,, \qquad (5.3)$$

where we factored out the main E dependence by setting $p''^2 = \frac{4}{3}Ex^2$. Apparently $A_{ij}(E)$ is less singular at the origin than the first-order amplitude, since it has a square root singularity for $l=0$. In order to obtain the left-hand projection of the second-order amplitude one subtracts the right-hand projection of (5.3). The right-hand singularities are due to the propagator $S_k \times (E(1-x^2))$. If one of the intermediate two-particle states supports a bound state then the corresponding S_k has a pole singularity which leads to a very simple discontinuity. The three-particle cut of S_k, however, leads to a more complex discontinuity, and to subtract out this contribution a two-dimensional integral has to be evaluated. The method for doing this is essentially the same as in the bound-state case and has been described before.[10]

The third-order Born term does not have a left-hand cut due to the propagators. Therefore, the only additional singularities are due to the form factors and combinations of them with the propagators. All these singularities lie far to the left, as their position is scaled by β^2.

The nondiagonal $2\rightarrow 3$ amplitudes do not have the simple pole or square root singularities at $E=0$, but have the usual left-hand cut which runs from the physical to the unphysical sheet (see Fig. 8). The position of the branch points is $E = -\frac{4}{3}\alpha^2$. These nondiagonal amplitudes will require a deformation of the integration contour if they occur in integrals running from $E=-\alpha^2$ to ∞. This occurs for example in the right-hand projection of the second-order amplitude (5.3) if k represents an intermediate two-body state (e.g. the nucleon-deuteron state). Deformations are also necessary

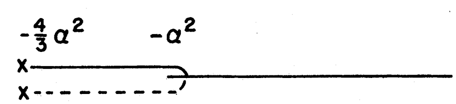

FIG. 8. Anomalous cut of the $2\rightarrow 3$ amplitude.

in the N/D equations, as we will discuss in the next section.

VI. MODIFIED MULTICHANNEL N/D EQUATIONS

In the N/D equations external two- and three-particle channels play a different role because the analytic structure of the $2\to 2$, $2\to 3$, and $3\to 3$ amplitudes is quite different. In order to retain the elegant matrix formulation of the multichannel equations we will, however, use the same notation for the two kinds of channels, and indicate the difference by referring either to the channel set 2 or 3. We will assume that the kinematic conditions are such that no anomalous thresholds arise from nondiagonal bound-state amplitudes. This seems to cover a fair number of cases, for example the $A=3$ and 4 systems can be described this way (at least if the occurring binding energies are close to the experimental values). Nondiagonal $2\to 3$ amplitudes necessarily give rise to anomalous thresholds. We treat these with the same methods as used earlier in the case of two-body bound-state channels (cf. Ref. 32). It is necessary to use numerator functions N_{ij} which for $i\in 2$ are discontinuous at a characteristic point $l_R^{(i)}$. These functions are finite at the two-body thresholds E_i, so that all the following integrals are well defined.

Generalizing the formulation of Ref. 32 to the case of more than two channels and nondiagonal phase-space matrices we obtain the following set of equations:

$$N_{ij}(E)=B^T_{ij}(E)-\sum_{k,m}\int_{E_K}^{\infty} dE' K^T_{ik}(E,E')\rho_{km}(E')N_{mj}(E'), \tag{6.1}$$

where

$$B^T_{ij}(E)=B^{\rm red}_{ij}(E)-L_{ij}(E)-M_{ij}(E)+J_{ij}(E). \tag{6.2}$$

The extra functions in (6.2) are integrals over the various anomalous intervals:

$$L_{ij}(E)=\sum_{k\in 2}\int_{l_>^{(k)}}^{l_R^{(k)}} dE' K_{ik}(E,E')\rho_k(E')\Delta B_{kj}(E'), \quad j\in 3 \tag{6.3}$$

where

$$K_{ik}(E,E')=\frac{B_{ik}(E)-B_{ik}(E')}{E-E'}, \tag{6.4}$$

and ΔB_{kj} is the discontinuity of B_{kj} over the anomalous cut

$$M_{ij}(E)=\sum_{k\in 2}\int_{l_>^{(k)}}^{l_R^{(k)}} dE'\Delta B_{ik}(E')\frac{\rho_k(E')}{E'-E}B_{kj}(E'), \quad i\in 3 \tag{6.5}$$

and

$$J_{ij}(E)=\sum_{k\in 2}\int_{l_>^{(k)}}^{l_R^{(k)}} dE'\Delta B_{ik}(E')\frac{\rho_k(E')}{E'-E}L_{kj}(E'), \quad i,j\in 3. \tag{6.6}$$

In the last integral $L_{kj}(E')$ has no singularities at the anomalous interval since $k\in 2$, so that the kernel in Eq. (6.3) is regular.

The functions $B_{kj}(E)$ and consequently the kernels $K_{kj}(E,E')$ are discontinuous at $E=l_R^{(k)}$, provided $k\in 2$ and $j\in 3$. The superscript "red" in Eq. (6.2) only pertains to second-order input functions $B^{(2)}_{ikj}$ with i, $j\in 3$, and $k\in 2$. These reduced amplitudes are defined as the average of the two possible continuations of $B^{(2)}_{ikj}$ from the normal to the anomalous case. This implies that $B^{(2)\rm red}_{ikj}$ does not involve integrals over the product $\Delta B_{ik}\Delta B_{kj}$. The introduction of "reduced," and discontinuous input functions is necessary to invoke certain cancellations in the anomalous N/D equations rigorously, so that the resulting integral equations can be solved numerically. For a detailed description of the analytic continuation of the N/D equations from the normal to the anomalous case we again refer the reader to Ref. 32.

The kernel in Eq. (6.1) can be expressed in terms of the modified input functions $B^T_{ij}(E)$ through the equation

$$K^T_{ik}(E,E')=\frac{B^T_{ik}(E)-B^T_{ik}(E')}{E-E'}+\frac{(M_{ij}(E)-B_{ij}(E))H_{jk}(E')-\{{}^{E\,\to\,E'}_{i\,\to\,k}\}}{E-E'}, \tag{6.7}$$

where j runs over the set 2. The function $H_{ij}(E)$ is defined by

$$H_{jk}(E)=\int_{l_>^{(j)}}^{l_R^{(j)}} dE'\,\frac{\rho_j(E')\Delta B_{jk}(E')}{E'-E}, \quad j\in 2,\quad k\in 3. \tag{6.8}$$

The functions L, H, and M satisfy the following relationship for $E>E_j$:

$$L_{ij}(E)=-\sum_k B_{ik}(E)H_{kj}(E)+M_{ji}(E), \quad j\in 3. \tag{6.9}$$

The anomalous integrals and the kernels are real in the region of interest. Equations (6.1) are, however, complex since the off-diagonal three-body phase-space factors ρ_{km} are complex in general. Since the three-body phase-space factors satisfy

$$\rho_{km}^*(E) = \rho_{mk}(E)\,, \tag{6.10}$$

the amplitudes A_{ij} are still real analytic as is easily inferred from Eq. (3.14) by using time reversal invariance. The complex nature of the N/D equations can be treated exactly, or in an iterative way if the imaginary part of the three-body phase-space factor is small with respect to the real part. Once we have solved for the numerator function N_{ij} in Eq. (6.1) we can determine the analytic continuation of $N_{ij}(E)$ below $E = l_R^{(i)}$ for $i \in 2$ through the relation $N_{ij}^{\mathrm{AC}}(E) = N_{ij}(E) + \Delta N_{ij}(E)$ (AC ≡ analytical continuation) where

$$\Delta N_{ij}(E) = \sum_{k\in 3} \Delta B_{ik}(E) D_{kj}(E)\,, \quad i \in 2\,. \tag{6.11}$$

The D functions are determined through a simple quadrature formula:

$$D_{ij}(E) = \delta_{ij} + \sum_k \int_{E_k} \frac{\rho_{ik}(E')}{E' - E} N_{kj}^{\mathrm{AC}}(E') + \int_{l_>^{(i)}}^{E_i} dE' \frac{\rho_i(E')}{E' - E} \Delta N_{ij}(E')\,, \tag{6.12}$$

where the last term only occurs if $i \in 2$. The time reversal and reality properties of the resulting amplitude give us a means of checking the numerical accuracy of the solution.

If a two-body channel gives rise to a bound-state amplitude with an anomalous threshold singularity (for example if the two-body channel contains a loosely bound particle) then the summations in Eqs. (6.3)–(6.6) should include this channel, in other words the channel is effectively a three-body channel. In writing down Eqs. (6.1)–(6.7) we assumed that the functions $B_{ij}(E)$, i, $j \in 2$ are nonsingular at the anomalous intervals where they have to be evaluated. We can generalize the formalism so as to allow for functions which do not have this property. However, since we think that these singularities are not very important, while they complicate the formalism considerably, we suggest circumventing these singularities in a way similar to the method used by Rinat and Stingl[9] for "removing" the anomalous threshold in the five-body problem. Instead of the unsubtracted equations presented here, one can also employ subtracted equations as in Ref. 32. *CDD* poles can also be included. The problem with *CDD* poles is that there is now a straightforward way to determine whether they should be included, and if so what their positions and residues are. Recent calculations[41] for doublet nucleon-deuteron scattering have shown that a *CDD* pole can be necessary in a one-channel N/D description of this system in order to reproduce the zero in the amplitude just above threshold. This special situation is due to the effective range anomaly in the doublet case, and we do not expect similar situations for other few-body systems. The effect of this *CDD* pole seems to be small both in the negative energy region and the positive energy region (where it only gives rise to an extra jump in the phase shift). Like other studies on *CDD* poles, this study[41] concludes that *CDD* poles can arise by neglecting important channels, and in fact can be shown to be absent in the two-channel approach to the three-nucleon system which has been investigated here.

VII. DISCUSSION

The inclusion of three-body unitarity within a dispersionlike description of three-body systems has an important impact on various properties of the system. For example in the zero-range limit we see the breakdown of the Thomas theorem[12] if the three-body system is described as an inert bound state plus a third particle. In fact, one can even show that the inclusion of the exchange part of the three-particle unitarity contribution does not suffice to produce the Thomas theory, since the relevant integral equations remain well defined. From recent studies we know[13] also that the number of bound states of the three-particle system becomes infinite if the scattering length of the two-body system goes to infinity. Again it seems that this so-called Efimov effect is absent if one includes only two-particle unitarity, and describes the input by a second-order box graph. Further investigations of these qualitative properties of the three-body system are currently performed and may shed more light on the role of three-body unitarity.

We know from various calculations[10,42] that three-particle intermediate states are very important in determining the binding energy of the triton. For example the binding energy of the three-boson system considered by Aaron, Amado, and Yam[42] reduces from 25.5 to 4.5 MeV if one neglects three-body intermediate states. Using two-body unitarity in the N/D equations for this system with pole and triangle input (neglecting intermediate three-body states), one also gets 4.5 MeV binding.[48] In the more realistic doublet calculations of Ref. 10 we found binding energies of respectively 7.340 and 2.227 MeV, the latter one only a few keV below the dueteron threshold. However, from the latter calculations it is clear that the increase in the binding energy comes nearly exclusively from the inclusion of the intermediate three-body states in the input. Still, it seems to be more consistent to treat the unitarity equations and the input to the same order, and therefore include three-particle unitarity in such calculations. It is also clear from the calculations in Ref. 10 that three-body unitarity has an important effect on the positive energy behavior of the amplitudes, especially on the absorption coefficients.

In Sec. IV we analyzed the off-shell functions which play such an important role in the microscopic calculation of the three-body contribution to the phase-space factor. It was hard to make simple and reliable calculations of these functions as they appeared to be quite sensitive to the details of the potentials used in the three-nucleon calculation. It therefore appears to be necessary to introduce "on-shell" amplitudes at $q=0$ for every channel. "On-shell" amplitudes at $p=(E/3)^{1/2}$ would help to further reduce the remaining uncertainty in the off-shell function although they require (standard) regularization schemes to handle the logarithmic singularities on the physical axis. We also mentioned a semiphenomenological approach in which one parametrizes the three-body phase-space factor directly. The latter approach was essentially taken by Bower,[17] and simplifies the calculations considerably, especially if the number of channels increases and the calculation of off-shell amplitudes—even in a zero-range approximation—becomes too messy. This approach is also more in the spirit of the dispersion method, which tries to concentrate on the physical mechanism and the dynamical aspects of the reaction, without going into details of the underlying potentials. Further numerical applications of the N/D method have to show whether such a semiphenomenological approach is successful. The three-nucleon system would still be the most obvious test system. Cases in which the N/D approach may be more useful than conventional potential approaches are the three-nucleon system, if one includes certain pionic diagrams, and few-body systems with $N \geq 4$, for which Faddeev calculations are still very complicated. The general framework developed in this paper seems equally well applicable to few-nucleon systems with $N \geq 4$. The analysis of Secs. III and IV has to be modified slightly, since the mass ratios and therefore the position of various singularities is different. For example, the angle θ introduced in Eq. (4.7) will equal $\sin^{-1}(1/\sqrt{3})$ in the four-body system.

The formalism in this paper was nonrelativistic, because we had in mind the application to the three-nucleon system. The generalization of N/D theory to the relativistic case is possible and was performed in the framework of the helicity formalism for aribtrary spins in a paper by Polikarpov.[43] The only important change occurs in the form of the relativistic phase-space factors. Until now practically all investigations of relativistic three-body systems have been confined to resonance-particle systems of a quasi-two-body type (cf. Refs. 15, 16, 23, and 44), with the exception of a recent schematic investigation of Brayshaw.[45] One of the most important applications of the relativistic N/D method—πd scattering in the resonance region—can be done in the present formalism with a minor change in the three-body phase-space factor due to relativistic pion kinematics.

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APPENDIX: PROPERTIES OF SPECIAL LEGENDRE FUNCTIONS

In Sec. IV we encountered special Legendre functions of the type

$$P_{i\tau-1/2}^{-l-1/2}(z) \text{ and } Q_{i\tau-1/2}^{-l-1/2}(z).$$

We only need explicit expressions for $l=0$ and 1 (Ref. 46, 8.6.8–8.6.11):

$$P_{i\tau-1/2}^{1/2}(\cosh\theta)=\left(\frac{2}{\pi}\right)^{1/2}\cos(\tau\theta)\sinh\theta^{-1/2}, \tag{A1}$$

$$P_{i\tau-1/2}^{-1/2}(\cosh\theta)=\left(\frac{2}{\pi}\right)^{1/2}\sin(\tau\theta)\sinh\theta^{-1/2}, \tag{A2}$$

$$Q_{-i\tau-1/2}^{1/2}(\cosh\theta)=\left(\frac{\pi}{2}\right)^{1/2}ie^{i\tau\theta}\sinh\theta^{-1/2}, \tag{A3}$$

$$Q_{-i\tau-1/2}^{-1/2}(\cosh\theta)=-\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\tau}e^{i\tau\theta}\sinh\theta^{-1/2}, \tag{A4}$$

where $|\mathrm{Im}\theta|<\pi$ and $|\mathrm{Im}(\sinh\theta)|<\pi$. For $-1\leq z\leq 1$ one uses Eqs. 8.6.12–8.6.14 in Ref. 46. From the recursion relations (Ref. 47, 8-733-3 and 8-731-3)

$$(l^2+\tau^2)P_{i\tau-1/2}^{-l-1/2}(\cos\theta)=P_{i\tau-1/2}^{-l+3/2}(\cos\theta) -(2l-1)\cot\theta P_{i\tau-1/2}^{-l+1/2}(\cos\theta), \tag{A5}$$

$$(l^2+\tau^2)P_{i\tau-1/2}^{-l-1/2}(\cosh\theta)=-P_{i\tau-1/2}^{-l+3/2}(\cosh\theta)+(2l-1) \times\coth\theta P_{i\tau-1/2}^{-l+1/2}(\cosh\theta), \tag{A6}$$

which also hold for the Q's, one can now derive the Legendre functions for any l value.

For example, it is easy to derive that

$$P^{-1/2-2n}_{i\tau-1/2}(0) = \frac{1}{(2^2+\tau^2)(4^2+\tau^2)\cdots(4n^2+\tau^2)} \times\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\tau}\sinh\left(\frac{\tau\pi}{2}\right),$$

and

$$P^{+1/2-2n}_{i\tau-1/2}(0) = \frac{1}{(1+\tau^2)(3^2+\tau^2)\cdots((2n-1)^2+\tau^2)} \times\left(\frac{2}{\pi}\right)^{1/2}\cosh\left(\frac{\tau\pi}{2}\right),$$

which functions are required in the evaluation of $L_l(\tau)$.

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