Semiempirical nuclear level density formula with shell effects

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(Received 15 June 1977)

A new semiempirical nuclear level density formula is proposed, which takes into account the influence of nuclear shell structure on level densities and the excitation energy dependence of shell effects. The ground state shell and pairing energies enter directly into this formula, which involves three mass-independent parameters characterizing the average single-particle level density near the Fermi level and the wavelength of shell oscillations. The present formulation is shown not only to give a good fit to the experimental data on neutron resonance spacings of spherical nuclei, but also to provide a reliable extrapolation to higher excitation energies. The present analysis has also brought out the need to include the dependence of level density parameter a on the surface to volume ratio of nuclei. The analysis of the data for deformed nuclei with the present formulation does not indicate an enhancement of the level densities of the magnitude suggestive of a rotation degree of freedom completely decoupled from intrinsic degrees at the excitation energy equal to neutron binding energies.

[NUCLEAR STRUCTURE Nuclear level densities, semiempirical formula, dependence on surface to volume ratio, shell effects, spherical, deformed nuclei.]

I. INTRODUCTION

Nuclear level densities play a central role in any statistical analysis of nuclear reactions. Experimental information on nuclear level densities as obtained from analysis of neutron and charged particle resonances, inelastic scattering, particle evaporation spectra, etc., extend neither over a wide range of excitation energies nor over all nucleon numbers over the Periodic Table. In view of this, in any practical calculation, one often resorts to theoretical estimates of nuclear level densities. Two different approaches have been employed in the past for theoretical estimates of nuclear level densities. In the first approach, one retains the traditional Bethe expression[1] for level density and the level density parameter a is obtained from available experimental data by suitable interpolation and extrapolation techniques as, for instance, given by Newton[2] and by Gilbert and Cameron.[3] However, since the constants of these formulas are determined from experimental data confined to a narrow range of excitation energies, extrapolation of these formulas to other regions of excitation energy is subject to large errors. It is in fact now well known[4] on the basis of numerical calculations starting from a shell model single-particle energy level scheme that the Bethe form of the level density formula cannot satisfactorily describe the excitation energy dependence of shell effects on nuclear level density by treating a as an energy independent free parameter. In particular, washing out of shell effects on the thermodynamic properties of excited nuclei is a feature, which is now well established but which is not incorporated properly in the existing level density formulas.

In the second and the more recent approach[5] one numerically computes the level density starting from a set of shell model single-particle energy level scheme. This approach takes into account in a natural way the influence of nuclear shell and pairing effects on the level density and its washing out with excitation energy and has found a wider use in the last few years. There exist, however, a few inherent drawbacks in this approach also. First is, of course, the requirement of detailed shell model calculations for all nuclei, resulting in a considerable computational effort. While this, in itself, is not a big constraint, because of a number of single-particle level schemes currently available in literature and easy accessibility of fast computers, this is a step which one will like to gladly dispense with for routine calculations. The second drawback of this procedure is more intrinsic. In any shell model calculation, the quantity on which the calculated level density crucially depends is the density of single-particle states near the Fermi level. This is not a quantity which is crucially adjusted in any calculation of shell model energy level scheme. In fact, differences to the extent of 10–20% are known to exist between the calculated average single-particle density near the Fermi level corresponding to various level schemes currently being used in literature for the calculation of nuclear shell correction energies. Even the ground state shell correction energies

evaluated from different single-particle level schemes are found to be appreciably different. It is therefore necessary to adopt a normalization procedure which ensures that the shell-independent part of the calculated level densities are consistent with a liquid drop model (LDM) estimate and the shell-dependent part is consistent with experimental shell correction energies. In the absence of such a normalization, the calculated level densities from different single-particle level schemes are expected to exhibit appreciable differences, as for example, found by Huizenga *et al.*[6] It may also be pointed out here that normalization to the LDM values is a well established procedure in deformation potential energy calculations and the need for such a normalization in microscopic calculations of nuclear moments of inertia parameters from shell model single-particle level scheme has also been emphasized earlier.[7] Another objection which can be raised[8] against the numerical calculations of level densities from shell model single-particle level scheme is that these calculations of level densities are carried out in the independent particle model approximation. Some earlier studies have in fact shown[8] that if one takes into consideration that the shell model potential is an effective potential generated by two body interactions the calculated level densities as a function of excitation energy differ significantly from those calculated in the independent particle approximation.

In the present work we propose a method of calculating nuclear level densities in a way which avoids the above mentioned inherent uncertainties of the numerical calculations. Furthermore, this method while retaining the simplicity of the earlier semiempirical methods does not have limitations of the earlier simpler prescriptions, since it takes into account nuclear shell effects in a realistic manner.

II. THEORETICAL OUTLINE OF THE PRESENT APPROACH

It is known that to a good degree of approximation, the spin-dependent level density $\rho(E_x, I)$ is related to the state density $W(E_x)$ by the following expression:

$$\rho(E_x, I) = \frac{(2I+1)\,W(E_x)}{2\sqrt{2\pi}\,\sigma^3(E_x)} \exp\left(\frac{-I(I+1)}{2\sigma^2(E_x)}\right), \tag{1}$$

where $\sigma^2(E_x)$ is spin cut-off factor. The state density $W(E_x)$ is mainly determined by the entropy $S(E_x)$ of the nucleus by the following relation:

$$W(E_x) = \frac{\exp[S(E_x)]}{(2\pi)^{3/2} D^{1/2}} = C \exp[S(E_x)]\ . \tag{2}$$

For a model single-particle scheme with equidistant single-particle states, the following well known relations hold:

$$\begin{aligned} S &= 2(aE_x)^{1/2}, \\ E_x &= aT^2, \\ a &= (\pi^2/6) g_0, \\ D &= (\pi^2/12) g_0^{\,3} T^5, \end{aligned} \tag{3}$$

and

$$C = \frac{\sqrt{\pi}}{12 a^{1/4} E_x^{\,5/4}}\ ,$$

where g_0 is density of the single-particle states.

Further, the spin cutoff factor σ^2 is related to the moment of inertia of the nucleus by the relation

$$\sigma^2 = \frac{JT}{\hbar^2}\ . \tag{4}$$

It is, however, known[3] that if Eqs. (1)–(4) of the equidistant model are used to fit the experimental data on the neutron resonance spacings, the parameter a fluctuates systematically in accordance with the known nuclear shell effects. Once the values of a as a function of mass number are empirically determined in this way, it may be possible to estimate, by a suitable interpolation of a values the neutron resonance spacing in unknown cases. However, since the shell dependence of the thermodynamic behavior of a nucleus is excitation energy dependent, the shell-dependent values of a applicable for excitation energies at neutron resonances cannot be used for the calculation of level densities at other excitation energies.

In the present formulation for the calculation of shell dependent level densities, the parameter a which enters into the expressions for entropy and excitation energy is a shell-independent parameter equal to that of the equivalent liquid-drop model nucleus, and the shell effects on the level density enter through the ground state shell correction energy and the wave length of shell oscillations as described below.

It is known that the single-particle states in nuclei exhibit appreciable fluctuations from the equidistant level scheme and these fluctuations are the source of the observed shell effects in nuclear masses and other observables. We first investigate the influence of these fluctuations on the nuclear level densities. Let us consider only one kind of nucleons, say neutrons. Let $G(\epsilon) = \sum_i \delta(\epsilon - \epsilon_i)$ represent the single-particle density for these nucleons, where ϵ_i is a suitable set of shell states. In the spirit of the macroscopic-microscopic approach[9,10] one can write $G(\epsilon)$ as

$$G(\epsilon) = g(\epsilon) + \delta g(\epsilon)\ ,$$

where $g(\epsilon)$ represents the overall smooth component of the single-particle level density and $\delta g(\epsilon)$ represents the local fluctuation. For a small range of energies around the Fermi energy, $g(\epsilon)$ can be well approximated by a constant value g_0. The fluctuation $\delta g(\epsilon)$ can be represented by a Fourier series expansion as follows:

$$\delta g(\epsilon) = \sum_m g_m \cos(m\omega\epsilon - \phi_m) . \tag{5}$$

An analytical expression for the entropy as a function of the temperature for such a level scheme has been obtained by Gilbert[11] and is given below:

$$S = \tfrac{1}{3}\pi^2 g_0 T + \sum_m \frac{g_m \cos(m\omega\mu - \phi_m)}{m^2\omega^2 T} \times \left(\frac{m^2\omega^2\pi^2 T^2 \cosh(m\pi\omega T)}{\sinh^2(m\pi\omega T)} - \frac{m\pi\omega T}{\sinh(m\pi\omega T)} \right) . \tag{6}$$

The corresponding expression for the excitation energy is

$$E_x = \tfrac{1}{6}\pi^2 g_0 T^2 + \sum_m \frac{g_m}{m^2\omega^2} \cos(m\omega\mu - \phi_m) \times \left(\frac{m^2\pi^2\omega^2 T^2 \cosh(m\pi\omega T)}{\sinh^2(m\pi\omega T)} - 1 \right) . \tag{7}$$

The temperature dependence of the entropy and excitation energy calculated numerically starting from a single-particle scheme has been analyzed earlier[12] with regard to Eqs. (6) and (7). It was found to be a good approximation to neglect the terms with $m > 1$ in Eqs. (6) and (7), since the contribution of the fundamental term is the most predominant. If one also assumes that in the range of temperatures of interest in nuclear reaction analysis the effect of the weak temperature dependence of the chemical potential μ is negligible on the calculation of entropy and excitation energy and μ can be treated as energy independent in Eqs. (6) and (7), one can write Eqs. (6) and (7) as follows:

$$S = \tfrac{1}{3}\pi^2 g_0 T + \frac{A_1}{T} \left(\frac{\pi^2\omega^2 T^2 \cosh(\pi\omega T)}{\sinh^2(\pi\omega T)} - \frac{\pi\omega T}{\sinh(\pi\omega T)} \right) , \tag{8}$$

$$E_x = \tfrac{1}{6}\pi^2 g_0 T^2 + A_1 \left(\frac{\pi^2\omega^2 T^2 \cosh(\pi\omega T)}{\sinh^2(\pi\omega T)} - 1 \right) , \tag{9}$$

where

$$A_1 = (g_1/\omega^2) \cos(\omega\mu - \phi_1) .$$

It follows from Eqs. (8) and (9), that in the limit of large value of T

$$S = \tfrac{1}{3}\pi^2 g_0 T$$

and

$$E_x = \tfrac{1}{6}\pi^2 g_0 T^2 - A_1 .$$

Thus, the quantity A_1 turns out to be the difference between the excitation energy of an equivalent smooth nucleus (i.e., without shell effects) and that of the actual nucleus in the asymptotic limit of high temperatures. The quantity A_1 is, therefore, also equal to the difference in the ground state energies of the actual nucleus and of the corresponding smooth system[12] and, hence, A_1 is to be identified as the ground state shell correction energy Δ_s.

The validity of the simplifying assumptions leading to Eqs. (8) and (9) have been examined by a comparison of the predictions of these equations with the exact numerical calculations. It is shown in the Appendix I that the simplifying assumptions are justified. Although Eqs. (8) and (9) were obtained for one component system, the same equations hold for a two component system like a nucleus provided S, E_x, g_0, and Δ_s refer to the nucleus as a whole, and the parameter ω refer to a suitable average for the neutron and proton components. The quantity g_0 is related to the LDM value of the level density parameter a as

$$a = \tfrac{1}{6}\pi^2 g_0 .$$

Equations (8) and (9) can therefore be written as

$$S = 2aT + \frac{\Delta_s}{T} \left(\frac{\pi^2\omega^2 T^2 \cosh(\pi\omega T)}{\sinh^2(\pi\omega T)} - \frac{\pi\omega T}{\sinh(\pi\omega T)} \right) \tag{10}$$

and

$$E_x = aT^2 + \Delta_s \left(\frac{\pi^2\omega^2 T^2 \cosh(\pi\omega T)}{\sinh^2(\pi\omega T)} - 1 \right) . \tag{11}$$

If one follows the usual assumption based on Fermi gas model then g_0 and, therefore, a is proportional to the mass number A of the nucleus, that is, $a = \alpha A$. The constant ω, which is the fundamental frequency of oscillation of $\delta g(\epsilon)$, is characteristic of the major shell spacing λ, in nuclei such that $\omega = 2\pi/\lambda$. It is also known that $\lambda = k/A^{1/3}$, where k is expected to be around 30 to 40 MeV. Hence one can write $\omega = \omega_0 A^{1/3}$ where $\omega_0 = 2\pi/k$ is a mass independent parameter.

Although the preexponential factor C in Eq. (3) can also be modified for shell effects in the same way as has been done above for S and E_x, it is not important since the level densities are predominantly determined by the entropy S. For the sake of simplicity, the value of C given by Eq. (3) can, therefore, be used for the calculations of level densities as has been done in the present work. It

is possible to include the shell dependence of the spin cut-off factor σ^2 in an approximate way by the following empirical relation:

$$\sigma^2 = \frac{\sigma_{\mathrm{LDM}}{}^2 S}{S_{\mathrm{LDM}}} = \frac{J_{\mathrm{RIGID}}}{\hbar^2}\left(\frac{2S}{a}\right). \quad (12)$$

In the above expression the liquid-drop model value of the spin cut-off parameter $\sigma_{\mathrm{LDM}}{}^2$ is multiplied by the ratio of single-particle level density near the Fermi level of the actual system and that of the equivalent smooth system to take into consideration the shell effects on the values of σ^2. It may be noted that Eq. (12) meets the requirement that in the asymptotic limit of high exciation energies, σ^2 should become equal to $\sigma_{\mathrm{LDM}}{}^2$.

Equations (10) and (11) form the basis of the present calculations of the level density versus excitation energy, in which the preexponential factors of Eqs. (1) and (2) are calculated with Eqs. (12) and (3), respectively. The effect of nucleon pairing is accounted by substitution of an effective excitation energy for the true excitation energy: $E_x' = E_x - \Delta_p$, where Δ_p is equal to the pairing energy values of Gilbert and Cameron.[3]

III. COMPARISON WITH EXPERIMENTAL DATA AND DISCUSSION

A calculation of level densities on the basis of Eqs. (10) and (11) requires that the ground state shell correction energies of nuclei be known. The two unknown mass-independent parameters α and ω_0 can then be determined by fitting the calculated level densities to the available experimental data. In the present calculations, "experimental" shell correction energies for different nuclei were substituted for Δ_s. These were obtained as the differences between the experimental binding energy

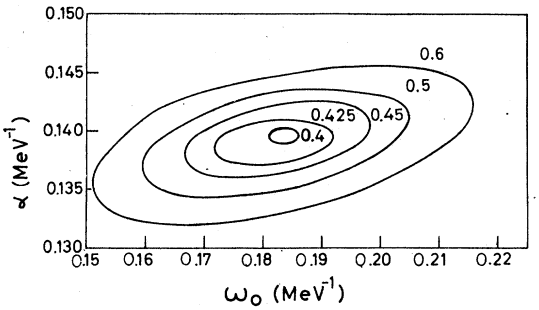


FIG. 1. Contour plot of the mean square deviation $\sigma^2 = \sum (\ln D_{\mathrm{th}}/D_{\mathrm{exp}})^2/N$ for about 100 spherical nuclei as a function α and ω_0. The best fits are obtained with $\alpha = 0.14$ and $\omega_0 = 0.185$.

and the LDM binding energy obtained with parameters of Seeger and Howard.[13] The level spacings were calculated for a range of values of the parameters α and ω_0 to determine the values for the best fit to the experimental level spacings deduced from neutron resonance studies. A value of $r_0 = 1.2$ fm was used for the calculation of J_{RIGID}. The level spacings in eV for one parity is defined by

$$D_{\mathrm{th}}(\mathrm{eV}) = \frac{2\times 10^6}{\rho(E_x, I_0 - \frac{1}{2}) + \rho(E_x, I_0 + \frac{1}{2})} \quad \text{for } I_0 > 0$$

$$= \frac{2\times 10^6}{\rho(E_x, I_0 + \frac{1}{2})} \quad \text{for } I_0 = 0. \quad (13)$$

The experimental level spacings D_{exp} were taken from the compilations of Lynn,[14] Baba,[15] and Dilg *et al.*[16] Since for several cases there is considerable scatter in the available values of D_{exp}, the most favorable value of D_{exp} for any nucleus was used in the least square fitting to theory.

A. Spherical nuclei

Figure 1 shows a contour plot of the calculated mean square deviation, defined as $\sum_N (\ln D_{\mathrm{th}}$

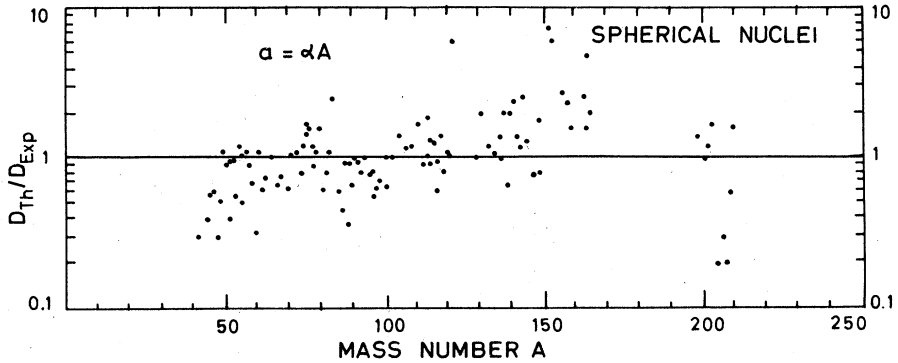


FIG. 2. Plot of $D_{\mathrm{th}}/D_{\mathrm{exp}}$ versus the mass number A. D_{th} was calculated with the expression $a = \alpha A$; and $\alpha = 0.14$ and $\omega_0 = 0.185$. The value of D_{exp} used is the most favorable value out of the three compilations (Refs. 14–16).

$-\ln D_{exp})^2/N$, where N is number of nuclei included in the fittings. The mean square deviations are plotted in this figure against the parameters α and ω_0 and about 100 spherical nuclei were included in the fitting. It is seen that a minimum mean square deviation is obtained for $\alpha = 0.14$ MeV^{-1} and $\omega_0 = 0.185$ MeV^{-1}. This value of ω_0 corresponds to a major shell spacing of about $34.5/A^{1/3}$ MeV in good agreement with the values used in literature.[13] The above value of α which corresponds to a value of a equal to $A/7$, is close to the values of a generally used in the literature.[3] Figure 2 shows a plot of D_{th}/D_{exp} for about 100 spherical nuclei based on the above values of α and ω_0. It is seen that a major fraction of the points are within the range of $0.5 \leq D_{th}/D_{exp} \leq 2$ and no significant systematic structure due to nuclear shell effects is evident, except in the region very close to the doubly magic nucleus ^{208}Pb where some systematic discrepancy seem to be present. It is, however, to be noted that this is precisely the mass region in which nuclei have large shell correction energies,

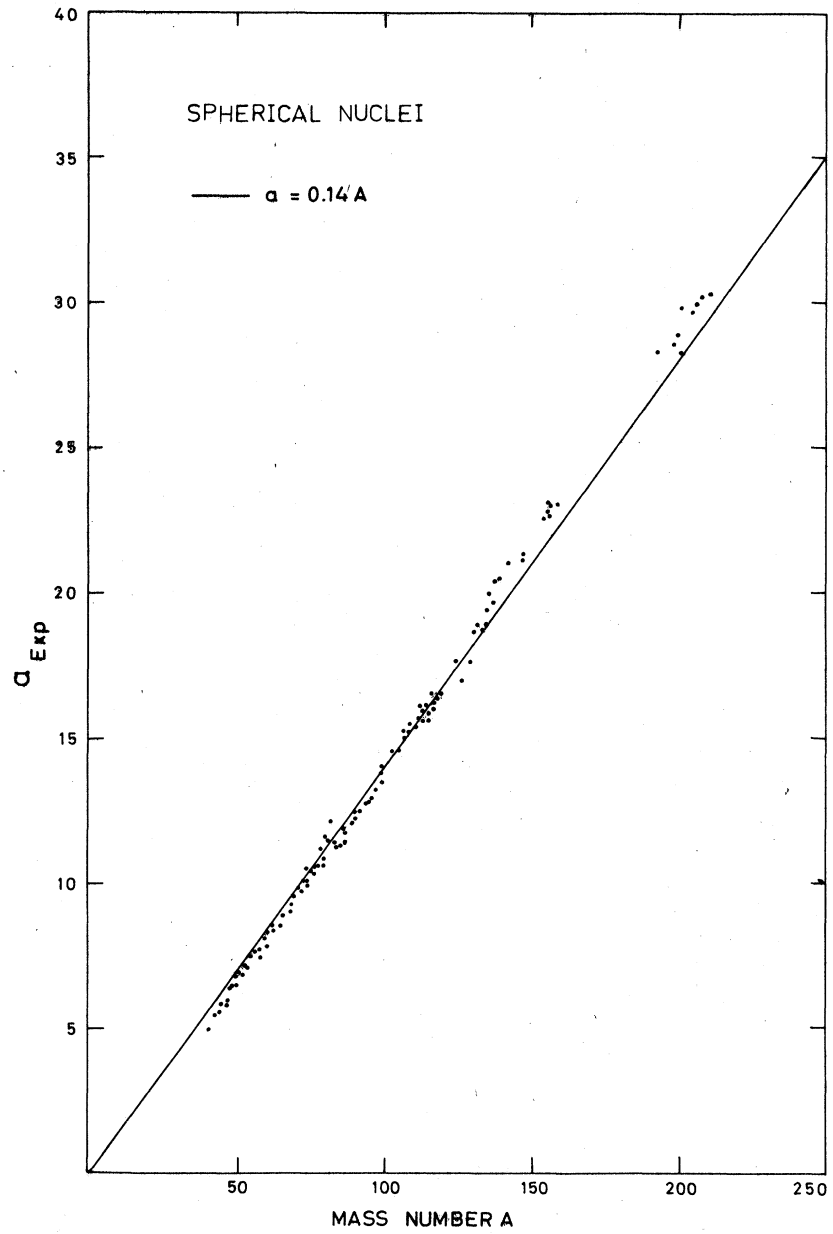


FIG. 3. Plot of the experimental value of a versus mass number A for spherical nuclei, deduced from the data as described in the text. The continuous line corresponds to the straight line $a = 0.14A$.

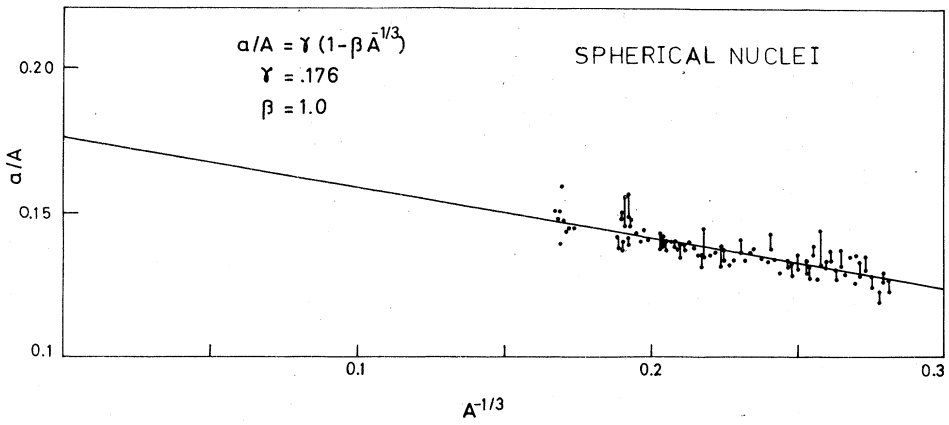


FIG. 4. Plot of a/A versus $A^{-1/3}$. The bars represent the uncertainty in deduced a, due to difference in the compiled values of D_{exp} for the same nucleus. The straight line is the least square fit to the points.

and where some difficulties in fitting even the nuclear masses are encountered.[13] The present simplified formulation is, therefore, adequate to incorporate nuclear shell effects on level densities and the small residual scatter in Fig. 2 can be attributed to the uncertainties in the experimental data and the shell and pairing correction energies estimates.

A small systematic increase in the values of D_{th}/D_{exp} with increasing mass number A is, however, evident in Fig. 2. This trend implies that the LDM part of the level density formula involving the assumption of $a = \alpha A$ *needs* reexamination. To study the mass dependence of a parameter, the values of a for each nucleus were deduced from the data on neutron resonance spacing D_{exp} on the basis of Eqs. (10) and (11), after substituting the experimental values of the shell correction energies, and the best value of ω_0, as determined earlier. The values of a thus deduced are shown in Fig. 3 as a function of mass number A. The straight line corresponds to the best fit value of $\alpha = 0.14$ MeV^{-1}, on the assumption of $a = \alpha A$. It can be seen that the values of a, thus deduced not only show a systematic deviation from the line, but also cannot be satisfactorily fitted to any other straight line passing through origin as required by the relation $a = \alpha A$. For a noninteracting Fermi gas confined to a finite volume, a can be expressed as (see Appendix II):

$$a = \gamma A (1 - \beta A^{-1/3} B_s), \tag{14}$$

where B_s is the usual nuclear surface area relative to spherical shape and γ and β are mass-independent constants. Taking guidance from this relation, the deduced values of a/A have been plotted versus $A^{-1/3}$ in Fig. 4, where it can be seen that the ratio a/A can be fitted to the functional form of Eq. (14). The best values of γ and β obtained by a least square fit to the points shown in Fig. 4 are found to be $\gamma = 0.176$ and $\beta = 1.0$. The requirement of best fit to the values of D_{exp} also gave the same values of γ and β, and also showed that the best value of ω_0 does not change significantly in going from one-parameter to two-parameter representation of a.

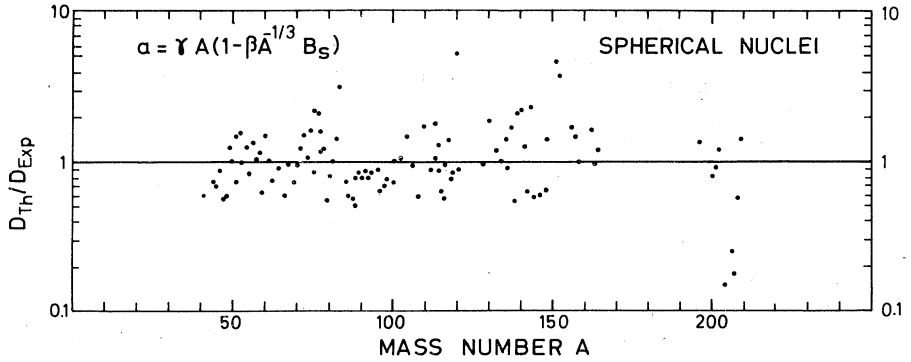


FIG. 5. Plot of D_{th}/D_{exp} for spherical nuclei with the expression $a = \gamma A(1 - \beta A^{-1/3})$. The values of parameters used are $\gamma = 0.176$, $\beta = 1.0$, and $\omega_0 = 0.185$.

Table I lists the calculated level spacings along with the available experimental data. Figure 5 shows a plot of D_{th}/D_{exp} in which D_{th} are calculated with the values of a obtained from Eq. (14). As expected, the residual systematics trend seen in Fig. 2 is not evident if a two-parameter representation for a is used to include nuclear surface effects. The above analysis thus brings out the need for the inclusion of finite size effects in the level density calculations.

TABLE I. Calculated and experimental level spacings of spherical nuclei.

Compound nucleus	Target spin, parity	E_x (MeV)	Δ_p (MeV)[a]	Δ_s (MeV)[b]	D_{exp} (eV) c	d	e	D_{th} (eV)
^{41}Ca	0^+	8.56	1.50	2.5	49 000	50 000	45 000	27 050
^{44}Ca	$\frac{7}{2}^-$	11.16	2.90	2.4	···	2 900	3 300	2 460
^{45}Ca	0^+	7.71	1.50	2.0	50 000	55 000	33 000	34 555
^{46}Sc	$\frac{7}{2}^-$	8.92	0.0	2.1	···	1 600	1 100	1 411
^{47}Ti	0^+	8.98	1.7	2.2	30 000	45 000	22 000	12 550
^{48}Ti	$\frac{5}{2}^-$	11.67	3.0	1.3	···	2 820	1 600	1 720
^{49}Ti	0^+	8.24	1.7	0.7	15 000	20 000	25 000	31 081
^{50}Ti	$\frac{7}{2}^-$	11.07	3.2	−0.6	···	3 600	6 000	6 050
^{51}Ti	0^+	6.48	1.7	−0.8	120 000	18 000	···	191 270
^{52}V	$\frac{7}{2}^-$	7.38	0.0	−0.8	3 600	4 390	4 900	7 478
^{51}Cr	0^+	9.44	1.4	0.2	16 500	19 000	21 000	11 985
^{53}Cr	0^+	8.11	1.4	−0.9	44 000	46 000	47 000	43 430
^{54}Cr	$\frac{3}{2}^-$	9.85	2.8	−0.5	···	3 200	5 700	7 168
^{56}Mn	$\frac{5}{2}^-$	7.37	0.0	−0.6	2 100	2 970	1 900	4 000
^{55}Fe	0^+	9.55	1.4	−1.5	25 000	21 000	20 000	17 000
^{57}Fe	0^+	7.97	1.4	−0.8	29 000	21 000	25 000	30 610
^{58}Fe	$\frac{1}{2}^-$	10.05	2.9	−0.3	···	5 900	1 500	6 766
^{60}Co	$\frac{7}{2}^-$	7.55	0.0	−0.7	960	1 530	1 300	2 316
^{59}Ni	0^+	9.3	1.4	−1.5	27 000	21 000	22 000	13 005
^{61}Ni	0^+	8.12	1.4	−0.7	23 000	21 000	17 000	16 934
^{62}Ni	$\frac{3}{2}^-$	10.6	2.8	−0.6	···	2 300	1 400	1 728
^{64}Cu	$\frac{3}{2}^-$	7.93	0.0	−0.1	1 200	1 060	580	946
^{66}Cu	$\frac{3}{2}^-$	7.08	0.0	0.3	2 000	1 170	1 000	1 185
^{67}Zn	0^+	7.06	1.1	1.5	5 000	5 600	6 000	5 370
^{69}Zn	0^+	6.6	1.1	1.4	···	20 000	10 000	7 200
^{70}Ga	$\frac{3}{2}^-$	7.66	0.0	1.0	340	320	320	303
^{72}Ga	$\frac{3}{2}^-$	6.52	0.0	1.5	170	190	370	562
^{71}Ge	0^+	7.43	1.4	2.1	1 700	2 000	1 330	2 460
^{73}Ge	0^+	6.8	1.4	2.0	2 100	3 900	1 550	3 990
^{74}Ge	$\frac{9}{2}^+$	10.2	2.8	1.9	···	77	62	123
^{75}Ge	0^+	6.5	1.4	1.1	8 500	8 500	3 900	7 384
^{77}Ge	0^+	6.03	1.4	1.3	8 000	8 000	4 200	9 330
^{76}As	$\frac{3}{2}^-$	7.33	0.0	1.4	87	87	87	186
^{75}Sc	0^+	8.05	1.4	2.3	250	200	370	846

TABLE I. (*Continued*)

Compound nucleus	Target spin, parity	E_x (MeV)	Δ_p (MeV)[a]	Δ_s (MeV)[b]	D_{exp} (eV) c	d	e	D_{th} (eV)
^{77}Se	0^+	7.42	1.4	1.6	1200	1200	700	1910
^{78}Se	$\frac{1}{2}^-$	10.5	2.9	2.1	100		120	143
^{79}Se	0^+	6.99	1.4	2.0	3700	4500	1000	2000
^{81}Se	0^+	6.71	1.4	0.6	4300	1600	1200	4400
^{83}Se	0^+	5.94	1.4	−0.7	7000	6900	6700	22400
^{80}Br	$\frac{3}{2}^-$	7.88	0.0	2.2	57	61	60	46
^{82}Br	$\frac{3}{2}^-$	7.6	0.0	0.6	51	52	80	113
^{86}Rb	$\frac{5}{2}^-$	8.65	0.0	−1.3	130	1100	130	77
^{88}Rb	$\frac{3}{2}^-$	6.08	0.0	−2.0	1200	1800	···	1430
^{85}Sr	0^+	8.53	1.2	1.2		350	400	260
^{87}Sr	0^+	8.43	1.2	−1.1		2100	1000	1200
^{88}Sr	$\frac{9}{2}^+$	11.11	2.5	−2.5		210	250	105
^{89}Sr	0^+	6.51	1.2	−1.9	55000	12000	12000	10300
^{90}Y	$\frac{1}{2}^-$	6.87	0	−2.5	1000	1600	3000	1270
^{91}Zr	0^+	7.24	1.0	−2.3	4500	3300	5000	4000
^{92}Zr	$\frac{5}{2}^+$	8.64	1.6	−1.8	315	250	110	251
^{93}Zr	0^+	6.78	1.0	−0.9	1200	3400	2500	3000
^{95}Zr	0^+	6.48	1.0	−0.2	2400	3300	2400	2150
^{96}Mo	$\frac{5}{2}^+$	9.15	1.9	−0.4		100	102	65
^{97}Mo	0^+	6.8	1.2	+0.6		1200	290	830
^{98}Mo	$\frac{5}{2}^+$	8.64	2.4	1.2	170	120	80	61
^{100}Tc	$\frac{9}{2}^+$	6.59	0.0	1.4	24	26	···	26
^{100}Ru	$\frac{5}{2}^+$	9.67	2.5	0.5	···	200	34	25
^{102}Ru	$\frac{5}{2}^+$	9.22	2.5	1.6	16	15	18	19
^{104}Rh	$\frac{1}{2}^-$	7.0	0.0	0.8	19	10	27	41
^{106}Pd	$\frac{5}{2}^+$	9.56	2.8	1.3	13	11	9	12
^{108}Ag	$\frac{1}{2}^+$	7.27	0.0	0.4	14	50	23	29
^{110}Ag	$\frac{1}{2}^+$	6.8	0.0	0.7	13	19	18	33
^{112}Cd	$\frac{1}{2}^+$	9.4	2.7	1.1	26	34	26	22
^{113}Cd	0^+	6.54	1.4	0.9	200	200	198	360
^{114}Cd	$\frac{1}{2}^+$	9.04	2.7	0.9	25	27	25	35
^{114}In	$\frac{9}{2}^+$	7.31	0.0	0.2	6.5	7.1	11	6.1
^{116}In	$\frac{9}{2}^+$	6.78	0.0	0.3	6.7	9.5	11	9.1
^{113}Sn	0^+	7.74	1.3	0.0	108	140	20	140
^{115}Sn	0^+	7.53	1.3	−0.2	150	320	300	190
^{116}Sn	$\frac{1}{2}^+$	9.57	2.6	0.0	50	50	···	29
^{117}Sn	0^+	6.94	1.3	−0.2	180	250	250	340
^{118}Sn	$\frac{1}{2}^+$	9.33	2.6	−0.1	25	65	45	35
^{119}Sn	0^+	6.48	1.3	−0.3	180	730	600	620

TABLE I. (*Continued*)

Compound nucleus	Target spin, parity	E_x (MeV)	Δ_p (MeV)[a]	Δ_s (MeV)[b]	D_{exp} (eV) c	d	e	D_{th} (eV)
^{120}Sn	$\frac{1}{2}^+$	9.10	2.6	0.6	30	62	70	62
^{128}I	$\frac{5}{2}^+$	6.83	0.0	−2.0	13	19	14	18
^{130}I	$\frac{7}{2}^+$	6.46	0.0	−2.9	18	21	···	38
^{132}Xe	$\frac{3}{2}^+$	8.94	2.0	−3.3	31	31	35	41
^{136}Xe	$\frac{3}{2}^+$	7.99	1.6	−7.3		500		710
^{134}Cs	$\frac{7}{2}^+$	6.89	0	−3.3	20	21	20	20
^{136}Ba	$\frac{3}{2}^+$	9.11	2.2	−3.5	51	35	36	31
^{137}Ba	0^+	6.9	1.1	−4.7	8 000	3 800	600	1 110
^{138}Ba	$\frac{3}{2}^+$	8.61	1.7	−5.7	200	460	230	110
^{139}Ba	0^+	4.82	1.1	−4.5	10 000	9 600	10 000	21 000
^{140}La	$\frac{7}{2}^+$	5.17	0.0	−3.9	73	110	260	241
^{141}Ce	0^+	5.43	1.2	−3.2	···	3 000	3 000	3 700
^{143}Ce	0^+	5.18	1.2	−1.9	1 000	1 000	1 000	2 300
^{142}Pr	$\frac{5}{2}^+$	5.85	0	−3.1	51	84	90	52.7
^{144}Nd	$\frac{7}{2}^-$	7.82	2.0	−2.0	72	19	36	20.8
^{146}Nd	$\frac{7}{2}^-$	7.57	2.2	−0.5	33	25	1.9	15.3
^{148}Pm	$\frac{7}{2}^+$	5.9	0.0	−0.9	5.2	5.7	···	8.4
^{148}Sm	$\frac{7}{2}^-$	8.14	1.9	−0.8	8	7.9	7.3	4.5
^{192}Ir	$\frac{3}{2}^+$	6.20	0	−5.1	3.1	3.2	2.8	5.9
^{194}Ir	$\frac{3}{2}^+$	6.07	0	−5.9	8.2	8.5	8.0	11.6
^{196}Pt	$\frac{1}{2}^-$	7.92	1.6	−6.5	18	19	12	17.6
^{200}Hg	$\frac{1}{2}^-$	8.03	1.3	−9.0	70	84	75	56
^{201}Hg	0^+	6.22	0.7	−9.4	2 200	1 300	1 300	1 180
^{202}Hg	$\frac{3}{2}^-$	7.76	1.6	−9.9	100	110	90	132
^{204}Tl	$\frac{1}{2}^+$	6.65	0.0	−13.1	2 000	2 200	2 000	290
^{206}Tl	$\frac{1}{2}^+$	6.54	0.0	−13.1	10 000	19 000	4 000	930
^{207}Pb	0^+	7.11	0.8	−13.6		24 000	50 000	3 700
^{208}Pb	0^+	4.84	0.8	−12.7		110 000	105 000	152 000

[a] Ground state pairing energy taken from Ref. 3.
[b] Experimental ground state shell correction energy (see text).
[c] Data compiled by Lynn (Ref. 14).
[d] Data compiled by Baba (Ref. 15).
[e] Data compiled by Dilg *et al.* (Ref. 16).

It should be mentioned that if one considers individual neutron and proton components, the expression for a will also have an additional parameter characterizing isospin. However, with the present experimental data confined to nuclei in the vicinity of β-stability line, and the experimental uncertainties present in the data, our investigations did not provide any definite evidence for the need for inclusion of such a term in the expression for a. Therefore although no evidence on isospin dependence of a could be deduced, the present analysis has presented evidence that a two-parameter representation for a to include both the volume and surface dependence provides a better fit to the experimental data on neutron resonances.

The proposed two-parameter form for a has some other interesting implications also. First, this would imply that extrapolations to unknown

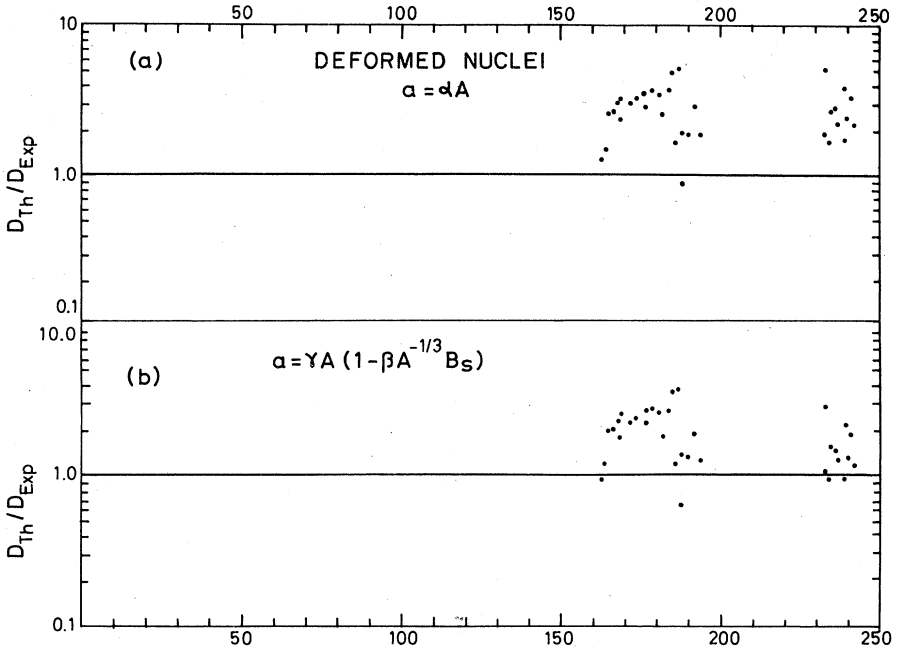


FIG. 6. Plot of D_{th}/D_{exp} for deformed nuclei: (a) with $a = 0.14A$ and $\omega_0 = 0.185$, (b) with $a = 0.176A(1 - 1.0A^{1/3}Bs)$ and $\omega_0 = 0.185$.

regions of mass numbers, or deformations based on the form $a = \alpha A$ can have considerable errors. For instance, the LDM level density parameter for the fission saddle point shape is expected to be smaller by about 5% as compared to that for the spherical shape—a fact which is important to take into account for a reliable analysis of the fission excitation functions. The evidence for such a decrease has also been seen[17] in the analysis of fission excitation function for the α particle induced fission of ^{208}Pb. Similarly, a reliable extrapolation of a values to higher mass numbers, particularly to the region of superheavy nuclei will require that the two-parameter form of a proposed here should be used.

We have also calculated level densities versus excitation energy for a few cases with the present formulation. The results for two typical cases of ^{56}Fe and ^{55}Mn compound nuclei are shown in Fig. 7 along with the available experimental data.[18] It can be seen that the present calculations provide a satisfactory fit to the experimental data at higher excitation energies as well without the use of any free parameters. At excitation energies lower than that corresponding to neutron binding energies, the observed small deviation particularly for the case of ^{56}Fe is understandable, since at these energies pairing effects cannot be taken into account simply by subtraction of Δ_p from the excitation energy E_x.

B. Deformed nuclei

The present calculations were also applied to deformed nuclei, for which level spacing data obtained from neutron resonance studies have also been compiled by Lynn,[14] Baba,[15] and Dilg *et al.*[16] The level spacings of these nuclei, were calculated on the basis of Eq. (1) with both one- and two-parameter representation for a and with the same values of the parameter ω_0 as determined from the analysis of spherical nuclei. The spin-cut-off factor σ^2 was calculated on the basis of Eq. (12), except that here the moment of inertia refers to the value parallel to the nuclear symmetry axis. As described in Appendix I, the fitting of the numerical calculation results with the functional forms of Eqs. (10) and (11) did not show a significant variation of the effective value of ω_0 with nuclear shapes up to the ground state deformed shapes. Therefore, the same value of ω_0 as deduced from analysis of spherical nuclei was used in the calculations for deformed nuclei. In the case of two-parameter representation for a, the values of B_s corresponding to known ground state deformations[13] were substituted, since it is known that at excitation energies of 6–10 MeV the nuclear deformation coincides with that of the ground state.[19]

The calculated values were compared with the experimental values to look for any significant enhancement of the level densities of deformed nuclei

TABLE II. Calculated and experimental level spacings for deformed nuclei.

Compound nucleus	Target spin, parity	E_x (MeV)	Δ_p (MeV)[a]	Δ_s (MeV)[b]	D_{exp} (eV) c	d	e	D_{th} (eV)
^{156}Gd	$\frac{3}{2}^-$	8.53	1.95	−2.0	2.1	1.9	2.0	3.7
^{157}Gd	0^+	6.37	1.00	−2.4		75	47	112.0
^{158}Gd	$\frac{3}{2}^-$	7.93	2.00	−2.4	12	6.1	6.0	11.8
^{162}Dy	$\frac{5}{2}^+$	8.2	1.96	−2.7	2.1	2.6	2.9	4.7
^{163}Dy	0^+	6.27	0.91	−3.1	130	42	72	126.5
^{164}Dy	$\frac{5}{2}^-$	7.66	1.97	−3.0	11.0	9.6	9.6	13.1
^{163}Er	0^+	6.91	0.92	−1.9		7.1	6.9	20.3
^{165}Er	0^+	6.66	0.92	−2.7		17	22	45.8
^{167}Er	0^+	6.44	0.92	−3.5		47	38	99.4
^{168}Er	$\frac{7}{2}^+$	7.77	1.99	−3.4	3	4	4.1	9.8
^{169}Er	0^+	6.00	0.92	−4.0		100	95	260.0
^{169}Yb	0^+	6.87	1.00	−3.0		20	···	35.9
^{177}Yb	0^+	5.56	1.00	−4.4		250	185	566.0
^{177}Hf	0^+	6.38	1.11	−3.9		41	32	113.0
^{179}Hf	0^+	6.1	1.11	−4.0		55	64	178.4
^{181}Hf	0^+	5.70	1.11	−4.2		140	125	381.0
^{182}Ta	$\frac{7}{2}^+$	6.06	0	−5.0	4	4	4	7.8
^{184}W	$\frac{1}{2}^-$	7.41	2.14	−4.1	15	16	12	43.9
^{185}W	0^+	5.76	1.23	−4.2		93	89	344.6
^{187}W	0^+	5.47	1.23	−4.3	150	87	123	582.6
^{186}Re	$\frac{5}{2}^+$	6.18	0	−4.7	4	3.2	3	4.8
^{188}Re	$\frac{5}{2}^+$	5.87	0	−4.9	5	6.4	4	8.3
^{188}Os	$\frac{1}{2}^-$	7.99	1.68	−4.7		14	9	9.1
^{190}Os	$\frac{3}{2}^-$	7.79	1.59	−5.1		5	4	6.7
^{233}Th	0^+	4.79	0.8	−3.0	17.5	12.4	16.7	52.0
^{233}U	0^+	5.74	0.81	−4.2	7.6	14.2	4.2	15.2
^{234}U	$\frac{5}{2}^+$	6.84	1.45	−4.1		0.99	0.61	0.94
^{235}U	0^+	5.31	0.81	−4.7	13	18	12.3	28.3
^{236}U	$\frac{7}{2}^-$	6.55	1.39	−3.9		0.67	0.53	1.0
^{237}U	0^+	5.12	0.81	−3.7	17	27	15.4	34.2
^{239}U	0^+	4.82	0.81	−3.3	17.7	18.1	20.8	47.0
^{239}U	0^+	5.66	0.69	−4.7	13	16	11.7	15.4
^{240}Pu	$\frac{1}{2}^+$	6.53	1.24	−4.5		2.3	2.4	3.05
^{241}Pu	0^+	5.24	69	−4.3	13	14	12.5	26.2
^{242}Pu	$\frac{5}{2}^+$	6.30	1.26	−3.10			1.2	1.4

[a] Ground state pairing energy taken from Ref. 3.
[b] Experimental ground state shell correction energy (see text).
[c] Data compiled by Lynn (Ref. 14).
[d] Data compiled by Baba (Ref. 15).
[e] Data compiled by Dilg *et al.* (Ref. 16).

over the corresponding single-particle values calculated on the basis of Eq. (1). Such an enhancement, has been indicated by an earlier analysis.[6] It has been pointed out[6] that this enhancement can arise as a result of additional rotational degrees of freedom available to deformed nuclei if one assumes that rotational degrees are totally decoupled from the intrinsic excitation. On the further assumption that the rotational energy is small as compared to the total excitation energy, it has been shown[6] that the enhancement factor is nearly equal to $\sigma_\perp^2 = J_\perp T/\hbar^2$, where $J_\perp$ is the moment of inertia about an axis perpendicular to nuclear symmetry axis. For mass numbers ranging from 150 to 250, rigid body moment of inertia and excitation energy equal to the neutron binding energy, $\sigma_\perp^2$ ranges from 35 to 65. Figures 6(a) and 6(b) show the present results on the ratio $D_{\text{th}}/D_{\text{exp}}$ for about 35 deformed nuclei with one- and two-parameter representation for a, respectively. We list in Table II the calculated level spacings with the two-parameter representation for a. It is seen that the calculated level spacings are systematically higher as compared with experimental values but only by an average factor of about 2 irrespective of the form chosen for a. Further, it was found that this factor of 2 has an uncertainty of almost another factor 2 in either direction, with a change of ±20% in the value of ω_0. The present analysis, therefore, does not indicate an enhancement of the magnitude $\sigma_\perp^2$. This may imply that the assumption of totally decoupled rotational degree is not valid at these excitation energies, leading to a lesser enhancement factor than indicated by the analysis of Huizenga *et al.*[6] The good fit obtained by them for deformed nuclei with the inclusion of rotational enhancement can be traced to the fact that the single-particle scheme of Nilsson *et al.*[20] used by them for deformed nuclei has about 15% lower value of the single-particle level density g_0 compared with the level scheme of Seeger and Perisho[21] used by them for spherical nuclei. In fact, it can be seen from Table V of Huizenga *et al.*[6] that the level densities calculated with the levels of Nilsson *et al.*[20] are systematically lower than those calculated with the levels of Seeger and Perisho.[21] If the single-particles level scheme of Nilsson *et al.*[20] is, therefore, used for calculations, the ratio $D_{\text{th}}/D_{\text{exp}}$ will be systematically overestimated almost by a factor of 5–10 even in the case of spherical nuclei. In view of the preceding discussion it can, therefore, be concluded that the present analysis of level densities of deformed nuclei does not provide sufficient evidence for the rotational enhancement of level densities of the magnitude comparable to $\sigma_\perp^2$.

The present discussion also brings out that for a reliable numerical calculation of level densities from single-particle levels there is a need for a suitable normalization of the values to an appropriate LDM average behavior. It is therfore apparent that a semiempirical approach similar to the well known macroscopic-microscopic approach for nuclear masses, is expected to be more reliable for the calculation of the level densities also.

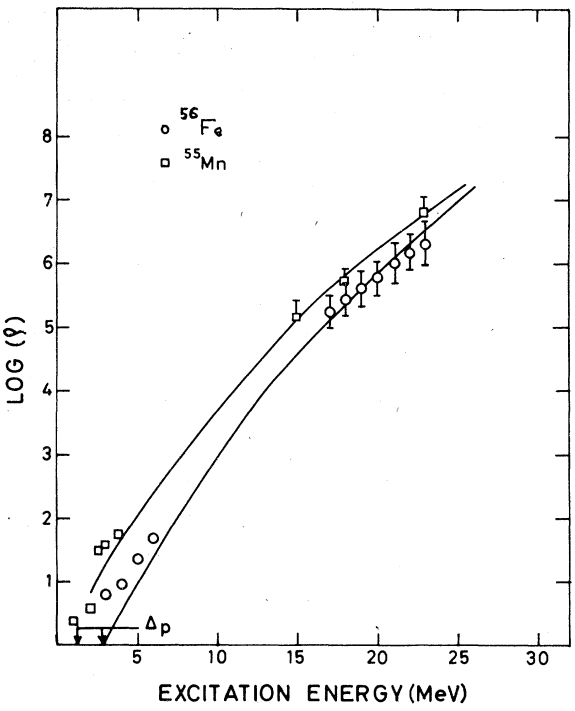


FIG. 7. Plot of calculated total level density versus excitation energy for the two nuclei ^{56}Fe and ^{55}Mn. The experimental points are from Ref. 18.

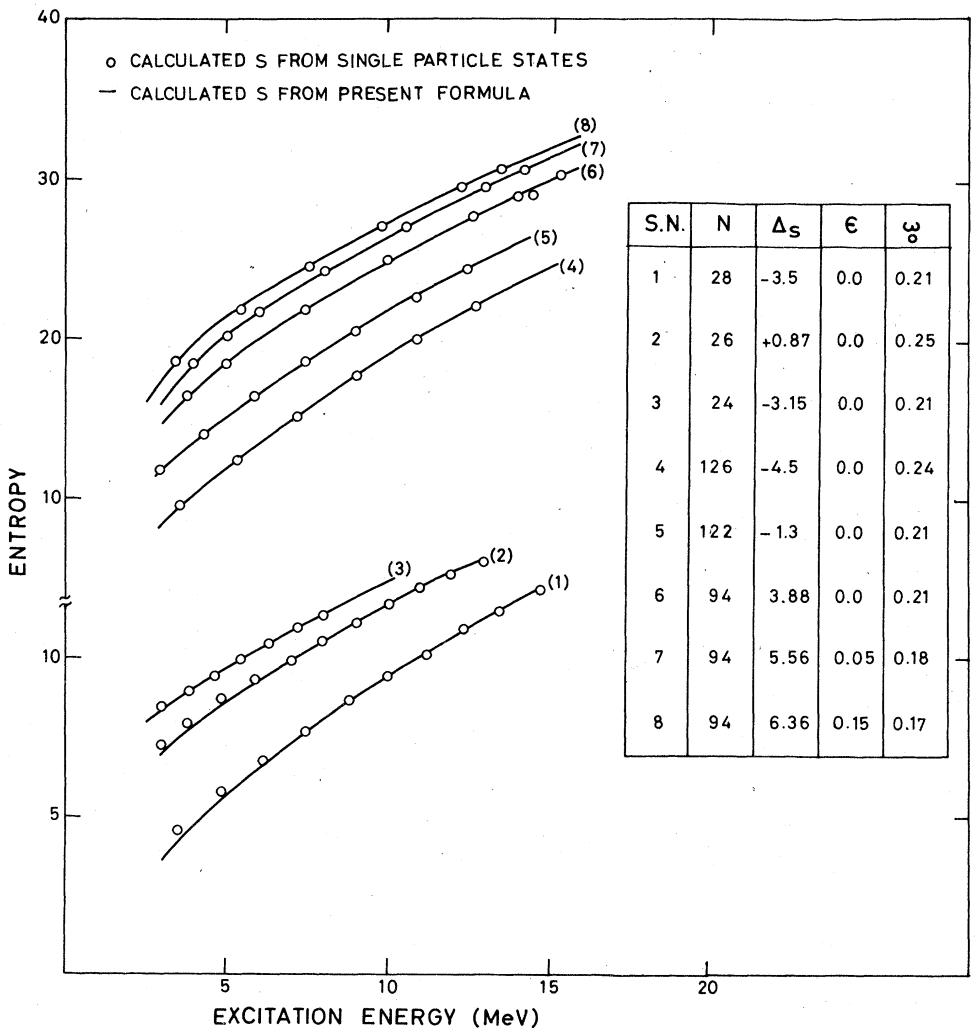
IV. SUMMARY

A semiempirical nuclear level density formula which takes into account nuclear shell effects in a realistic way has been proposed. The formulation is based on the results of earlier investigations of the thermodynamic properties of nuclei and has a built-in excitation energy dependence of shell effects on the level densities. The formula involves three mass-independent parameters, γ, β, and ω_0, characterizing the average single-particle level density near the Fermi level and the wavelength of the shell oscillations in the single-particle level density near the Fermi level. The other inputs required are the ground state shell and pairing correction energies of the nucleus. The best values of the parameters γ, β, and ω_0 have been obtained by fitting the formula to the experimental level spacings for spherical nuclei obtained from neutron resonance data. The level densities versus ex-

citation energy calculated with the present formula are also found to be in good agreement with experimental data for the cases examined. A notable feature of the formula is the explicit inclusion of a surface term in the LDM value of the parameter a, evidence for which was found to exist in the experimental data. This hitherto neglected surface dependence should be included for a reliable extrapolation of level densities for unknown cases involving superheavy nuclei and highly deformed nuclear shapes encountered in fission. An analysis of available experimental level spacing for deformed nuclei on the basis of the present formula has shown that though a small enhancement of the level densities of deformed nuclei is not ruled out, the magnitude of enhancement is significantly smaller at the excitation energies equal to neutron binding energies, than what is demanded on the assumption of complete decoupling of rotational degree of freedom from the intrinsic degrees.

APPENDIX I

Equations (10) and (11) were derived under the simplifying assumptions that (i) at finite temperatures the predominant contribution comes from the fundamental term in the expression for the shell oscillations $\delta g(E)$ and (ii) a weak temperature dependence of the chemical potential μ can be neglected. The validity of these assumptions can be ascertained by finding out whether Eqs. (10) and (11) can be made to fit the numerical calculations of S versus E_x. Starting from the single-particle level scheme of Seeger and Perisho,[21] numerical calculations of S versus E_x were carried out for a few typical cases of a given number of particles



S.N.	N	Δ_S	ϵ	ω_0
1	28	-3.5	0.0	0.21
2	26	+0.87	0.0	0.25
3	24	-3.15	0.0	0.21
4	126	-4.5	0.0	0.24
5	122	-1.3	0.0	0.21
6	94	3.88	0.0	0.21
7	94	5.56	0.05	0.18
8	94	6.36	0.15	0.17

FIG. 8. Comparison of the calculated entropy S versus excitation energy E_x from numerical calculations using harmonic oscillator single-particle level scheme and the calculation based on Eqs. (10) and (11).

and for deformed shapes. The ground state shell correction energies Δ_s for these cases were also obtained by the standard procedure. With these values of Δ_s, Eqs. (10) and (11) were fitted to the numerically calculated S versus E_x and these fits are shown in Fig. 8. It is seen that in all the cases studied, at excitation energies above 3.0 MeV, the maximum error introduced by the use of Eqs. (10) and (11) *vis a vis* numerical calculations is less than 0.5 in entropy. This corresponds to a maximum error of less than a factor of 2 in level spacings. The inset in the figure shows parameter values obtained for the different cases. Considering that the uncertainty in the derived values of ω_0 is about ± 0.03, one can conclude that no systematic variation of ω_0 exists with respect to either the sign and magnitude of the shell correction energies or the size and shape of the nuclei.

APPENDIX II

Following Hill and Wheeler,[22] the density of single-particle states of one kind of particles, say, neutrons, in an infinite square well potential confined to volume V with surface area S, and mean curvature C, can be written as

$$\frac{dN}{dk} = 2\left(\frac{V}{2\pi^2}k^2 - \frac{S}{8\pi}k + \frac{C}{8\pi^2}\right),$$

where the factor 2 has been included to account for spin degeneracy. Considering only the predominant volume and surface energy terms, the density g_0^n of the single-particle states at the Fermi energy, ϵ_F, is given by

$$g_0^n = \left.\frac{dN}{d\epsilon}\right|_{\epsilon=\epsilon_F} \simeq \frac{2m}{\hbar^2}\left(\frac{V}{2\pi^2}k_F - \frac{S}{8\pi}\right),$$

where m is the nucleon mass. The Fermi momentum k_F can be obtained from the number equation

$$N = V\frac{k_F^3}{3\pi^2} - \frac{S}{8\pi}k_F^2$$

that is

$$k_F \simeq \left(\frac{3\pi^2}{V}\right)^{1/3}\left[N + \frac{B_s S_0}{8\pi}\left(\frac{3\pi^2 N}{V}\right)^{2/3}\right]^{1/3}$$
$$\simeq \left(\frac{3\pi^2 N}{V}\right)^{1/3}(1 + 0.6\,B_s N^{-1/3}),$$

where surface area S has been expressed in units of the surface area of a sphere S_0 of the same volume V as $S = B_s S_0$. Substituting this value of k_F, we get

$$g_0^n = \frac{2m}{\hbar^2}\left[\frac{V}{2\pi^2}\left(\frac{3\pi^2 N}{V}\right)^{1/3}(1 + 0.6\,B_s N^{-1/3}) - \frac{B_s S_0}{8\pi}\right].$$

With $V = \frac{4}{3}\pi r_0^3 A$ and $S = 4\pi r_0^2 A^{2/3}$ and $A = 2N$ assuming equal proton and neutron contributions to g_0, one obtains the following expression for the LDM value of a

$$a = \tfrac{1}{6}\pi^2(g_0^n + g_0^p)$$
$$= \tfrac{1}{3}\pi^2 g_0$$
$$= \frac{A}{13.5}(1 - 0.95\,B_s A^{-1/3}) \quad \text{for } r_0 = 1.2 \text{ fm}.$$

Although the numerical constants obtained above for the simple case of a Fermi gas are not expected to apply for the case of an actual nucleus, the above deductions show that a functional form of the following type for a is expected:

$$a = \gamma A(1 - \beta B_s A^{-1/3}).$$

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