# Theory of radiative muon capture with applications to nuclear spin and isospin doublets*

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A theory of radiative muon capture, with applications to nuclear spin and isospin doublets, is formulated on the basis of the conservation of the hadronic electromagnetic current, the conservation of the hadronic weak polar current, the partial conservation of the hadronic weak axial-vector current, the SU(2) × SU(2) current algebra for the various hadronic currents, and a simplifying dynamical approximation for the hadron-radiating part of the transition amplitude—the "linearity hypothesis." The resultant total transition amplitude, which also includes the muon-radiating part, is worked out explicitly and applied to treat the processes $\mu^- p \to \nu_\mu n \gamma$ and $\mu^- {}^3\text{He} \to \nu_\mu {}^3\text{H}\gamma$.

[RADIOACTIVITY $\mu^- p \to \nu_\mu n\gamma$ and $\mu^- {}^3\text{He} \to \nu_\mu {}^3\text{H}\gamma$; general theory of radiative muon capture with applications to nuclear spin and isospin doublets.]

## INTRODUCTION

The theory of radiative muon capture by a nucleus: $\mu^- N_i \to \nu_\mu N_f \gamma$,[1-11] even in the simplest case when the nucleus is a proton, is not altogether satisfactory because of the uncertainties involved in the determination of the hadron-radiating part of the transition amplitude $\mathcal{T}^{(h)}$, i.e., the part containing contributions from $\mu^- p \to \mu^- X\gamma \to \nu_\mu n\gamma$ and $\mu^- p \to \nu_\mu Y \to \nu_\mu n\gamma$ (where $X = p, \Delta^+, \ldots$ and $Y = n, \Delta^0, \ldots$) and from $\mu^- p \to (\nu_\mu \gamma) n$ (where the $\nu_\mu$ and $\gamma$ emerge "together" from the weak vertex). Clearly, these uncertainties become even more serious in radiative muon capture by a complex nucleus. In addition, a formalism is desirable which permits *explicit* verification of the constraints imposed on $\mathcal{T}^{(h)}$ which arise from the conservation of the hadronic electromagnetic current (CEC), the conservation of the hadronic weak polar current (CVC), and the partial conservation of the hadronic weak axial-vector current (PCAC).

Our approach is based on hypothesizing a simplifying dynamical approximation for $\mathcal{T}^{(h)}$—we call this approximation the "linearity hypothesis" (LH). The linearity hypothesis, which is fully described below, together with the constraints arising from CEC, CVC, and PCAC, enables us to express $\mathcal{T}^{(h)}$ entirely in terms of the form factors characterizing the nonradiative muon capture between $N_i$ and $N_f$ and the electromagnetic properties of these nuclei. Since the muon-radiating part of the transition amplitude $\mathcal{T}^{(l)}$, i.e., the part containing the contribution from $\mu^- N_i \to \mu^- \gamma N_i \to \nu_\mu N_f \gamma$, can be similarly expressed entirely in terms of the form factors characterizing the nonradiative muon capture between $N_i$ and $N_f$ and the charge on the muon, the whole transition amplitude $\mathcal{T}^{(l)} + \mathcal{T}^{(h)}$ is determined and applications to specific processes, i.e., $\mu^- p \to \nu_\mu n\gamma$ and $\mu^- {}^3\text{He} \to \nu_\mu {}^3\text{H}\gamma$, can be made.

## FORMULATION

We define the various relevant form factors in the case of the nuclear spin and isospin doublets by the expressions immediately below. Further, we assume the absence of second-class currents which, if they exist, can easily be taken into account on the phenomenological level on which we operate. Thus, with $V_\lambda(x)$ and $A_\lambda(x)$ the hadronic weak polar and axial-vector currents,

$$\langle N_f(p^{(f)}, s^{(f)}) | V_\lambda(0) | N_i(p^{(i)}, s^{(i)}) \rangle = \bar{u}^{(f)}(p^{(f)}, s^{(f)}) \left( F_V(q^2)\gamma_\lambda - F_M(q^2) \frac{\sigma_{\lambda\eta} q_\eta}{2m_p} \right) u^{(i)}(p^{(i)}, s^{(i)}), \tag{1a}$$

$$\langle N_f(p^{(f)}, s^{(f)}) | A_\lambda(0) | N_i(p^{(i)}, s^{(i)}) \rangle = \bar{u}^{(f)}(p^{(f)}, s^{(f)}) \left( F_A(q^2)\gamma_\lambda\gamma_5 + F_P(q^2) i \frac{2M q_\lambda \gamma_5}{m_\pi^2} \right) u^{(i)}(p^{(i)}, s^{(i)}), \tag{1b}$$

$$\langle N_f(p^{(f)}, s^{(f)}) | \partial_\lambda A_\lambda(0) | N_i(p^{(i)}, s^{(i)}) \rangle = \bar{u}^{(f)}(p^{(f)}, s^{(f)}) \left[ \left( \frac{F_D(q^2)}{1 + q^2/m_\pi^2} \right) \right] 2M\gamma_5\, u^{(i)}(p^{(i)}, s^{(i)}), \tag{1c}$$

where

$$\bar{u} \equiv i u^\dagger \gamma_4, \quad q_\lambda \equiv (p^{(f)} - p^{(i)})_\lambda, \quad Q_\lambda \equiv (p^{(f)} + p^{(i)})_\lambda, \quad M \equiv \tfrac{1}{2}(M_f + M_i) \cong M_f \cong M_i,$$

 

and

$$M_{f,i}=[-(p^{(f),(i)})^2]^{1/2}$$

($\Delta \equiv M_f - M_i$ is neglected consistently below), and where $F_{V,M,A,P}(q^2)$ and $F_D(q^2)/(1+q^2/m^2_\pi)$ are, respectively, the vector, weak magnetism, axial-vector, pseudoscalar, and axial-divergence weak nuclear form factors characterizing the nonradiative muon capture $\mu^- N_i \rightarrow \nu_\mu N_f$. Equations (1b) and (1c) yield

$$F_A(q^2)+\frac{q^2}{m_\pi^2}F_P(q^2)=\frac{F_D(q^2)}{1+q^2/m_\pi^2} \tag{1d}$$

which is an equation that, together with PCAC, yields the generalized Goldberger-Treiman relation [see Eqs. (8c)–(8f) below].

Making use of standard reduction formulas for the outgoing photon, we can write the transition amplitude $\mathcal{T}$ for the radiative muon capture

$$\begin{aligned}&\mu^-(p^{(\mu)},s^{(\mu)})+N_i(p^{(i)},s^{(i)})\rightarrow \nu_\mu(p^{(\nu)},s^{(\nu)})+N_f(p^{(f)},s^{(f)})+\gamma(k,\epsilon)\,,\\&p^{(\mu)}+p^{(i)}=p^{(\nu)}+p^{(f)}+k=p^{(\nu)}+p^{(i)}+q+k=p^{(\nu)}-p^{(i)}+Q+k\,.\end{aligned} \tag{2}$$

as follows:

$$\mathcal{T}=\mathcal{T}^{(l)}+\mathcal{T}^{(h)}\,, \tag{3}$$

where

$$\begin{aligned}\mathcal{T}^{(l)}=&-\frac{Ge}{\sqrt{2}}\,\langle N_f(p^{(f)},s^{(f)})|[V_\lambda(0)+A_\lambda(0)]|N_i(p^{(i)},s^{(i)})\rangle\\&\times 1/(2k_0)^{1/2}\,\bar{u}^{(\nu)}(p^{(\nu)},s^{(\nu)})\gamma_\lambda(1+\gamma_5)i\frac{m_\mu-i(\not{p}^{(\mu)}-\not{k})}{m_\mu^2+(p^{(\mu)}-k)^2}\,\not{\epsilon}^*u^{(\mu)}(p^{(\mu)},s^{(\mu)})\,,\end{aligned} \tag{4}$$

$$\mathcal{T}^{(h)}=-\frac{Ge}{\sqrt{2}}\,\bar{u}^{(\nu)}(p^{(\nu)},s^{(\nu)})\gamma_\lambda(1+\gamma_5)u^{(\mu)}(p^{(\mu)},s^{(\mu)})\frac{1}{m_p}\frac{\epsilon_\mu^*}{(2k_0)^{1/2}}[V_{\mu\lambda}(k,q,Q)+A_{\mu\lambda}(k,q,Q)]\,, \tag{5}$$

$$V_{\mu\lambda}(k,q,Q)\equiv -im_p\int d^4x e^{-ik\cdot x}\langle N_f(p^{(f)},s^{(f)})|\,T(J_\mu(x)V_\lambda(0))\,|N_i(p^{(i)},s^{(i)})\rangle\,, \tag{5a}$$

$$A_{\mu\lambda}(k,q,Q)\equiv -im_p\int d^4x e^{-ik\cdot x}\langle N_f(p^{(f)},s^{(f)})|\,T(J_\mu(x)A_\lambda(0))\,|N_i(p^{(i)},s^{(i)})\rangle \tag{5b}$$

with $\epsilon^*=(\vec{\epsilon}^*,i\epsilon_0^*)$, $J_\mu(x)$ the hadronic electromagnetic current, and

$$\begin{aligned}T(J_\mu(x)K_\lambda(y))&\equiv J_\mu(x)K_\lambda(y):\;\; x_0>y_0\\&\equiv K_\lambda(y)J_\mu(x):\;\; x_0<y_0\,.\end{aligned}$$

Here $\mathcal{T}^{(l)}$ describes the contribution to $\mathcal{T}$ arising from radiation by the muon, while $\mathcal{T}^{(h)}$ describes the contribution to $\mathcal{T}$ arising from radiation by the initial and final nuclei, by the intermediate hadrons, and by any charged particle ($W^\pm$) that transmits the weak interaction. The Lorentz gauge is used so that

$$\epsilon_\mu^* k_\mu=(\vec{\epsilon}^*\cdot\hat{k}-\epsilon_0^*)k_0=0\,. \tag{5c}$$

To obtain the constraints on $V_{\mu\lambda}(k,q,Q)$ and $A_{\mu\lambda}(k,q,Q)$ due to CEC, CVC, and PCAC, we assume

$$J_\lambda(x)=I_\lambda^{(3)}(x)+\tfrac{1}{2}Y_\lambda(x)\,, \tag{6a}$$

$$V_\lambda(x)=I_\lambda^{(1)}(x)-iI_\lambda^{(2)}(x)\,, \tag{6b}$$

$$A_\lambda(x)=I_\lambda^{5(1)}(x)-iI_\lambda^{5(2)}(x)\,, \tag{6c}$$

where $I_\lambda^{(i)}(x)$ $(i=1,2,3)$ and $Y_\lambda(x)$ are, respectively, isovector (isospin) and isoscalar (hypercharge) polar currents, while $I_\lambda^{5(i)}(x)$ $(i=1,2,3)$ are isovector axial-vector currents "conjugate" to $I_\lambda^{(i)}(x)$ $(i=1,2,3)$ so that the SU(2)×SU(2) current algebra (CA) is valid:

$$[I_4^{(i)}(x),I_\lambda^{(j)}(y)]_{x_0=y_0}=-\epsilon_{ijk}I_\lambda^{(k)}(x)\delta^3(\vec{x}-\vec{y})\,, \tag{7a}$$

$$[I_4^{(i)}(x),I_\lambda^{5(j)}(y)]_{x_0=y_0}=-\epsilon_{ijk}I_\lambda^{5(k)}(x)\delta^3(\vec{x}-\vec{y})\,. \tag{7b}$$

Further, in the evaluation of matrix elements of functionals of currents between hadronic states, $J_\lambda(x)$, $V_\lambda(x)$, and $A_\lambda(x)$ can be assumed to satisfy the differential conservation equations

$$\text{CEC}:\;\; \partial_\lambda J_\lambda(x)=0\,, \tag{8a}$$

$$\text{CVC}:\;\; \partial_\lambda V_\lambda(x)=0\,, \tag{8b}$$

$$\text{PCAC}:\;\; \partial_\lambda A_\lambda(x)=a_\pi m_\pi^3(-\partial_\lambda\partial_\lambda+m_\pi^2)^{-1}\mathcal{J}^{(\pi)}(x)\,, \tag{8c}$$

where $\mathcal{J}^{(\pi)}(x)$ is the pion-source current and $a_\pi$ the pion-decay constant. As a first example of the use of these differential conservation equations, we note that Eqs. (8c) and (1c) give

$$\frac{1}{a_\pi} F_D(q^2) = \frac{m_\pi}{2M} \frac{\langle N_f(p^{(f)}, s^{(f)}) | \mathcal{J}^{(\pi)}(0) | N_i(p^{(i)}, s^{(i)})\rangle}{\bar{u}^{(f)}(p^{(f)}, s^{(f)})\gamma_5 u^{(i)}(p^{(i)}, s^{(i)})}$$

$$\equiv f_{\pi N_f N_i}(q^2) \tag{8d}$$

with $f_{\pi N_f N_i}(q^2)$ the $\pi^- + N_i \leftrightarrow N_f$ vertex function or form factor; combining Eq. (8d) with Eq. (1d) yields

$$F_A(q^2) + \frac{q^2}{m_\pi^2} F_P(q^2) = \frac{a_\pi f_{\pi N_f N_i}(q^2)}{1 + q^2/m_\pi^2},$$

$$F_A(0) = a_\pi f_{\pi N_f N_i}(0) \tag{8e}$$

which is the generalized Goldberger-Treiman relation and which, in the $p \leftrightarrow n$ case, corresponds to the validity of pion-pole dominance of $F_D(q^2)/(1+q^2/m_\pi^2)$ [since $f_{\pi n p}(-m_\pi^2) \cong f_{\pi n p}(0) \cong f_{\pi n p}(+m_\mu^2)$]. Rearrangement of Eq. (8e) yields

$$F_P(q^2) = -\frac{F_A(q^2)}{1+q^2/m_\pi^2} \times \left[1 + \frac{m_\pi^2}{q^2}\left(1 - \frac{f_{\pi N_f N_i}(q^2)/f_{\pi N_f N_i}(0)}{F_A(q^2)/F_A(0)}\right)\right]$$

$$\cong -\frac{F_A(q^2)}{1+q^2/m_\pi^2} \tag{8f}$$

with the second equality known to be approximately valid both for the $p \leftrightarrow n$ case and the $^3He \leftrightarrow {}^3H$ case.

We next apply Eqs. (8a)–(8c) and (7a) and (7b) to Eqs. (5a) and (5b). This yields the following constraints on $V_{\mu\lambda}(k,q,Q)$ and $A_{\mu\lambda}(k,q,Q)$:

$$\text{CEC:}\quad \frac{k_\mu}{m_p} V_{\mu\lambda}(k,q,Q) = \langle N_f(p^{(f)}, s^{(f)}) | V_\lambda(0) | N_i(p^{(i)}, s^{(i)})\rangle, \tag{9a}$$

$$\frac{k_\mu}{m_p} A_{\mu\lambda}(k,q,Q) = \langle N_f(p^{(f)}, s^{(f)}) | A_\lambda(0) | N_i(p^{(i)}, s^{(i)})\rangle; \tag{9b}$$

$$\text{CVC:}\quad \frac{(k_\lambda + q_\lambda)}{m_p} V_{\mu\lambda}(k,q,Q) = \langle N_f(p^{(f)}, s^{(f)}) | V_\mu(0) | N_i(p^{(i)}, s^{(i)})\rangle; \tag{10}$$

$$\text{PCAC:}\quad \frac{(k_\lambda + q_\lambda)}{m_p} A_{\mu\lambda}(k,q,Q) = \langle N_f(p^{(f)}, s^{(f)}) | A_\mu(0) | N_i(p^{(i)}, s^{(i)})\rangle + D_\mu(k,q,Q) \tag{11}$$

with

$$D_\mu(k,q,Q) = \int d^4x\, e^{-ik\cdot x} \times \langle N_f(p^{(f)}, s^{(f)}) | T(J_\mu(x)\partial_\lambda A_\lambda(0)) | N_i(p^{(i)}, s^{(i)})\rangle \tag{11a}$$

and [using again Eq. (8a)]

$$\begin{aligned} k_\mu D_\mu(k,q,Q) &= i\langle N_f(p^{(f)}, s^{(f)}) | \partial_\lambda A_\lambda(0) | N_i(p^{(i)}, s^{(i)})\rangle \\ &= \bar{u}^{(f)}(p^{(f)}, s^{(f)})\left[\left(\frac{a_\pi f_{\pi N_f N_i}(q^2)}{1+q^2/m_\pi^2}\right) 2M\gamma_5\right] u^{(i)}(p^{(i)}, s^{(i)}) \\ &= \bar{u}^{(f)}(p^{(f)}, s^{(f)})\{[F_A(q^2) + (q^2/m_\pi^2)F_P(q^2)]2M\gamma_5\} \\ &\quad \times u^{(i)}(p^{(i)}, s^{(i)}), \end{aligned} \tag{11b}$$

where we have also used Eqs. (1c), (8d), and (8e). We note that Eqs. (5)—(5b) and (9a)—(11b), which are basic for our calculation of $\mathcal{T}^{(h)}$, are independent of the possible presence of "seagull" terms [in Eqs. (5a) and (5b)] and Schwinger terms [in Eqs. (7a) and (7b)].[12] We also note that the first equality in Eq. (11b) is simply a consequence of Eqs. (9b) and (11).

We proceed to construct the most general Lorentz-covariant expressions for $V_{\mu\lambda}(k,q,Q)$, $A_{\mu\lambda}(k,q,Q)$, and $D_\mu(k,q,Q)$ from which approximate expressions for $\epsilon_\mu^* V_{\mu\lambda}(k,q,Q)$ and $\epsilon_\mu^* A_{\mu\lambda}(k,q,Q)$ will be obtained with the aid of the linearity hypothesis and the CEC, CVC, and PCAC constraint equations.

We have

$$\begin{aligned} V_{\mu\lambda}(k,q,Q) = \bar{u}^{(f)}(p^{(f)}, s^{(f)}) \Bigg[ & \sigma_{\mu\lambda} F_{00}^{(a)} + i\sigma_{\mu\lambda}\frac{\not{k}}{m_p} F_{00}^{(b)} + \frac{\gamma_\mu}{m_p}[k_\lambda F_{11}^{(a)} + Q_\lambda F_{12}^{(a)} + q_\lambda F_{13}^{(a)}] + \frac{\gamma_\lambda}{m_p}[k_\mu F_{11}^{(b)} + Q_\mu F_{12}^{(b)} + q_\mu F_{13}^{(b)}] \\ & + \frac{\sigma_{\mu\nu}k_\nu}{m_p^2}[k_\lambda F_{21}^{(a)} + Q_\lambda F_{22}^{(a)} + q_\lambda F_{23}^{(a)}] + \frac{\sigma_{\lambda\nu}k_\nu}{m_p^2}[k_\mu F_{21}^{(b)} + Q_\mu F_{22}^{(b)} + q_\mu F_{23}^{(b)}] \\ & + \Bigg(\delta_{\mu\lambda} F_{00}^{(c)} + \frac{k_\mu}{m_p^2}(k_\lambda F_{11}^{(c)} + Q_\lambda F_{12}^{(c)} + q_\lambda F_{13}^{(c)}) + \frac{Q_\mu}{m_p^2}(k_\lambda F_{21}^{(c)} + Q_\lambda F_{22}^{(c)} + q_\lambda F_{23}^{(c)}) \\ & \quad + \frac{q_\mu}{m_p^2}(k_\lambda F_{31}^{(c)} + Q_\lambda F_{32}^{(c)} + q_\lambda F_{33}^{(c)})\Bigg) \\ & + i\frac{\not{k}}{m_p}\Bigg(\delta_{\mu\lambda} F_{00}^{(d)} + \frac{k_\mu}{m_p^2}(k_\lambda F_{11}^{(d)} + Q_\lambda F_{12}^{(d)} + q_\lambda F_{13}^{(d)}) + \frac{Q_\mu}{m_p^2}(k_\lambda F_{21}^{(d)} + Q_\lambda F_{22}^{(d)} + q_\lambda F_{23}^{(d)}) \\ & \quad + \frac{q_\mu}{m_p^2}(k_\lambda F_{31}^{(d)} + Q_\lambda F_{32}^{(d)} + q_\lambda F_{33}^{(d)})\Bigg)\Bigg] u^{(i)}(p^{(i)}, s^{(i)}), \end{aligned} \tag{12}$$

$$A_{\mu\lambda}(k,q,Q)=\bar{u}^{(f)}(p^{(f)},s^{(f)})\gamma_5\Big[\sigma_{\mu\lambda}G_{00}^{(a)}+i\sigma_{\mu\lambda}\frac{\not k}{m_p}G_{00}^{(b)}+\frac{\gamma_\mu}{m_p}[k_\lambda G_{11}^{(a)}+Q_\lambda G_{12}^{(a)}+q_\lambda G_{13}^{(a)}]+\frac{\gamma_\lambda}{m_p}[k_\mu G_{11}^{(b)}+Q_\mu G_{12}^{(b)}+q_\mu G_{13}^{(b)}]$$
$$+\frac{\sigma_{\mu\nu}k_\nu}{m_p^2}[k_\lambda G_{21}^{(a)}+Q_\lambda G_{22}^{(a)}+q_\lambda G_{23}^{(a)}]+\frac{\sigma_{\lambda\nu}k_\nu}{m_p^2}[k_\mu G_{21}^{(b)}+Q_\mu G_{22}^{(b)}+q_\mu G_{23}^{(b)}]$$
$$+\Big(\delta_{\mu\lambda}G_{00}^{(c)}+\frac{k_\mu}{m_p^2}(k_\lambda G_{11}^{(c)}+Q_\lambda G_{12}^{(c)}+q_\lambda G_{13}^{(c)})+\frac{Q_\mu}{m_p^2}(k_\lambda G_{21}^{(c)}+Q_\lambda G_{22}^{(c)}+q_\lambda G_{23}^{(c)})$$
$$+\frac{q_\mu}{m_p^2}(k_\lambda G_{31}^{(c)}+Q_\lambda G_{32}^{(c)}+q_\lambda G_{33}^{(c)})\Big)$$
$$+i\frac{\not k}{m_p}\Big(\delta_{\mu\lambda}G_{00}^{(d)}+\frac{k_\mu}{m_p^2}(k_\lambda G_{11}^{(d)}+Q_\lambda G_{12}^{(d)}+q_\lambda G_{13}^{(d)})+\frac{Q_\mu}{m_p^2}(k_\lambda G_{21}^{(d)}+Q_\lambda G_{22}^{(d)}+q_\lambda G_{23}^{(d)})$$
$$+\frac{q_\mu}{m_p^2}(k_\lambda G_{31}^{(d)}+Q_\lambda G_{32}^{(d)}+q_\lambda G_{33}^{(d)})\Big)\Big]u^{(i)}(p^{(i)},s^{(i)}) \qquad (13)$$

$$D_\mu(k,q,Q)=\bar{u}^{(f)}(p^{(f)},s^{(f)})\Big(f_A\gamma_\mu\gamma_5+f_P i\frac{2Mq_\mu\gamma_5}{m_\pi^2}-f_E\frac{\gamma_5\sigma_{\mu\nu}q_\nu}{2m_p}+\tilde{f}_P i\frac{2Mk_\mu\gamma_5}{m_\pi^2}-\tilde{f}_E\frac{\gamma_5\sigma_{\mu\nu}k_\nu}{2m_p}$$
$$+i\frac{\gamma_5\not k}{m_p^2}(k_\mu f_1+Q_\mu f_2+q_\mu f_3)\Big)u^{(i)}(p^{(i)},s^{(i)}). \qquad (14)$$

Here, each of the weak radiative nuclear form factors $R\equiv F_{ij}^{(a),(b),(c),(d)}$, $G_{ij}^{(a),(b),(c),(d)}$, $f_{A,P,E}$, $\tilde{f}_{P,E}$, and $f_i$, is, in general, a function of the three Lorentz invariants $q^2$, $Q\cdot k$, and $q\cdot k$.

We now apply CEC, as described by Eqs. (9a), (9b) and (1a), (1b) to $V_{\mu\lambda}(k,q,Q)$ and $A_{\mu\lambda}(k,q,Q)$ as given by Eqs. (12) and (13). This yields

$$F_{22}^{(b)}Q\cdot k+F_{23}^{(b)}q\cdot k=m_p^2F_{00}^{(a)}, \qquad (15a)$$
$$F_{12}^{(b)}Q\cdot k+F_{13}^{(b)}q\cdot k=m_p^2\Big(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\Big), \qquad (15b)$$
$$F_{21}^{(d)}Q\cdot k+F_{31}^{(d)}q\cdot k=m_p^2[i(F_{00}^{(b)}+F_{11}^{(a)})-F_{00}^{(d)}], \qquad (15c)$$
$$F_{22}^{(d)}Q\cdot k+F_{32}^{(d)}q\cdot k=im_p^2F_{12}^{(a)}, \qquad (15d)$$
$$F_{23}^{(d)}Q\cdot k+F_{33}^{(d)}q\cdot k=im_p^2F_{13}^{(a)}, \qquad (15e)$$
$$F_{21}^{(c)}Q\cdot k+F_{31}^{(c)}q\cdot k=-m_p^2F_{00}^{(c)}, \qquad (15f)$$
$$F_{22}^{(c)}Q\cdot k+F_{32}^{(c)}q\cdot k=\tfrac{1}{2}im_p^2F_M(q^2), \qquad (15g)$$
$$F_{23}^{(c)}Q\cdot k+F_{33}^{(c)}q\cdot k=0; \qquad (15h)$$
$$G_{22}^{(b)}Q\cdot k+G_{23}^{(b)}q\cdot k=m_p^2G_{00}^{(a)}, \qquad (16a)$$
$$G_{12}^{(b)}Q\cdot k+G_{13}^{(b)}q\cdot k=-m_p^2F_A(q^2), \qquad (16b)$$
$$G_{21}^{(d)}Q\cdot k+G_{31}^{(d)}q\cdot k=m_p^2[i(G_{00}^{(b)}+G_{11}^{(a)})-G_{00}^{(d)}], \qquad (16c)$$
$$G_{22}^{(d)}Q\cdot k+G_{32}^{(d)}q\cdot k=im_p^2G_{12}^{(a)}, \qquad (16d)$$
$$G_{23}^{(d)}Q\cdot k+G_{33}^{(d)}q\cdot k=im_p^2G_{13}^{(a)}, \qquad (16e)$$
$$G_{21}^{(c)}Q\cdot k+G_{31}^{(c)}q\cdot k=-m_p^2G_{00}^{(c)}, \qquad (16f)$$
$$G_{22}^{(c)}Q\cdot k+G_{23}^{(c)}q\cdot k=0, \qquad (16g)$$
$$G_{23}^{(c)}Q\cdot k+G_{33}^{(c)}q\cdot k=i\frac{2Mm_p^3}{m_\pi^2}F_P(q^2). \qquad (16h)$$

Next, we apply CVC, as described by Eqs. (10) and (1a) to $V_{\mu\lambda}(k,q,Q)$ as given by Eq. (12). This yields

$$F_{21}^{(a)}q\cdot k+F_{22}^{(a)}Q\cdot k+F_{23}^{(a)}q\cdot(q+k)+m_p^2F_{00}^{(a)}=0, \qquad (17a)$$
$$(F_{11}^{(a)}+F_{00}^{(b)})q\cdot k+(F_{12}^{(a)}-F_{00}^{(b)})Q\cdot k+F_{13}^{(a)}q\cdot(q+k)$$
$$-2Mm_pF_{00}^{(a)}=m_p^2\Big(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\Big), \qquad (17b)$$
$$m_p^2F_{00}^{(c)}+F_{11}^{(c)}q\cdot k+F_{12}^{(c)}Q\cdot k+F_{13}^{(c)}q\cdot(q+k)$$
$$+iF_{21}^{(b)}Q\cdot k=0, \qquad (17c)$$
$$F_{21}^{(c)}q\cdot k+F_{22}^{(c)}Q\cdot k+F_{23}^{(c)}q\cdot(q+k)+iF_{22}^{(b)}Q\cdot k$$
$$-im_p^2F_{00}^{(a)}=\tfrac{1}{2}im_p^2F_M(q^2), \qquad (17d)$$
$$m_p^2F_{00}^{(c)}+F_{31}^{(c)}q\cdot k+F_{32}^{(c)}Q\cdot k+F_{33}^{(c)}q\cdot(q+k)$$
$$+iF_{23}^{(b)}Q\cdot k=0, \qquad (17e)$$
$$m_p^2F_{00}^{(d)}+F_{11}^{(d)}q\cdot k+F_{12}^{(d)}Q\cdot k+F_{13}^{(d)}q\cdot(q+k)$$
$$-i2Mm_pF_{21}^{(b)}-im_p^2F_{11}^{(b)}+im_p^2F_{00}^{(b)}=0, \qquad (17f)$$
$$F_{21}^{(d)}q\cdot k+F_{22}^{(d)}Q\cdot k+F_{23}^{(d)}q\cdot(q+k)$$
$$-i2Mm_pF_{22}^{(b)}-im_p^2F_{12}^{(b)}-im_p^2F_{00}^{(b)}=0, \qquad (17g)$$
$$m_p^2F_{00}^{(d)}+F_{31}^{(d)}q\cdot k+F_{32}^{(d)}Q\cdot k+F_{33}^{(d)}q\cdot(q+k)$$
$$-i2Mm_pF_{23}^{(b)}-im_p^2F_{13}^{(b)}=0. \qquad (17h)$$

Finally, we apply PCAC, as described by Eqs. (11), (11a), (11b), and (1b) to $A_{\mu\lambda}(k,q,Q)$ and $D_\mu(k,q,Q)$ as given by Eqs. (13) and (14). This yields

$$G_{21}^{(a)}q\cdot k+G_{22}^{(a)}Q\cdot k+G_{23}^{(a)}q\cdot(q+k)+m_p^2G_{00}^{(a)}$$
$$-2Mm_pG_{00}^{(b)}=-\tfrac{1}{2}m_p^2\tilde{f}_E, \qquad (18a)$$

$$(G_{11}^{(a)}+G_{00}^{(b)})q\cdot k+(G_{12}^{(a)}-G_{00}^{(b)})Q\cdot k+G_{13}^{(a)}q\cdot(q+k)=-m_p^2[F_A(q^2)+f_A],\quad (18b)$$

$$m_p^2G_{00}^{(c)}+G_{11}^{(c)}q\cdot k+G_{12}^{(c)}Q\cdot k+G_{13}^{(c)}q\cdot(q+k)+iG_{21}^{(b)}Q\cdot k-i2Mm_p(G_{11}^{(b)}-G_{00}^{(b)})=i\frac{2Mm_p^3}{m_\pi^2}\tilde{f}_P,\quad (18c)$$

$$G_{21}^{(c)}q\cdot k+G_{22}^{(c)}Q\cdot k+G_{23}^{(c)}q\cdot(q+k)+iG_{22}^{(b)}Q\cdot k-i2Mm_pG_{12}^{(b)}-im_p^2G_{00}^{(a)}=\tfrac{1}{2}im_p^2f_E,\quad (18d)$$

$$m_p^2G_{00}^{(c)}+G_{31}^{(c)}q\cdot k+G_{32}^{(c)}Q\cdot k+G_{33}^{(c)}q\cdot(q+k)+iG_{23}^{(b)}Q\cdot k-i2Mm_pG_{13}^{(b)}=i\frac{2Mm_p^3}{m_\pi^2}[F_P(q^2)+f_P],\quad (18e)$$

$$m_p^2G_{00}^{(d)}+G_{11}^{(d)}q\cdot k+G_{12}^{(d)}Q\cdot k+G_{13}^{(d)}q\cdot(q+k)-im_p^2G_{11}^{(b)}+im_p^2G_{00}^{(b)}=m_p^2f_1,\quad (18f)$$

$$G_{21}^{(d)}q\cdot k+G_{22}^{(d)}Q\cdot k+G_{23}^{(d)}q\cdot(q+k)-im_p^2G_{12}^{(b)}-im_p^2G_{00}^{(b)}=m_p^2f_2,\quad (18g)$$

$$m_p^2G_{00}^{(d)}+G_{31}^{(d)}q\cdot k+G_{32}^{(d)}Q\cdot k+G_{33}^{(d)}q\cdot(q+k)-im_p^2G_{13}^{(b)}=m_p^2f_3,\quad (18h)$$

$$f_2Q\cdot k+f_3q\cdot k=-im_p^2F_A,\quad (18i)$$

$$\frac{1}{2M}\frac{1}{2m_p}f_EQ\cdot k+\frac{1}{m_\pi^2}f_Pq\cdot k=F_A(q^2)+\frac{q^2}{m_\pi^2}F_P(q^2)\quad (18j)$$

We note that the CVC constraint is consistent with the CEC constraint since Eq. (17b) follows from Eqs. (17g) and (17h) and Eqs. (15a)—(15e), while Eqs. (17d) and (17e) are compatible with Eqs. (15a) and (15f)—(15h). It is to be noted that such consistency can be obtained only if $\Delta\equiv M_f-M_i$ is neglected. Further, the PCAC constraint is also consistent with the CEC constraint since Eq. (18i) follows from Eqs. (18b), (18g), and (18h), and Eqs. (16b)—(16e) while Eq. (18j) is a consequence of Eqs. (18d) and (18e) and Eqs. (16a), (16b), (16f), and (16h).

We now proceed to determine, at least approximately, the various weak radiative nuclear form factors $R(q^2,Q\cdot k,q\cdot k)$ as functions of $q^2$, $Q\cdot k$, and $q\cdot k$. To do this, we abstract certain general properties of these form factors which follow from perturbation theory and from several simple physical assumptions, viz:

(I) The dependence of the $R(q^2,Q\cdot k,q\cdot k)$ on $Q\cdot k$ and $q\cdot k$ enters predominantly through the propagators of the initial and final nuclei: $[M_i^2+(p^{(i)}-k)^2]^{-1}=-[(Q-q)\cdot k]^{-1}$ and $[M_f^2+(p^{(f)}+k)^2]^{-1}=+[(Q+q)\cdot k]^{-1}$.

(II) The $R(q^2,Q\cdot k,q\cdot k)$ are primarily determined by the weak nonradiative nuclear form factors $F_{V,M,A,P}$ [Eqs. (1a), (1b)] and by the electric-charge and anomalous-magnetic-moment form factors of the initial and final nuclei $e_i, e_f, \mu_i, \mu_f$.

(III) More specifically, the $R(q^2,Q\cdot k,q\cdot k)$ are at most *linear* in the $\pm[(Q\pm q)\cdot k]^{-1}$, *linear* in the $F_{V,M,A,P}$, and *linear* in the $e_i, e_f, \mu_i, \mu_f$. Also wherever $[(Q-q)\cdot k]^{-1}$ appears it is multiplied by $e_i$ or $\mu_i$ and wherever $[(Q+q)\cdot k]^{-1}$ appears it is multiplied by $e_f$ or $\mu_f$.

Assumptions I, II, and III indicate that any $R(q^2,Q\cdot k,q\cdot k)$ may be expressed as

$$R(q^2,Q\cdot k,q\cdot k)=\frac{R^+}{(Q+q)\cdot k}+\frac{R^-}{(Q-q)\cdot k}+R^0,\quad (19)$$

where $R^-$ is linear in the $F_{V,M,A,P}((q+k)^2, (p^{(i)}-k)^2, (p^{(f)})^2)$ and linear in the $e_i(k^2,(p^{(i)})^2,(p^{(i)}-k)^2)$ and $\mu_i(k^2,(p^{(i)})^2,(p^{(i)}-k)^2)$ and where entirely analogous expressions hold for $R^+$ and $R^0$.

It is further seen that the $R(q^2,Q\cdot k,q\cdot k)$ of Eq. (19) cannot satisfy the CEC, CVC, and PCAC constraint equations (15a)–(18j) unless one sets

$$\begin{aligned}F_{V,M,A,P}((q+k)^2,(p^{(i)}-k)^2,(p^{(f)})^2)&=F_{V,M,A,P}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\\&\cong F_{V,M,A,P}(q^2,-M_i^2,-M_f^2)\\&\equiv F_{V,M,A,P}(q^2),\end{aligned}$$

$$\begin{aligned}e_i(k^2,(p^{(i)})^2,(p^{(i)}-k)^2)&=e_i(0,-M_i^2,(p^{(i)}-k)^2)\\&\cong e_i(0,-M_i^2,-M_i^2)\equiv e_i,\ \text{etc.}\end{aligned}$$

This last restriction together with Eq. (19) suggests the form of the "linearity hypothesis" (LH) for the $R(q^2,Q\cdot k,q\cdot k)$, viz:

$$R(q^2,Q\cdot k,q\cdot k)=\frac{R^+(q^2)}{(Q+q)\cdot k}+\frac{R^-(q^2)}{(Q-q)\cdot k}+R^{(0)}(q^2),\quad (20)$$

where $R^+(q^2)$ is linear in $F_{V,M,A,P}(q^2)$ and in $e_f$, $\mu_f$, $R^-(q^2)$ is linear in $F_{V,M,A,P}(q^2)$ and in $e_i,\mu_i$, and $R^0(q^2)$ is linear in $F_{V,M,A,P}(q^2)$ and in $e_i, e_f, \mu_i, \mu_f$. We note that expressions of the form of Eq. (20) for the $R(q^2,Q\cdot k,q\cdot k)$ can always be made to satisfy the CEC, CVC, and PCAC constraint equations (15a)–(18j) identically by proper choice of constant parameters. For example, the CEC constraint equation (15b) is identically satisfied by

$$F_{12}^{(b)}(q^2,Q\cdot k,q\cdot k)=m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right),\quad (21a)$$

$$F_{13}^{(b)}(q^2, Q\cdot k, q\cdot k) = -m_p^2\left(F_V(q^2) + \frac{M}{m_p}F_M(q^2)\right) \times\left(\frac{e_i}{(Q-q)\cdot k} + \frac{e_f}{(Q+q)\cdot k}\right) \tag{21b}$$

since $e_f = e_i - 1$.

The meaning of LH may be explored a little further by comparing the $R(q^2, Q\cdot k, q\cdot k)$ obtained from LH with the $R(q^2, Q\cdot k, q\cdot k)$ calculated by perturbation theory. Recalling that in perturbation theory contributions to $R(q^2, Q\cdot k, q\cdot k)$ come not only from diagrams corresponding to $\mu^- N_i \to \mu^- X\gamma \to \nu_\mu N_f\gamma$ and $\mu^- N_i \to \nu_\mu Y \to \nu_\mu N_f\gamma$ where the photon is emitted "before" and "after" the neutrino but also from the "box" diagram (BD) corresponding to $\mu^- N_i \to \nu_\mu N_f\gamma$ where the photon and neutrino are emitted "simultaneously," we obtain, for example,

$$\begin{aligned} F_{12}^{(b)}(q^2, Q\cdot k, q\cdot k) = {} & m_p^2\sum_X\left(F_V^{(X\to N_f)}((q+k)^2, (p^{(i)}-k)^2, -M_f^2) + \frac{M}{m_p}F_M^{(X\to N_f)}((q+k)^2, (p^{(i)}-k)^2, -M_f^2)\right) \\ & \times\left(\frac{e^{(N_i\to X)}(0, -M_i^2, (p^{(i)}-k)^2)}{-M_X^2 + M_i^2 + (Q-q)\cdot k}\right) \\ & - m_p^2\sum_Y\left(F_V^{(N_i\to Y)}((q+k)^2, -M_i^2, (p^{(f)}+k)^2) + \frac{M}{m_p}F_M^{(N_i\to Y)}((q+k)^2, -M_i^2, (p^{(f)}+k)^2)\right) \\ & \times\left(\frac{e^{(Y\to N_f)}(0, (p^{(f)}+k)^2, -M_f^2)}{M_Y^2 - M_f^2 + (Q+q)\cdot k}\right) + [F_{12}^{(b)}(q^2, Q\cdot k, q\cdot k)]_{\rm BD}\,, \end{aligned} \tag{22a}$$

$$\begin{aligned} F_{13}^{(b)}(q^2, Q\cdot k, q\cdot k) = {} & -m_p^2\sum_X\left(F_V^{(X\to N_f)}((q+k)^2, (p^{(i)}-k)^2, -M_f^2) + \frac{M}{m_p}F_M^{(X\to N_f)}((q+k)^2, (p^{(i)}-k)^2, -M_f^2)\right) \\ & \times\left(\frac{e^{(N_i\to X)}(0, -M_i^2, (p^{(i)}-k)^2)}{-M_X^2 + M_i^2 + (Q-q)\cdot k}\right) \\ & - m_p^2\sum_Y\left(F_V^{(N_i\to Y)}((q+k)^2, -M_i^2, (p^{(f)}+k)^2) \right.\\ & \left. + \frac{M}{m_p}F_M^{(N_i\to Y)}((q+k)^2, -M_i^2, (p^{(f)}+k)^2)\right) \\ & \times\left(\frac{e^{(N_i\to Y)}(0, (p^{(f)}+k)^2, -M_f^2)}{M_Y^2 - M_f^2 + (Q+q)\cdot k}\right) + [F_{13}^{(b)}(q^2, Q\cdot k, q\cdot k)]_{\rm BD}\,, \end{aligned} \tag{22b}$$

where the sum over $X\,[Y]$ includes contributions from the nucleus $N_i\,[N_f]$ and from all other hadronic systems with the same electric charge as $N_i\,[N_f]$ and where

$$F_{V,M}^{(N_i\to N_f)}((q+k)^2, (p^{(i)}-k)^2, -M_f^2) \equiv F_{V,M}((q+k)^2, (p^{(i)}-k)^2, -M_f^2),$$

$$e^{(N_i\to N_i)}(0, -M_i^2, (p^{(i)}-k)^2) \equiv e_i(0, -M_i^2, (p^{(i)}-k)^2)\,[F_{V,M}^{(N_i\to N_f)}((q+k)^2, -M_i^2, (p^{(f)}+k)^2)$$

$$\equiv F_{V,M}((q+k)^2, -M_i^2, (p^{(f)}+k)^2),$$

$$e^{(N_f\to N_f)}(0, (p^{(f)}+k)^2, -M_f^2) \equiv e_f(0, (p^{(f)}+k)^2, -M_f^2)]\,.$$

With these expressions for $F_{12}^{(b)}(q^2, Q\cdot k, q\cdot k)$, $F_{13}^{(b)}(q^2, Q\cdot k, q\cdot k)$, the CEC constraint equation (15b) becomes

$$\left(F_V((q+k)^2,(p^{(i)}-k)^2,-M_f^2)+\frac{M}{m_p}F_M((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\right)e_i(0,-M_i^2,(p^{(i)}-k)^2)$$
$$-\left(F_V((q+k)^2,-M_i^2,(p^{(f)}+k)^2)+\frac{M}{m_p}F_M((q+k)^2,-M_i^2,(p^{(f)}+k)^2)\right)e_f(0,(p^{(f)}+k)^2,-M_f^2)$$
$$+\sum_{X(X\neq N_i)}\left(F_V^{(X\to N_f)}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)+\frac{M}{m_p}F_M^{(X\to N_f)}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\right)$$
$$\times\left(\frac{e^{(N_i\to X)}(0,-M_i^2,(p^{(i)}-k)^2)(Q-q)\cdot k}{-M_X^2+M_i^2+(Q-q)\cdot k}\right)$$
$$-\sum_{Y(Y\neq N_f)}\left(F_V^{(N_i\to Y)}((q+k)^2,-M_i^2,(p^{(f)}+k)^2)+\frac{M}{m_p}F_M^{(N_i\to Y)}((q+k)^2,-M_i^2,(p^{(f)}+k)^2)\right)$$
$$\times\left(\frac{e^{(Y\to N_f)}(0,(p^{(f)}+k)^2,-M_f^2)(Q+q)\cdot k}{M_Y^2-M_f^2+(Q+q)\cdot k}\right)$$
$$+[F_{12}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}\frac{Q\cdot k}{m_p^2}+[F_{13}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}\frac{q\cdot k}{m_p^2}=F_V(q^2)+\frac{M}{m_p}F_M(q^2)\qquad(22c)$$

and is not identically or manifestly satisfied if only because

$$[F_{12}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}\quad\text{and}\quad[F_{13}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}$$

are not easily calculable; alternatively, Eq. (22c) can be viewed as a sum rule which permits determination of

$$[F_{12}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}\frac{Q\cdot k}{m_p^2}+[F_{13}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}\frac{q\cdot k}{m_p^2}$$

in terms of

$$F_{V,M}(q^2),\ F_{V,M}^{(X\to N_f)}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\quad\text{and}\quad F_{V,M}^{(N_i\to Y)}((q+k)^2,-M_i^2,(p^{(f)}+k)^2).$$

On the other hand, comparison of Eqs. (22a) and (22b) with Eqs. (21a) and (21b) yields

$$\left(F_V((q+k)^2,(p^{(i)}-k)^2,-M_f^2)+\frac{M}{m_p}F_M((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\right)\left(\frac{e_i(0,-M_i^2,(p^{(i)}-k)^2)}{(Q-q)\cdot k}\right)$$
$$+\sum_{X(X\neq N_i)}\left(F_V^{(X\to N_f)}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)+\frac{M}{m_p}F_M^{(X\to N_f)}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\right)$$
$$\times\left(\frac{e^{(N_i\to X)}(0,-M_i^2,(p^{(i)}-k)^2)}{-M_X^2+M_i^2+(Q-q)\cdot k}\right)+\frac{1}{2m_p^2}\{[F_{12}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}-[F_{13}^{(b)}(q^2,Q\cdot k,q\cdot k]_{\mathrm{BD}}\}$$
$$=[F_V(q^2)+\frac{M}{m_p}F_M(q^2)]\left(\frac{e_i}{(Q-q)\cdot k}\right),\qquad(22d)$$

$$\left(F_V((q+k)^2,-M_i^2,(p^{(f)}+k)^2)+\frac{M}{m_p}F_M((q+k)^2,-M_i^2,(p^{(f)}+k)^2)\right)\left(\frac{e_f(0,(p^{(f)}+k)^2,-M_f^2)}{(Q+q)\cdot k}\right)$$
$$+\sum_{Y(Y\neq N_f)}\left(F_V^{(N_i\to Y)}((q+k)^2,-M_i^2,(p^{(f)}+k)^2)+\frac{M}{m_p}F_M((q+k)^2,-M_i^2,(p^{(f)}+k)^2)\right)$$
$$\times\left(\frac{e^{(Y\to N_f)}(0,(p^{(f)}+k)^2,-M_f^2)}{M_Y^2-M_f^2+(Q+q)\cdot k}\right)-\frac{1}{2m_p^2}\{[F_{12}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}+[F_{13}^{(b)}(q^2,Q\cdot k,q\cdot k)]_{\mathrm{BD}}\}$$
$$=[F_V(q^2)+\frac{M}{m_p}F_M(q^2)]\left(\frac{e_f}{(Q+q)\cdot k}\right)\qquad(22e)$$

which can be viewed as "approximate sum rules." It is interesting that from the point of view of Eqs. (22d) and (22e) the whole contribution of the $\sum_{X(X\neq N_i)}\cdots$, $\sum_{Y(Y\neq N_f)}\cdots$, and BD terms corresponds to a shift in the values of the variables on which $F_V$, $F_M$, $e_i$, and $e_f$ depend [i.e., $(q+k)^2\to q^2$, $(p^{(i)}-k)^2\to(p^{(i)})^2=-M_i^2$, $(p^{(f)}+k)^2\to(p^{(f)})^2=-M_f^2$].

We go on to calculate explicitly all of the "relevant" $R(q^2, Q\cdot k, q\cdot k)$, i.e., all of the $R(q^2,Q\cdot k, q\cdot k)$ which contribute to $\mathcal{T}^{(h)}$; these are all of the $F_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)$ and $G_{ij}^{(a),(b),(c),(d)}(q^2, Q\cdot k,q\cdot k)$ except for $F_{11}^{(b)}(q^2,Q\cdot k,q\cdot k)$, $F_{21}^{(b)}(q^2, Q\cdot k,q\cdot k)$, $F_{1j}^{(c)}(q^2, Q\cdot k, q\cdot k)$, $F_{1j}^{(d)}(q^2, Q\cdot k,q\cdot k)$, $G_{11}^{(b)}(q^2, Q\cdot k, q\cdot k)$, $G_{21}^{(b)}(q^2, Q\cdot k, q\cdot k)$, $G_{1j}^{(c)}(q^2, Q\cdot k, q\cdot k)$, and $G_{1j}^{(d)}(q^2, Q\cdot k, q\cdot k)$ which, in view of Eqs. (12), (13), and (5c), do not contribute to $\epsilon_\mu^* V_{\mu\lambda}(k,Q,q)$ and $\epsilon_\mu^* A_{\mu\lambda}(k,Q,q)$ and so do not contribute to $\mathcal{T}^{(h)}$ [Eq. (5)]. We do this by working out the perturbation-theory expressions for an "appropriate" set of $F_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)$ or $G_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)$ in the approximation:

$$\begin{aligned}
&F_{V,M,A,P}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\\
&\qquad\cong F_{V,M,A,P}((q+k)^2,-M_i^2,(p^{(f)}+k)^2)\\
&\qquad\cong F_{V,M,A,P}(q^2),\\
&F_{V,M,A,P}(q^2)\gg F_{V,M,A,P}^{(X\to N_f)}((q+k)^2,(p^{(i)}-k)^2,-M_f^2)\quad (X\neq N_i),\\
&F_{V,M,A,P}(q^2)\gg F_{V,M,A,P}^{(N_i\to Y)}((q+k)^2,-M_i^2,(p^{(f)}+k)^2)\quad (Y\neq N_f),\\
&F_{V,M}(q^2)\gg [F_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)]_{\rm BD},\\
&F_{A,P}(q^2)\gg [G_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)]_{\rm BD},\\
&e_i(0,-M_i^2,(p^{(i)}-k)^2)\cong e_i,\\
&\mu_i(0,-M_i^2,(p^{(i)}-k)^2)\cong \mu_i,\\
&e_f(0,(p^{(f)}+k)^2,-M_f^2)\cong e_f,\\
&\mu_f(0,(p^{(f)}+k)^2,-M_f^2)\cong \mu_f.
\end{aligned}\tag{23}$$

We proceed to calculate all the relevant $F_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)$. We first note that any $F_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)$ calculated by perturbation theory in the approximation of Eq. (23) is of the form required by the LH of Eq. (20) [see, e.g., Eqs. (24a) and (24b) below] and we define an appropriate set of relevant $F_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)$ as a set of relevant $F_{ij}^{(a),(b),(c),(d)}(q^2,Q\cdot k,q\cdot k)$ whose specification via perturbation theory in the approximation of Eq. (23) is sufficient to determine all the other relevant $F_{ij}^{(a),(b),(c),(d)}(q^2, Q\cdot k,q\cdot k)$ by means of the CEC and CVC constraint equations (15a)–(15h) and (17a)–(17h) and the assumption that these other relevant $F_{ij}^{(a),(b),(c),(d)}(q^2, Q\cdot k,q\cdot k)$ are also of the form required by the LH of Eq. (20). Thus choosing $F_{00}^{(b)}(q^2,Q\cdot k,q\cdot k)$, $F_{11}^{(a)}(q^2,Q\cdot k,q\cdot k)$, and $F_{23}^{(a)}(q^2,Q\cdot k,q\cdot k)$ as the members of an appropriate set and performing the perturbation theory calculation in the approximation of Eq. (23), we obtain

$$F_{00}^{(b)}=-m_p^2\left[\frac{F_M(q^2)}{2m_p}\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right)\right],\tag{24a}$$

$$F_{11}^{(a)}=m_p^2\left\{\frac{F_M(q^2)}{2m_p}\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left[\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right)-\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}+\frac{e_f+\mu_f}{(Q+q)\cdot k}\right)\right]\right\},\tag{24b}$$

$$F_{23}^{(a)}=0\tag{24c}$$

whence, determining the other relevant $F_{ij}^{(a),(b),(c),(d)}$ by means of Eqs. (15a)–(15h), (17a)–(17h), and (20) (see Appendix A), we obtain

$$F_{00}^{(a)}=m_p\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right),\tag{25a}$$

$F_{00}^{(b)}$ as given by Eq. (24a), (25b)

$F_{11}^{(a)}$ as given by Eq. (24b), (25c)

$$F_{12}^{(a)}=-\tfrac{1}{2}m_pF_M(q^2)\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right),\tag{25d}$$

$$F_{13}^{(a)}=0,\tag{25e}$$

$$F_{12}^{(b)}=m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right),$$

[as in Eq. (21a)] (25f)

$$F_{13}^{(b)}=-m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right),$$

[as in Eq. (21b)] (25g)

$$F_{21}^{(a)}=m_p^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25h}$$

$$F_{22}^{(a)}=-m_p^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25i}$$

$F_{23}^{(a)}$ as given by Eq. (24c), (25j)

$$F_{22}^{(b)}=m_p^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25k}$$

$$F_{23}^{(b)}=-m_p^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25l}$$

$$F_{00}^{(c)}=im_p\left[\frac{F_M(q^2)}{2m_p}(e_i+e_f)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right)\right], \tag{25m}$$

$$F_{21}^{(c)}=-im_p^3\left[\frac{F_M(q^2)}{2m_p}\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right)\right], \tag{25n}$$

$$F_{22}^{(c)}=\tfrac{1}{2}im_p^2F_M(q^2)\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \tag{25o}$$

$$F_{23}^{(c)}=0, \tag{25p}$$

$$F_{31}^{(c)}=im_p^3\left[\frac{F_M(q^2)}{2m_p}\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right)\right], \tag{25q}$$

$$F_{32}^{(c)}=-\tfrac{1}{2}im_p^2F_M(q^2)\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right), \tag{25r}$$

$$F_{33}^{(c)}=0, \tag{25s}$$

$$F_{00}^{(d)}=-im_p^2\left[\frac{F_M(q^2)}{2m_p}\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}+\frac{e_f+\mu_f}{(Q+q)\cdot k}\right)\right], \tag{25t}$$

$$F_{21}^{(d)}=\tfrac{1}{2}im_p^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25u}$$

$$F_{22}^{(d)}=-\tfrac{1}{2}im_p^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25v}$$

$$F_{23}^{(d)}=0, \tag{25w}$$

$$F_{31}^{(d)}=-\tfrac{1}{2}im_p^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25x}$$

$$F_{32}^{(d)}=\tfrac{1}{2}im_p^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i}-\frac{1}{(Q-q)\cdot k}\frac{\mu_f}{2M_f}\right), \tag{25y}$$

$$F_{33}^{(d)}=0, \tag{25z}$$

Substitution of these values of the $F_{ij}^{(a),(b),(c),(d)}$ into Eq. (12) for $V_{\mu\lambda}(k,q,Q)$ and calculation of the quantity $(1/m_p)\epsilon_\mu^* V_{\mu\lambda}(k,q,Q)$ which enters linearly into $\mathcal{T}^{(h)}$ [Eq. (5)] yields the remarkably simple expression:

$$\frac{1}{m_p}\epsilon_\mu^*V_{\mu\lambda}(k,q,Q)=-\bar{u}^{(f)}(p^{(f)},s^{(f)})\left[\left(F_V(q^2)\gamma_\lambda-F_M(q^2)\frac{\sigma_{\lambda\eta}(q+k)_\eta}{2m_p}\right)i\frac{M_i-i(\not p^{(i)}-\not k)}{M_i^2+(p^{(i)}-k)^2}\left(e_i+\frac{\mu_i}{2M_i}i\not k\right)\not\epsilon^* + \not\epsilon^*\left(e_f-\frac{\mu_f}{2M_f}i\not k\right)\frac{M_f-i(\not p^{(f)}+\not k)}{M_f^2+(p^{(f)}+k)^2}i\left(F_V(q^2)\gamma_\lambda-F_M(q^2)\frac{\sigma_{\lambda\eta}(q+k)_\eta}{2m_p}\right)+\frac{F_M(q^2)}{2m_p}\epsilon_\mu^*\sigma_{\mu\lambda}\right]u^{(i)}(p^{(i)},s^{(i)}). \tag{26}$$

We emphasize that had we chosen another appropriate set of "relevant" $F_{ij}^{(a),(b),(c),(d)}(q^2, Q\cdot k, q\cdot k)$, say, $\{F_{00}^{(b)}, F_{12}^{(a)}, F_{13}^{(a)}, F_{23}^{(a)}\}$ or $\{F_{11}^{(a)}, F_{12}^{(a)}, F_{22}^{(b)}, F_{23}^{(a)}\}$ or $\cdots$ (all of which exclude $F_{00}^{(a)}$ from membership), to be calculated by perturbation theory in the approximation of Eq. (23) and then, as previously, determined the other relevant $F_{ij}^{(a),(b),(c),(d)}$ by means of Eqs. (15a)–(15h), (17a)–(17h), and (20), we would have obtained again the expressions in Eqs. (25a)–(25z). In fact, any set of relevant $F_{ij}^{(a),(b),(c),(d)}$ (not containing $F_{00}^{(a)}$ as a member) which is sufficient to determine all the other relevant $F_{ij}^{(a),(b),(c),(d)}$ may serve equally well as the "appropriate" set. Thus our results for the $F_{ij}^{(a),(b),(c),(d)}$ in Eqs. (25a)–(25z) are independent of which particular appropriate set of $F_{ij}^{(a),(b),(c),(d)}$ is chosen initially. We also emphasize that the $F_{ij}^{(a),(b),(c),(d)}(q^2, Q\cdot k, q\cdot k)$ in Eqs. (25a)–(25z) with but one exception agree with what one would calculate by perturbation theory in the approximation of Eq. (23). The exception, $F_{00}^{(a)}$ in Eq. (25a), must therefore contain a non-negligible net contribution from

$$F_{V,M}^{(X\to N_f)}, \quad F_{V,M}^{(N_i\to Y)} \quad (X\neq N_i, Y\neq N_f),$$

and $(F_{00}{}^{(a)})_{\rm BD}$ which is automatically included through our use of the CEC constraint equation (15a); this net contribution can be obtained by comparing Eq. (25a) with the result of perturbation theory in the approximation of Eq. (23) and is just $-\frac{1}{2}F_M(q^2)$. Finally we note that perturbation-theory calculations in the approximation of Eq. (23) for $F_{11}^{(b)}$, $F_{21}^{(b)}$, $F_{1j}^{(c)}$, and $F_{1j}^{(d)}$ together with the $F_{ij}^{(a),(b),(c),(d)}$ of Eqs. (25a)–(25z) are inconsistent with the CVC constraint equations (17a)–(17h)—however, this is of no consequence since, as noted above, $F_{11}^{(b)}, F_{21}^{(b)}, F_{1j}^{(c)}$, and $F_{1j}^{(d)}$ do not contribute to $\mathcal{T}^{(h)}$.

In a completely analogous way, we can, starting with an "appropriate" set of "relevant" $G_{ij}^{(a),(b),(c),(d)}(q^2, Q\cdot k, q\cdot k)$, determine all the other relevant $G_{ij}^{(a),(b),(c),(d)}(q^2, Q\cdot k, q\cdot k)$; the members of such an appropriate set are taken to be

$$G_{00}^{(a)} = -m_\rho F_A(q^2)\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right), \tag{27}$$

$$G_{00}^{(b)} = m_\rho{}^2 F_A(q^2)\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right), \tag{28}$$

$$G_{13}^{(a)} = -m_\rho{}^2 F_P(q^2)\,\frac{2M}{m_\pi{}^2}\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right), \tag{29}$$

$$G_{22}^{(a)} = 0, \tag{30}$$

$$G_{23}^{(a)} = -m_\rho{}^3 F_P(q^2)\,\frac{2M}{m_\pi^2}\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right), \tag{31}$$

where the expressions on the right-hand sides of the equations are calculated from perturbation theory in the approximation of Eq. (23). Then, determining the other relevant $G_{ij}^{(a),(b),(c),(d)}$ by means of Eqs. (16a)–(16h), (18a)–(18j), and (20) (see Appendix A), we obtain

$G_{00}^{(a)}$ as given by Eq. (27), (32a)

$G_{00}^{(b)}$ as given by Eq. (28), (32b)

$$G_{11}^{(a)} = 2m_\rho{}^2 F_A(q^2)\,\frac{e_f+\mu_f}{(Q+q)\cdot k} + m_\rho{}^2 F_P(q^2)\,\frac{2M}{m_\pi{}^2}\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2m_f}\right), \tag{32c}$$

$$G_{12}^{(a)} = 0, \tag{32d}$$

$G_{13}^{(a)}$ as given by Eq. (29), (32e)

$$G_{12}^{(b)} = -m_\rho{}^2 F_A(q^2)\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \tag{32f}$$

$$G_{13}^{(b)} = m_\rho{}^2 F_A(q^2)\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right), \tag{32g}$$

$$G_{21}^{(a)} = m_\rho{}^3 F_P(q^2)\,\frac{2M}{m_\pi^2}\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right), \tag{32h}$$

$G_{22}^{(a)}$ as given by Eq. (30), (32i)

$G_{23}^{(a)}$ as given by Eq. (31), (32j)

$$G_{22}^{(b)} = -m_\rho{}^3 F_A(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{32k}$$

$$G_{23}^{(b)} = m_\rho{}^3 F_A(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{32l}$$

$$G_{00}^{(c)} = -i\,m_\rho\left[F_A(q^2)\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right)-F_P(q^2)\,\frac{2M}{m_\pi^2}\right], \tag{32m}$$

$$G_{21}^{(c)} = i\,m_\rho{}^3\left[F_A(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right)-F_P(q^2)\,\frac{2M}{m_\pi{}^2}\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right)\right], \tag{32n}$$

$$G_{22}^{(c)} = 0, \tag{32o}$$

$$G_{23}^{(c)} = i\,m_\rho{}^3 F_P(q^2)\,\frac{2M}{m_\pi{}^2}\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \tag{32p}$$

$$G_{31}^{(c)} = -i m_p^3 \Big[ F_A(q^2) \left( \frac{1}{(Q-q)\cdot k} \frac{\mu_i}{2M_i} + \frac{1}{(Q+q)\cdot k} \frac{\mu_f}{2M_f} \right) - F_P(q^2) \frac{2M}{m_\pi^2} \left( \frac{e_i}{(Q-q)\cdot k} + \frac{e_f}{(Q+q)\cdot k} \right) \Big] , \quad (32q)$$

$$G_{32}^{(c)} = 0 , \quad (32r)$$

$$G_{33}^{(c)} = -i m_p^3 F_P(q^2) \frac{2M}{m_\pi^2} \left( \frac{e_i}{(Q-q)\cdot k} + \frac{e_f}{(Q+q)\cdot k} \right) , \quad (32s)$$

$$G_{00}^{(d)} = i m_p^2 F_A(q^2) \left( \frac{e_i + \mu_i}{(Q-q)\cdot k} + \frac{e_f + \mu_f}{(Q+q)\cdot k} \right) , \quad (32t)$$

$$G_{21}^{(d)} = i m_p^4 F_P(q^2) \frac{2M}{m_\pi^2} \left( \frac{1}{(Q-q)\cdot k} \frac{\mu_i}{2M_i} - \frac{1}{(Q+q)\cdot k} \frac{\mu_f}{2M_f} \right) , \quad (32u)$$

$$G_{22}^{(d)} = 0 , \quad (32v)$$

$$G_{23}^{(d)} = -i m_p^4 F_P(q^2) \frac{2M}{m_\pi^2} \left( \frac{1}{(Q-q)\cdot k} \frac{\mu_i}{2M_i} - \frac{1}{(Q+q)\cdot k} \frac{\mu_f}{2M_f} \right) , \quad (32w)$$

$$G_{31}^{(d)} = -i m_p^4 F_P(q^2) \frac{2M}{m_\pi^2} \left( \frac{1}{(Q-q)\cdot k} \frac{\mu_i}{2M_i} + \frac{1}{(Q+q)\cdot k} \frac{\mu_f}{2M_f} \right) , \quad (32x)$$

$$G_{32}^{(d)} = 0 , \quad (32y)$$

$$G_{33}^{(d)} = i m_p^4 F_P(q^2) \frac{2M}{m_\pi^2} \left( \frac{1}{(Q-q)\cdot k} \frac{\mu_i}{2M_i} + \frac{1}{(Q+q)\cdot k} \frac{\mu_f}{2M_f} \right) . \quad (32z)$$

In addition, we obtain

$$f_A = \left( F_A(q^2) + \frac{q^2}{m_\pi^2} F_P(q^2) \right) (\mu_i - \mu_f) , \quad (32a')$$

$$f_P = -m_\pi^2 \left( F_A(q^2) + \frac{q^2}{m_\pi^2} F_P(q^2) \right) \times \left( \frac{e_i}{(Q-q)\cdot k} + \frac{e_f}{(Q+q)\cdot k} \right) , \quad (32b')$$

$$f_E = 4Mm_p \left( F_A(q^2) + \frac{q^2}{m_\pi^2} F_P(q^2) \right) \times \left( \frac{e_i}{(Q-q)\cdot k} - \frac{e_f}{(Q+q)\cdot k} \right) , \quad (32c')$$

$$f_2 = -i m_p^2 \left( F_A(q^2) + \frac{q^2}{m_\pi^2} F_P(q^2) \right) \times \left( \frac{\mu_i}{(Q-q)\cdot k} - \frac{\mu_f}{(Q+q)\cdot k} \right) , \quad (32d')$$

$$f_3 = i m_p^2 \left( F_A(q^2) + \frac{q^2}{m_\pi^2} F_P(q^2) \right) \times \left( \frac{\mu_i}{(Q-q)\cdot k} + \frac{\mu_f}{(Q+q)\cdot k} \right) , \quad (32e')$$

$$\tilde{f}_E = 4Mm_p \left( F_A(q^2) + \frac{q^2}{m_\pi^2} F_P(q^2) \right) \times \left( \frac{e_i + \mu_i}{(Q-q)\cdot k} - \frac{e_f + \mu_f}{(Q+q)\cdot k} \right) + 2m_p F_A(q^2) \left( \frac{\mu_i}{2M_i} + \frac{\mu_f}{2M_f} \right) , \quad (32f')$$

Equation (13) together with Eqs. (32a)–(32z) yields the remarkably simple expression:

$$\frac{1}{m_p} \epsilon_\mu^* A_{\mu\lambda}(k,q,Q) = -\bar{u}^{(f)}(p^{(f)}, s^{(f)}) \times \Bigg[ \left( F_A(q^2)\gamma_\lambda\gamma_5 + F_P(q^2) i \frac{2M(q_\lambda - k_\lambda)\gamma_5}{m_\pi^2} \right) i \frac{M_i - i(\not p^{(i)} - \not k)}{M_i^2 + (p^{(i)} - k)^2} \left( e_i + \frac{\mu_i}{2M_i} i\not k \right) \not\epsilon^* + \not\epsilon^* \left( e_f - \frac{\mu_f}{2M_f} i\not k \right) \frac{M_f - i(\not p^{(f)} + \not k)}{M_f^2 + (p^{(f)} + k)^2} i \left( F_A(q^2)\gamma_\lambda\gamma_5 + F_P(q^2) i \frac{2M(q_\lambda - k_\lambda)\gamma_5}{m_\pi^2} \right) - i\gamma_5 \epsilon_\mu^* \delta_{\mu\lambda} F_P(q^2) \frac{2M}{m_\pi^2} \Bigg] u^{(i)}(p^{(i)}, s^{(i)}) . \quad (33)$$

This expression, together with that for $(1/m_p) \times \epsilon_\mu^* V_{\mu\lambda}(k,q,Q)$ [Eq. (26)], completely determines the $\mathcal{T}^{(h)}$ of Eq. (5).

We emphasize that all the relevant $G_{ij}^{(a),(b),(c),(d)}$ in Eqs. (32a)–(32z) except $G_{00}^{(c)}$, $G_{11}^{(a)}$, $G_{21}^{(a)}$, $G_{21}^{(c)}$, $G_{31}^{(c)}$, $G_{21}^{(d)}$, and $G_{31}^{(d)}$ agree with what one would calculate by perturbation theory in the approximation of Eq. (23). We note that the difference between the values of $G_{11}^{(a)}$, $G_{21}^{(a)}$, $G_{21}^{(c)}$, $G_{31}^{(c)}$, $G_{21}^{(d)}$, and $G_{31}^{(d)}$ obtained by perturbation theory in the approximation of Eq. (23) and the values of $G_{11}^{(a)}$, $G_{21}^{(a)}$, $G_{21}^{(c)}$, $G_{31}^{(c)}$, $G_{21}^{(d)}$, and $G_{31}^{(d)}$ in Eqs. (32c), (32h), (32n), (32q), (32u), and (32x) correspond to a switch in sign of all the $F_P(q^2)$ terms. We further note that the value of $G_{00}^{(c)}$ in Eq. (32m) differs from the value of $G_{00}^{(c)}$ obtained by perturbation theory in the approximation of Eq. (23) by the term $(i2Mm_p/m_\pi{}^2)F_P(q^2)$.

We also emphasize that, as in the case of the $F_{ij}^{(a),(b),(c),(d)}$, our results for the relevant $G_{ij}^{(a),(b),(c),(d)}$ [Eqs. (32a)–(32z)] are independent of which particular set of $G_{ij}^{(a),(b),(c),(d)}$ is chosen initially as appropriate [i.e., chosen initially to be calculated by perturbation theory in the approximation of Eq. (23)]. It is, however, clear that an appropriate set must exclude every one of the $G_{00}^{(c)}$, $G_{11}^{(a)}$, $G_{21}^{(a)}$, $G_{21}^{(c)}$, $G_{31}^{(c)}$, $G_{21}^{(d)}$, and $G_{31}^{(d)}$, from membership.

As an overall check, we can test our results for gauge invariance (GI) by replacing $\epsilon_\mu^*$ by $k_\mu$ in the $\mathcal{T}$ of Eqs. (3)–(5); this gives zero, as required by GI, if

$$\frac{k_\mu}{m_p}[V_{\mu\lambda}(k,q,Q)+A_{\mu\lambda}(k,q,Q)] = \langle N_f(p^{(f)},s^{(f)})\,|[V_\lambda(0)+A_\lambda(0)]\,|N_i(p^{(i)},s^{(i)})\rangle \tag{34}$$

a result which is guaranteed by Eqs. (9a) and (9b). Further, we can perform the GI test on the explicit $\mathcal{T}$ of Eqs. (3)–(5) *and* Eqs. (1a), (1b), (26), and (33). Replacement of $\epsilon_\mu^*$ by $k_\mu$ in this $\mathcal{T}$ again gives zero.

Finally we note that the $\mathcal{T}$ of Eqs. (3)–(5), (1a), (1b), (26), and (33) can be written as

$$\mathcal{T}=\mathcal{T}^{(l)}+\mathcal{T}^{(h)} = \frac{e}{(2k_0)^{1/2}}\left(\frac{A}{k_0}+\frac{B}{m_\mu}+\frac{Ck_0}{m_\mu{}^2}\right), \tag{35}$$

where

$$A\equiv k_0\left(\frac{p^{(\mu)}\cdot\epsilon^*}{p^{(\mu)}\cdot k}-e_i\frac{p^{(i)}\cdot\epsilon^*}{p^{(i)}\cdot k}+e_f\frac{p^{(f)}\cdot\epsilon^*}{p^{(f)}\cdot k}\right)\mathcal{T}' \tag{36a}$$

with

$$\begin{aligned}\mathcal{T}'\equiv{}&\frac{G}{\sqrt{2}}\,\bar u^{(\nu)}(p^{(\nu)},s^{(\nu)})\gamma_\lambda(1+\gamma_5)u^{(\mu)}(p^{(\mu)},s^{(\mu)})\bar u^{(f)}(p^{(f)},s^{(f)})\\&\times\left(F_V(q^2)\gamma_\lambda-F_M(q^2)\frac{\sigma_{\lambda\eta}q_\eta}{2m_p}+F_A(q^2)\gamma_\lambda\gamma_5+F_P(q^2)\,i\,\frac{2Mq_\lambda\gamma_5}{m_\pi{}^2}\right)u^{(i)}(p^{(i)},s^{(i)}),\\ q_\lambda\equiv{}&(p^{(f)}-p^{(i)})_\lambda=(p^{(\mu)}-p^{(\nu)}-k)_\lambda,\end{aligned} \tag{36b}$$

$$\begin{aligned}B\equiv{}&-\frac{G}{\sqrt{2}}\,m_\mu\bar u^{(\nu)}(p^{(\nu)},s^{(\nu)})\gamma_\lambda(1+\gamma_5)\frac{\not k\not\epsilon^*}{2p^{(\mu)}\cdot k}\,u^{(\mu)}(p^{(\mu)},s^{(\mu)})\bar u^{(f)}(p^{(f)},s^{(f)})\\&\times\left(F_V(q^2)\gamma_\lambda-F_M(q^2)\frac{\sigma_{\lambda\eta}q_\eta}{2m_p}+F_A(q^2)\gamma_\lambda\gamma_5+F_P(q^2)\,i\,\frac{2Mq_\lambda\gamma_5}{m_\pi{}^2}\right)u^{(i)}(p^{(i)},s^{(i)})\\&+\frac{G}{\sqrt{2}}\,m_\mu\bar u^{(\nu)}(p^{(\nu)},s^{(\nu)})\gamma_\lambda(1+\gamma_5)u^{(\mu)}(p^{(\mu)},s^{(\mu)})\bar u^{(f)}(p^{(f)},s^{(f)})\\&\times\Bigg[\left(F_V(q^2)\gamma_\lambda-F_M(q^2)\frac{\sigma_{\lambda\eta}q_\eta}{2m_p}+F_A(q^2)\gamma_\lambda\gamma_5+F_P(q^2)\,i\,\frac{2Mq_\lambda\gamma_5}{m_\pi{}^2}\right)\\&\times\left(-\frac{\mu_i}{2M_i}\,i\not\epsilon^*+\frac{\mu_i}{2M_i}\frac{p^{(i)}\cdot\epsilon^*}{p^{(i)}\cdot k}\,i\not k+(e_i+\mu_i)\frac{\not k\not\epsilon^*}{2p^{(i)}\cdot k}\right)\\&+\left(-\frac{\mu_f}{2M_f}\,i\not\epsilon^*+\frac{\mu_f}{2M_f}\frac{p^{(f)}\cdot\epsilon^*}{p^{(f)}\cdot k}\,i\not k+(e_f+\mu_f)\frac{\not\epsilon^*\not k}{2p^{(f)}\cdot k}\right)\\&\times\left(F_V(q^2)\gamma_\lambda-F_M(q^2)\frac{\sigma_{\lambda\eta}q_\eta}{2m_p}+F_A(q^2)\gamma_\lambda\gamma_5+F_P(q^2)\,i\,\frac{2Mq_\lambda\gamma_5}{m_\pi^2}\right)\\&+\left(e_i\frac{p^{(i)}\cdot\epsilon^*}{p^{(i)}\cdot k}-e_f\frac{p^{(f)}\cdot\epsilon^*}{p^{(f)}\cdot k}\right)\left(F_M(q^2)\frac{\sigma_{\lambda\eta}k_\eta}{2m_p}+F_P(q^2)\,i\,\frac{2Mk_\lambda\gamma_5}{m_\pi^2}\right)\\&-\left(F_M(q^2)\frac{\sigma_{\lambda\mu}\epsilon_\mu^*}{2m_p}+F_P(q^2)\,i\,\frac{2M\epsilon_\lambda^*\gamma_5}{m_\pi^2}\right)\Bigg]u^{(i)}(p^{(i)},s^{(i)})\end{aligned} \tag{37a}$$

and

$$C \equiv -\frac{G}{\sqrt{2}}\frac{m_\mu^2}{k_0}\bar{u}^{(\nu)}(p^{(\nu)},s^{(\nu)})\gamma_\lambda(1+\gamma_5)u^{(\mu)}(p^{(\mu)},s^{(\mu)})\bar{u}^{(f)}(p^{(f)},s^{(f)})$$
$$\times\Big[\Big(F_M(q^2)\frac{\sigma_{\lambda\eta}k_\eta}{2m_p}+F_P(q^2)i\frac{2Mk_\lambda\gamma_5}{m_\pi^2}\Big)\Big(-\frac{\mu_i}{2M_i}i\not\epsilon^*+\frac{\mu_i}{2M_i}\frac{p^{(i)}\cdot\epsilon^*}{p^{(i)}\cdot k}i\not k+(e_i+\mu_i)\frac{\not k\not\epsilon^*}{2p^{(i)}\cdot k}\Big)$$
$$+\Big(-\frac{\mu_f}{2M_f}i\not\epsilon^*+\frac{\mu_f}{2M_f}\frac{p^{(f)}\cdot\epsilon^*}{p^{(f)}\cdot k}i\not k+(e_f+\mu_f)\frac{\not\epsilon^*\not k}{2p^{(f)}\cdot k}\Big)\Big(F_M(q^2)\frac{\sigma_{\lambda\eta}k_{\eta'}}{2m_p}+F_P(q^2)i\frac{2Mk_\lambda\gamma_5}{m_\pi^2}\Big)\Big]u^{(i)}(p^{(i)},s^{(i)}) \tag{37b}$$

and where $\lim_{k_0\to 0}\mathcal{T}'=\mathcal{T}_{\rm nonrad}$, the transition amplitude for the nonradiative muon capture $\mu^- N_i\to\nu_\mu N_f$. Equations (35)–(37b) show that $A$, $B$, and $C$ are (1) determined completely in terms of the nonradiative weak nuclear form factors $F_{V,M,A,P}$ and the electromagnetic form factors $e_i, e_f, \mu_i, \mu_f$, (2) depend on $k_0$ only through the $k_\lambda$ dependence of $q_\lambda$, and (3) become independent of $k_0$ for $k_0\ll m_\mu$; these results are consistent with Low's theorem.[13] Further, with $\mathcal{T}$ specified by Eq. (35), the photon energy spectrum

$$\frac{d\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)}{dk_0}$$

becomes, taking

$$\vec{p}^{(\mu)}=\vec{p}^{(i)}=0$$

and with

$$\hat{\nu}\equiv\vec{p}^{(\nu)}/|\vec{p}^{(\nu)}|=\vec{p}^{(\nu)}/E^{(\nu)},$$

$$\frac{d\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)}{dk_0}=(2\pi)^4\int\frac{d^3p^{(f)}}{(2\pi)^3}\int\frac{d^3p^{(\nu)}}{(2\pi)^3}\int\frac{k_0^2d\Omega^{(\gamma)}}{(2\pi)^3}\,\delta^4(p^{(f)}+p^{(\nu)}+k-p^{(i)}-p^{(\mu)})\frac{1}{4}\sum_{\rm spins}|\mathcal{T}|^2,$$

$\Big(\sum_{\rm spins}\cdots\equiv$ sum over final and initial spin orientations$\Big)$

$$=(2\pi)^{-5}\int d\Omega^{(\gamma)}\int d\Omega^{(\nu)}\int dE^{(\nu)}(E^{(\nu)})^2k_0^2\,\delta\Big(\frac{(E^{(\nu)}\hat{\nu}+k_0\hat{k})^2}{2M_f}+E^{(\nu)}+k_0-m_\mu\Big)\frac{1}{4}\sum_{\rm spins}|\mathcal{T}|^2$$
$$\cong(2\pi)^{-5}\int d\Omega^{(\gamma)}\int d\Omega^{(\nu)}(m_\mu-k_0)^2k_0^2\,\frac{1}{4}\sum_{\rm spins}\frac{e^2}{2k_0}\Big[\frac{|A|^2}{k_0^2}+\frac{2}{m_\mu k_0}{\rm Re}(A^*B)+\frac{1}{m_\mu^2}[|B|^2+2\,{\rm Re}(A^*C)]$$
$$+\frac{2k_0}{m_\mu^3}{\rm Re}(B^*C)+\frac{k_0^2}{m_\mu^4}|C|^2\Big]. \tag{38}$$

Thus, since in general

$$\lim_{k_0\to 0}|A|^2=|\mathcal{T}_{\rm nonrad}|^2\lim_{k_0\to 0}\Big[k_0\Big(\frac{p^{(\mu)}\cdot\epsilon^*}{p^{(\mu)}\cdot k}-e_i\frac{p^{(i)}\cdot\epsilon^*}{p^{(i)}\cdot k}+e_f\frac{p^{(f)}\cdot\epsilon^*}{p^{(f)}\cdot k}\Big)\Big]^2\neq 0$$

the $|A|^2$ term in Eq. (38) gives rise to an infrared divergence in the total radiative muon capture rate

$$\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)\cong\int_0^{m_\mu}\Big(\frac{d\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)}{dk_0}\Big)dk_0.$$

Such an infrared divergence is clearly another manifestation of Low's theorem[13] and can be treated quantitatively in the standard manner.[14] A general estimate of the importance of this infrared divergence is easily obtained if one remembers that $e_i=e_f+1$, $\epsilon_0^*=\vec{\epsilon}^*\cdot\hat{k}$ [Eq. (5)], and $\vec{p}^{(\mu)}=\vec{p}^{(i)}=0$. Equation (36a) then becomes

$$A=-e_f\frac{\vec{\epsilon}^*\cdot[\hat{k}\times(\hat{k}\times\hat{\nu})](E^{(\nu)}/E^{(f)})}{1-(\vec{p}^{(f)}/E^{(f)})\cdot\hat{k}}\mathcal{T}'$$
$$\cong -e_f\frac{\vec{\epsilon}^*\cdot[\hat{k}\times(\hat{k}\times\hat{\nu})][(m_\mu-k_0)/M_f]}{1+\{[(m_\mu-k_0)\hat{\nu}+k_0\hat{k}]/M_f\}\cdot\hat{k}}\mathcal{T}'. \tag{39}$$

Thus, in the case $\mu^-p\to\nu_\mu n\gamma$, $e_f=e_n=0$, so that $A=0$ and there is no infrared divergence. On the other hand, in the case $\mu^-\,{}^3{\rm He}\to\nu_\mu\,{}^3{\rm H}\gamma$, $e_f=e_{^3{\rm H}}=1$, so that $|A|\approx(m_\mu/M_f)|\mathcal{T}'|\neq 0$. Thus the term $\sim|A|^2$ will here yield an (eventually made finite) infrared-divergence contribution $\approx(m_\mu/M_f)^2|\mathcal{T}'|^2$; however, this contribution is so small that it need not be considered and, in fact, only the terms $\sim|B|^2$, $2\,{\rm Re}\,B^*C$, and $|C|^2$ need be taken into account in the various applications.

## APPLICATIONS

With the aid of the transition amplitude $\mathcal{T}=\mathcal{T}^{(l)}+\mathcal{T}^{(h)}$ obtained in the previous section [Eqs. (3)–(5), (1a)–(2), (26), (33)] we proceed to evaluate the various observable quantities associated with $\mu^-N_i\to\nu_\mu N_f\gamma$ taking into account the possible total-spin configurations, i.e., $S=1$, $S_z=1,0,-1$ and $S=0$, $S_z=0$ of the $(\mu^- N_i)$ state. Our results are obtained using the fact that the numerical values of the nonradiative form factors $|F_V|$, $|F_A|$, $|F_M(m_\mu/2m_p)|$, and $|F_P|$ are all of the same order so that the expansion of the expressions for the various observable quantities in powers of $m_\mu/M$ is rapidly convergent. Thus, using the gauge $\epsilon_0^* = \vec{\epsilon}^* \cdot \hat{k} = 0$ and remembering that $e_i = e_f + 1$ and $\vec{p}^{(\mu)} = \vec{p}^{(i)} = 0$, we can express the $\mathcal{T}^{(l)}$ and $\mathcal{T}^{(h)}$ of Eqs. (4), (1a), (1b), (5), (26), and (33) to zeroth order in $m_\mu/M$ as

$$\mathcal{T}^{(l)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z) \cong \frac{Ge}{\sqrt{2}} \frac{1}{(2k_0)^{1/2}} \frac{1}{2m_\mu} \left( v^{(f)} \frac{v^{(\nu)}}{\sqrt{2}} \right)^\dagger_{s_z^{(\nu)}, s_z^{(f)}} (1 - \vec{\sigma}^{(L)} \cdot \hat{\nu})$$
$$\times \left( F_V(q^2) - \frac{m_\mu}{2m_p} F_M(q^2) i(\vec{\sigma}^{(N)} \times \vec{\sigma}^{(L)}) \cdot \frac{\vec{p}^{(f)}}{m_\mu} - F_A(q^2) \vec{\sigma}^{(N)} \cdot \vec{\sigma}^{(L)} \right.$$
$$\left. - \frac{m_\mu^2}{m_\pi^2} F_P(q^2) \vec{\sigma}^{(N)} \cdot \frac{\vec{p}^{(f)}}{m_\mu} \vec{\sigma}^{(L)} \cdot \frac{\vec{p}^{(f)}}{m_\mu} \right) (1 - \vec{\sigma}^{(L)} \cdot \hat{k})(\vec{\sigma}^{(L)} \cdot \vec{\epsilon}^*)(v^{(i)} v^{(\mu)})_{S, S_z}, \tag{40a}$$

$$\mathcal{T}^{(h)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z) \cong \frac{Ge}{\sqrt{2}} \frac{1}{(2k_0)^{1/2}} \frac{1}{2m_\mu} \left( v^{(f)} \frac{v^{(\nu)}}{\sqrt{2}} \right)^\dagger_{s_z^{(\nu)}, s_z^{(f)}} (1 - \vec{\sigma}^{(L)} \cdot \hat{\nu})$$
$$\times \left[ \frac{m_\mu}{2m_p} F_M(q^2) 2[i(\vec{\sigma}^{(N)} \times \vec{\sigma}^{(L)}) \cdot \vec{\epsilon}^* - i(\vec{\sigma}^{(N)} \times \vec{\epsilon}^*) \cdot \hat{k}] \right.$$
$$\left. + \frac{m_\mu^2}{m_\pi^2} F_P(q^2) 2 \left( 1 - \frac{2k_0}{m_\mu} \vec{\sigma}^{(N)} \cdot \vec{\epsilon}^* (1 + \vec{\sigma}^{(L)} \cdot \hat{k}) - \vec{\sigma}^{(N)} \cdot \frac{\vec{p}^{(f)}}{m_\mu} \vec{\sigma}^{(L)} \cdot \vec{\epsilon}^* \right) \right] (v^{(i)} v^{(\mu)})_{S, S_z}, \tag{40b}$$

where $v^{(f)}, v^{(i)}$ and $v^{(\nu)}, v^{(\mu)}$ are two-component Pauli spinors for $N_f, N_i$ and $\nu_\mu$, $\mu^-$, $\vec{\sigma}^{(N)}$ and $\vec{\sigma}^{(L)}$ are two-by-two Pauli matrices to be sandwiched between $v^{(f)\dagger}, v^{(i)}$ and $v^{(\nu)\dagger}, v^{(\mu)}$, respectively, and $\vec{p}^{(f)} = -(\vec{p}^{(\nu)} + \vec{k}) \cong -[(m_\mu - k_0)\hat{\nu} + k_0 \hat{k}]$. We note that inclusion of terms $\sim F_V(m_\mu/M)$, $[F_M(m_\mu/2m_p)](m_\mu/M)$, $F_A(m_\mu/M)$, and $F_P(m_\mu/M)$ in Eqs. (40a) and (40b) affects the (eventually calculated) $\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)$ by only a few percent—a detailed (and tedious) estimate, in the case of a uniform distribution of the $(\mu^- N_i)$ over their possible total-spin configurations, yields a modification of $\lesssim 3\%$ for $\Gamma(\mu^- p \to \nu_\mu n \gamma)$ and $\lesssim 2\%$ for $\Gamma(\mu^- {}^3He \to \nu_\mu {}^3H\gamma)$.[15]

We proceed to calculate

$$\sum_{\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}} |\mathcal{T}^{(l)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z) + \mathcal{T}^{(h)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z)|^2 .$$

This quantity is of the form

$$(v^{(i)} v^{(\mu)})^\dagger_{S, S_z} (A + B \vec{\sigma}^{(L)} \cdot \vec{\sigma}^{(N)} + \vec{\sigma}^{(L)} \cdot \vec{C} \vec{\sigma}^{(N)} \cdot \vec{D} + \vec{\sigma}^{(L)} \cdot \vec{E} + \vec{\sigma}^{(N)} \cdot \vec{F})(v^{(i)} v^{(\mu)})_{S, S_z}$$
$$= A + B[2S(S+1) - 3] + \vec{C} \cdot \vec{D}[S(S+1) - S_z^2 - 1] + \vec{C} \cdot \hat{z} \vec{D} \cdot \hat{z}[-S(S+1) + 3S_z^2] + (\vec{E} \cdot \hat{z} + \vec{F} \cdot \hat{z}) S_z , \tag{41}$$

where $A$, $B$, $\vec{C}$, $\vec{D}$, $\vec{E}$, and $\vec{F}$ depend on the $F_{V,M,A,P}(q^2)$ and on $k_0$, $\hat{\nu}$, and $\hat{k}$. We thus have

$$\sum_{\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}} |\mathcal{T}^{(l)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z) + \mathcal{T}^{(h)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z)|^2$$
$$= \left( \frac{Ge}{\sqrt{2}} \frac{1}{(2k_0)^{1/2}} \frac{1}{m_\mu} \right)^2 \{ W_1 + W_2[2S(S+1) - 3] + W_3[S(S+1) - S_z^2 - 1]$$
$$+ [W_4(\hat{\nu} \cdot \hat{z})^2 + W_5(\hat{\nu} \cdot \hat{z})(\hat{k} \cdot \hat{z}) + W_6(\hat{k} \cdot \hat{z})^2][-S(S+1) + 3S_z^2] + [W_7 \hat{\nu} \cdot \hat{z} + W_8 \hat{k} \cdot \hat{z}] S_z \} , \tag{42a}$$

$$W_i = W_i^{(0)} + W_i^{(1)}(\hat{\nu} \cdot \hat{k}) + W_i^{(2)}(\hat{\nu} \cdot \hat{k})^2, \quad i = 1, 2, 3, \ldots, 8, \tag{42b}$$

where the $W_i^{(n)}$ are structure functions which are homogeneous and quadratic in the $F_{V,M,A,P}(q^2)$ and which also depend on $k_0$. All the $W_i^{(n)}$ can be decomposed according to

$$W_i^{(n)} = W_i^{(n);(ll)} + W_i^{(n);(lh)} + W_i^{(n);(hh)} \tag{42c}$$

with the $W_i^{(n);(ll)}$, $W_i^{(n);(lh)}$, and $W_i^{(n);(hh)}$ originating, respectively, from

$$\sum_{\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}} |\mathcal{T}^{(l)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z)|^2,$$

$$\sum_{\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}} 2\,\mathrm{Re}\,\mathcal{T}^{(l)*}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z)\mathcal{T}^{(h)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z),$$

and

$$\sum_{\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}} |\mathcal{T}^{(h)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z)|^2.$$

For purposes of illustration, we append here certain of our results regarding the $W_i^{(n);(ll)}$, $W_i^{(n);(lh)}$, and $W_i^{(n);(hh)}$, viz.,

$$W_1^{(0);(ll)} = [F_V(q^2)]^2 + 3[F_A(q^2)]^2 + 2F_A(q^2)F_P(q^2)\frac{m_\mu^2}{m_\pi^2} + \left(F_P(q^2)\frac{m_\mu^2}{m_\pi^2}\right)^2\left[1 - 2\frac{k_0}{m_\mu} + 2\left(\frac{k_0}{m_\mu}\right)^2\right]$$
$$+ 4F_A(q^2)F_M(q^2)\frac{m_\mu}{2m_p}\left(1 - 2\frac{k_0}{m_\mu}\right) + 2\left[F_M(q^2)\frac{m_\mu}{2m_p}\right]^2\left[1 - 3\frac{k_0}{m_\mu} + 3\left(\frac{k_0}{m_\mu}\right)^2\right], \quad (42d)$$

$$W_3^{(0);(\xi)} = W_6^{(0);(\xi)}, \quad W_3^{(1);(\xi)} = W_5^{(0);(\xi)} + W_6^{(1);(\xi)}, \quad W_3^{(2);(\xi)} = W_5^{(1);(\xi)},$$
$$W_4^{(0);(\xi)} = W_4^{(1);(\xi)} = W_4^{(2);(\xi)} = 0,$$
$$W_5^{(2);(\xi)} = W_6^{(2);(\xi)} = W_7^{(2);(\xi)} = 0: \quad \xi = ll, lh, hh. \quad (42e)$$

and refer the reader to Eqs. (B1)–(B58) in Appendix B where all of the $W_i^{(n);(ll)}$, $W_i^{(n);(lh)}$, and $W_i^{(n);(hh)}$ are given explicitly. We also evaluate the average of

$$\sum_{\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}} |\mathcal{T}^{(l)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z) + \mathcal{T}^{(h)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z)|^2$$

with respect to the distribution of the $(\mu^- N_i)$ over their possible total-spin configurations, i.e., evaluate the quantity

$$\langle|\mathcal{T}|^2\rangle = \sum_{S, S_z}\left(\sum_{\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}} |\mathcal{T}^{(l)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z) + \mathcal{T}^{(h)}(\vec{\epsilon}^*, s_z^{(\nu)}, s_z^{(f)}; S, S_z)|^2\right)P(S, S_z), \quad (42f)$$

where $P(S,S_z)$ is the probability of finding $(\mu^- N_i)$ at the instant of muon capture in the total-spin configuration specified by $S,S_z$, i.e.,

$$P(1, \pm1) = \tfrac{1}{4}(1 \pm P_\mu \pm P_{N_i} + P_\mu P_{N_i}),$$
$$P(1,0) = P(0,0) = \tfrac{1}{4}(1 - P_\mu P_{N_i}): \text{ no } S=1 \leftrightarrow S=0 \text{ conversion in a time} \approx \tau(\mu^- - \text{decay}), \quad (42g)$$
$$P(1,\pm1) = P(1,0) = 0, P(0,0) = 1: \text{ complete } S=1 \to S=0 \text{ conversion in a time} \approx \tau(\mu^- - \text{decay})$$

$P_\mu \hat{z}$ and $P_{N_i}\hat{z}$ being the $\mu^-$ and $N_i$ polarizations at the instant of arrival of the $\mu^-$ in the lowest Bohr orbit around the $N_i$. Substitution of Eqs. (42a)–(42c) into Eq. (42f) yields

$$\langle|\mathcal{T}|^2\rangle = \left(\frac{Ge}{\sqrt{2}}\frac{1}{(2k_0)^{1/2}}\frac{1}{m_\mu}\right)^2\{W_1 + W_2[P(1,1) + P(1,0) + P(1,-1) - 3P(0,0)] + W_3[P(1,0) - P(0,0)]$$
$$+ [W_4(\hat{\nu}\cdot\hat{z})^2 + W_5(\hat{\nu}\cdot\hat{z})(\hat{k}\cdot\hat{z}) + W_6(\hat{k}\cdot\hat{z})^2][P(1,1) + P(1,-1) - 2P(1,0)]$$
$$+ [W_7\hat{\nu}\cdot\hat{z} + W_8\hat{k}\cdot\hat{z}][P(1,1) - P(1,-1)]\},$$

$$W_i = \sum_{n=0}^{2} W_i^{(n)}(\hat{\nu}\cdot\hat{k})^n = \sum_{n=0}^{2}(W_i^{(n);(ll)} + W_i^{(n);(lh)} + W_i^{(n);(hh)})(\hat{\nu}\cdot\hat{k})^n. \quad (42h)$$

Equation (42h), together with Eqs. (B1)–(B58) for the $W_i^{(n);(ll)}$, $W_i^{(n);(lh)}$, $W_i^{(n);(hh)}$ and the specification of appropriate $P(S,S_z)$ [Eq. (42g)], constitutes the basis of our further calculations.

To perform the calculations we first note that the $W_i^{(n)}(q^2, k_0/m_\mu)$, which depend on $q^2$ through the $F_{V,M,A,P}(q^2)$, themselves have a dependence on $\hat{\nu}\cdot\hat{k}$ since

$$q^2 = (p^{(f)} - p^{(i)})^2 = (p^{(\mu)} - p^{(\nu)} - k)^2 = -m_\mu^2 + 2(E^{(\nu)} + k_0)m_\mu - 2E^{(\nu)}k_0(1 - \hat{\nu}\cdot\hat{k})$$
$$\cong m_\mu^2 - 2(m_\mu - k_0)k_0(1 - \hat{\nu}\cdot\hat{k}) \quad (43a)$$

so that

$$F_{V,M,A,P}(q^2) \cong F_{V,M,A,P}(m_\mu^2 - 2(m_\mu - k_0)k_0) + [2(m_\mu - k_0)k_0(\hat{\nu}\cdot\hat{k})]\left(\frac{\partial F_{V,M,A,P}(q^2)}{\partial q^2}\right)_{q^2=m_\mu^2-2(m_\mu-k_0)k_0} \tag{43b}$$

In view of the relatively slow variation of $F_{V,M,A}(q^2)$ with $q^2$ (in both the $p \leftrightarrow n$ case and the $^3He \leftrightarrow {}^3H$ case) the ratio of the second to the first term on the right side of Eq. (43b) is quite small so that the second term can be safely neglected. As regards $F_P(q^2)$, one has from Eq. (8f) (which holds in both the $p \leftrightarrow n$ case and the $^3He \leftrightarrow {}^3H$ case)

$$\left|[2(m_\mu - k_0)k_0(\hat{\nu}\cdot\hat{k})]\left(\frac{1}{F_P(q^2)}\frac{\partial F_P(q^2)}{\partial q^2}\right)_{q^2=m_\mu^2-2(m_\mu-k_0)k_0}\right| \cong \frac{(m_\mu^2/m_\pi^2)2(1-k_0/m_\mu)(k_0/m_\mu)|(\hat{\nu}\cdot\hat{k})|}{1+(m_\mu^2/m_\pi^2)(1-2(1-k_0/m_\mu)k_0/m_\mu)} \tag{43c}$$

with the expression on the right side of Eq. (43c) varying between 0 and 0.2 $|\hat{\nu}\cdot\hat{k}| < 0.2$ as $k_0/m_\mu$ varies between 0 and $\frac{1}{2}$ or between 1 and $\frac{1}{2}$ [in fact, suitable averages of $(\hat{\nu}\cdot\hat{k})$ are numerically $\ll 1$ (see below) so that 0.2 $|\hat{\nu}\cdot\hat{k}|$ is effectively $\ll 0.2$]. Such a variation of $F_P(q^2)$ does not affect the values of the $W_i^{(n)}(q^2, k_0/m_\mu)$ sufficiently to modify the $\langle|\mathcal{T}|^2\rangle$ of Eq. (42h) in a significant way so that the second term on the right side of Eq. (43b) can be neglected even in the case of $F_P(q^2)$. Further, a similar replacement of $F_{V,M,A,P}[m_\mu^2 - 2(m_\mu - k_0)k_0]$ by $F_{V,M,A,P}(\frac{3}{5}m_\mu^2)$, where $\frac{3}{5}m_\mu^2$ is the average of $m_\mu^2 - 2(m_\mu - k_0)k_0$ over the "phase-space" photon energy spectrum: $12(k_0/m_\mu)(1 - k_0/m_\mu)^2$ [see Eq. (46a) below], can again be justified on the basis of the relatively slow variation of $F_{V,M,A}(q^2)$ with $q^2$ and of the expression for

$$\left|(\tfrac{2}{5}m_\mu^2 - 2(m_\mu - k_0)k_0)\left(\frac{1}{F_P(q^2)}\frac{\partial F_P(q^2)}{\partial q^2}\right)_{q^2=(3/5)m_\mu^2}\right| \cong \left|\frac{(m_\mu^2/m_\pi^2)\{\frac{2}{5} - 2[(1-k_0/m_\mu)k_0/m_\mu]\}}{1+\frac{3}{5}(m_\mu^2/m_\pi^2)}\right| \tag{43d}$$

which varies between 0.17 and 0.04 as $k_0/m_\mu$ varies between 0 and $\frac{1}{2}$ or between 1 and $\frac{1}{2}$. Thus the $W_i^{(n)}(q^2, k_0/m_\mu)$ in Eq. (42h) for $\langle|\mathcal{T}|^2\rangle$ can be replaced by $W_i^{(n)}(\frac{3}{5}m_\mu^2, k_0/m_\mu)$, and become, as can be seen from Eq. (B1)–(B58), simple quadratic functions of $k_0/m_\mu \equiv x$ with $0 \le x \lesssim 1$.

With these $W_i^{(n)}(\frac{3}{5}m_\mu^2, x) \equiv W_i^{(n)}(x)$ we can immediately proceed to calculate the various observable quantities associated with $\mu^- N_i \to \nu_\mu N_f \gamma$. First of all, the neutrino-photon angular correlation

$$\mathcal{C}(\hat{\nu}\cdot\hat{k}) = \frac{\langle|\mathcal{T}|^2\rangle}{\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle} \tag{44a}$$

is given by Eqs. (42h) and (42e) and corresponds to the forward-backward asymmetry

$$\mathcal{A}_{\hat{\nu},\hat{k}} \equiv \frac{\mathcal{C}(\hat{\nu}\cdot\hat{k}=1) - \mathcal{C}(\hat{\nu}\cdot\hat{k}=-1)}{\mathcal{C}(\hat{\nu}\cdot\hat{k}=1) + \mathcal{C}(\hat{\nu}\cdot\hat{k}=-1)} = \frac{N_{\hat{\nu},\hat{k}}}{D_{\hat{\nu},\hat{k}}}, \tag{44b}$$

with

$$\begin{aligned} N_{\hat{\nu},\hat{k}} = {} & W_1^{(1)}(x) + W_2^{(1)}(x)[P(1,1) + P(1,0) + P(1,-1) - 3P(0,0)] \\ & + W_3^{(1)}(x)\{[P(1,0) - P(0,0)] + (\hat{k}\cdot\hat{z})^2[P(1,1) + P1,-1) - 2P(1,0)]\} \\ & + [W_7^{(0)}(x) + W_8^{(1)}(x)]\hat{k}\cdot\hat{z}[P(1,1) - P(1,-1)], \\ D_{\hat{\nu},\hat{k}} = {} & [W_1^{(0)}(x) + W_1^{(2)}(x)] + [W_2^{(0)}(x) + W_2^{(2)}(x)][P(1,1) + P(1,0) + P(1,-1) - 3P(0,0)] \\ & + [W_3^{(0)}(x) + W_3^{(2)}(x)]\{(P(1,0) - P(0,0) + (\hat{k}\circ\hat{z})^2[P(1,1) + P(1,-1) - 2P(1,0)]\} \\ & + [W_7^{(1)}(x) + W_8^{(0)}(x) + W_8^{(2)}(x)]\hat{k}\cdot\hat{z}[P(1,1) - P(1,-1)], \end{aligned}$$

where $\hat{\nu}\cdot\hat{k} = \pm 1$ corresponds to $\hat{p}^{(f)}\cdot\hat{k} = -1$, $(m_\mu - 2k_0)/|m_\mu - 2k_0|$. Further, the angular correlation between the photon momentum and the polarization of the $\mu^-$ or/and the $N_i$

$$\mathcal{C}(\hat{k}\cdot\hat{z}) = \frac{\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle}{\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle} \tag{44c}$$

corresponds to the forward-backward asymmetry

$$\alpha_{\hat{k},\hat{z}} \equiv \frac{\mathcal{C}(\hat{k}\cdot\hat{z}=1)-\mathcal{C}(\hat{k}\cdot\hat{z}=-1)}{\mathcal{C}(\hat{k}\cdot\hat{z}=1)+\mathcal{C}(\hat{k}\cdot\hat{z}=-1)}=\frac{N_{\hat{k},\hat{z}}}{D_{\hat{k},\hat{z}}}, \tag{44d}$$

with

$$N_{\hat{\nu},\hat{k}}=(\tfrac{1}{3}W_7^{(1)}(x)+W_8^{(0)}(x)+\tfrac{1}{3}W_8^{(2)}(x))[P(1,1)-P(1,-1)],$$
$$D_{\hat{\nu},\hat{k}}=[W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)]+[W_2^{(0)}(x)+\tfrac{1}{3}W_2^{(0)}(x)][P(1,1)+P(1,0)+P(1,-1)-3P(0,0)],$$
$$+[W_3^{(0)}(x)+\tfrac{1}{3}W_3^{(2)}(x)][P(1,1)+P(1,-1)-P(1,0)-P(0,0)],$$

where we have used

$$\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle=\left(\frac{Ge}{\sqrt{2}}\frac{1}{(2k_0)^{1/2}}\frac{1}{m_\mu}\right)^2([W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)]+[W_2^{(0)}(x)+\tfrac{1}{3}W_2^{(2)}(x)][P(1,1)+P(1,0)+P(1,-1)-3P(0,0)]$$
$$+[W_3^{(0)}(x)+\tfrac{1}{3}W_3^{(2)}(x)]\{[P(1,0)-P(0,0)]+(\hat{k}\cdot\hat{z})^2[P(1,1)+P(1,-1)-2P(1,0)]\}$$
$$+[\tfrac{1}{3}W_7^{(1)}(x)+W_8^{(0)}(x)+\tfrac{1}{3}W_8^{(2)}(x)]\hat{k}\cdot\hat{z}[P(1,1)-P(1,-1)]) \tag{44e}$$

as obtained from Eqs. (42h) and (42e). Finally, we have from Eqs. (44e)

$$\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle=\left(\frac{Ge}{\sqrt{2}}\frac{1}{(2k_0)^{1/2}}\frac{1}{m_\mu}\right)^2([W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)]+\{[W_2^{(0)}(x)+\tfrac{1}{3}W_2^{(2)}(x)]+\tfrac{1}{3}[W_3^{(0)}(x)+\tfrac{1}{3}W_3^{(2)}(x)]\}$$
$$\times[P(1,1)+P(1,0)+P(1,-1)-3P(0,0)]) \tag{44f}$$

a quantity which determines the photon energy spectrum (see below). As examples of the relations in Eqs. (44a), (44d), and (44f), we consider the values of $\alpha_{\hat{\nu},\hat{k}}$, $\alpha_{\hat{k},\hat{z}}$, and

$$\frac{\displaystyle\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle}{\left[\dfrac{Ge}{\sqrt{2}}\dfrac{1}{(2k_0)^{1/2}}\dfrac{1}{m_\mu}\right]^2}$$

for three set of values of the $P(S,S_z)$ which occur in practice:

Case (a): No $S=1\to S=0$ conversion: $P(S,S_z)$ as in Eq. (42g)

$$\alpha_{\hat{\nu},\hat{k}}=\frac{N_{\hat{\nu},\hat{k}}}{D_{\hat{\nu},\hat{k}}}, \tag{45a}$$

with

$$N_{\hat{\nu},\hat{k}}=W_1^{(1)}(x)+[W_2^{(1)}(x)+W_3^{(1)}(x)(\hat{k}\cdot\hat{z})^2]P_\mu P_{N_i}+[W_7^{(0)}(x)+W_8^{(1)}(x)]\hat{k}\cdot\hat{z}\tfrac{1}{2}(P_\mu+P_{N_i})$$
$$D_{\hat{\nu},\hat{k}}=[W_1^{(0)}(x)+W_1^{(2)}(x)]+\{[W_2^{(0)}(x)+W_2^{(2)}(x)]+[W_3^{(0)}(x)+W_3^{(2)}(x)](\hat{k}\cdot\hat{z})^2\}P_\mu P_{N_i}$$
$$+[W_7^{(1)}(x)+W_8^{(0)}(x)+W_8^{(2)}(x)]\hat{k}\cdot\hat{z}\tfrac{1}{2}(P_\mu+P_{N_i})$$

$$\alpha_{\hat{k},\hat{z}}=\frac{[\tfrac{1}{3}W_7^{(1)}(x)+W_8^{(0)}(x)+\tfrac{1}{3}W_8^{(2)}(x)]\tfrac{1}{2}(P_\mu+P_{N_i})}{[W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)]+\{[W_2^{(0)}(x)+\tfrac{1}{3}W_2^{(2)}(x)]+[W_3^{(0)}(x)+\tfrac{1}{3}W_3^{(2)}(x)]\}P_\mu P_{N_i}}, \tag{45b}$$

$$\frac{\displaystyle\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle}{\left(\dfrac{Ge}{\sqrt{2}}\dfrac{1}{(2k_0)^{1/2}}\dfrac{1}{m_\mu}\right)^2}=[W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)]+\{[W_2^{(0)}(x)+\tfrac{1}{3}W_2^{(2)}(x)]+\tfrac{1}{3}[W_3^{(0)}(x)+\tfrac{1}{3}W_3^{(2)}(x)]\}P_\mu P_{N_i} \tag{45c}$$

Case (b): Complete $S=1\to S=0$ conversion: $P(S,S_z)$ as in Eq. (42g)

$$\alpha_{\hat{\nu},\hat{k}}=\frac{W_1^{(1)}(x)-3W_2^{(1)}(x)-W_3^{(1)}(x)}{[W_1^{(0)}(x)+W_1^{(2)}(x)]-3[W_2^{(0)}(x)+W_2^{(2)}(x)]-[W_3^{(0)}(x)+W_3^{(2)}(x)]}, \tag{45d}$$

$$\alpha_{\hat{k}\cdot\hat{z}}=0, \tag{45e}$$

$$\frac{\displaystyle\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle}{\left(\dfrac{Ge}{\sqrt{2}}\dfrac{1}{(2k_0)^{1/2}}\dfrac{1}{m_\mu}\right)^2}=[W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)]+\{[W_2^{(0)}(x)+\tfrac{1}{3}W_2^{(2)}(x)]+\tfrac{1}{3}[W_3^{(0)}(x)+\tfrac{1}{3}W_3^{(2)}(x)]\}\cdot(-3). \tag{45f}$$

Case (c): $\mu^- p \to \nu_\mu n \gamma$ from the total-spin $\frac{1}{2}$ ortho-$(p\mu^- p)$ molecule: $P(1,\pm 1) = P(1,0) = \frac{1}{12}$, $P(0,0) = \frac{3}{4}$

$$\mathcal{A}_{\hat{\nu},\hat{k}} = \frac{W_1^{(1)}(x) - 2W_2^{(1)}(x) - \frac{2}{3}W_3^{(1)}(x)}{[W_1^{(0)}(x) + W_1^{(2)}(x)] - 2[W_2^{(0)}(x) + W_2^{(2)}(x)] - \frac{2}{3}[W_3^{(0)}(x) + W_3^{(2)}(x)]}, \tag{45g}$$

$$\mathcal{A}_{\hat{k},\hat{z}} = 0, \tag{45h}$$

$$\frac{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} \langle |\mathcal{T}|^2 \rangle}{\left(\frac{Ge}{\sqrt{2}} \frac{1}{(2k_0)^{1/2}} \frac{1}{m_\mu}\right)^2} = [W_1^{(0)}(x) + \tfrac{1}{3}W_1^{(2)}(x)] + \{[W_2^{(0)}(x) + \tfrac{1}{3}W_2^{(2)}(x)] + \tfrac{1}{3}[W_3^{(0)}(x) + W_3^{(2)}(x)]\} \cdot (-2). \tag{45i}$$

We are now ready to specify $d\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)/dk_0$, the photon energy spectrum, and $\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)$, the radiative muon capture rate. We have, using Eq. (44f),

$$\frac{d\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)}{dx} \cong \frac{\alpha}{\pi} \Gamma_0 \left(\frac{k_0}{m_\mu}\right)^2 \left(1 - \frac{k_0}{m_\mu}\right)^2 \frac{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} \langle |\mathcal{T}|^2 \rangle}{\left(\frac{Ge}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{m_\mu^{3/2}}\right)^2}$$

$$= \frac{\alpha}{12\pi} \Gamma_0 (12x(1-x)^2) \{ (W_1^{(0)}(x) + \tfrac{1}{3}W_1^{(2)}(x)) + \{[W_2^{(0)}(x) + \tfrac{1}{3}W_2^{(2)}(x)] + \tfrac{1}{3}[W_3^{(0)}(x) + \tfrac{1}{3}W_3^{(2)}(x)]\}$$
$$\times [P(1,1) + P(1,0) + P(1,-1) - 3P(0,0)]\},$$

$$\Gamma_0 \equiv \frac{G^2 m_\mu^5}{2\pi^2} \left(1 - 2\frac{m_\mu}{m_\mu + M_f}\right) C_i \left(e_i \alpha \frac{M_i}{m_\mu + M_i}\right)^3,$$

$C_i = 1.00 : N_i = p$

$= 0.96 : N_i = {}^3\text{He}$ (see Ref. 17) (46a)

so that

$$\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma) \cong \int_0^1 \frac{d\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)}{dx} dx$$

$$= \frac{\alpha}{12\pi} \Gamma_0 \int_0^1 12x(1-x)^2 ([W_1^{(0)}(x) + \tfrac{1}{3}W_1^{(2)}(x)] + \{[W_2^{(0)}(x) + \tfrac{1}{3}W_2^{(2)}(x)] + \tfrac{1}{3}[W_3^{(0)}(x) + \tfrac{1}{3}W_3^{(2)}(x)]\}$$
$$\times [P(1,1) + P(1,0) + P(1,-1) - 3P(0,0)]) dx. \tag{46b}$$

It is also convenient to introduce the branching ratio for radiative muon capture

$$R \equiv \frac{\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)}{\Gamma(\mu^- N_i \to \nu_\mu N_f)}, \tag{46c}$$

where the nonradiative muon capture rate is given by[16,17]

$$\Gamma(\mu^- N_i \to \nu_\mu N_f)$$
$$= \Gamma_0 \{ (G_V^2 + 3G_A^2 - 2G_A G_P + G_P^2)$$
$$+ (-2G_A^2 + 2G_V G_A - \tfrac{2}{3} G_V G_P + \tfrac{4}{3} G_A G_P)$$
$$\times [P(1,1) + P(1,0) + P(1,-1) - 3P(0,0)]\} \tag{46d}$$

with

$$G_V \equiv F_V(q_{nr}^2)\left(1 + \frac{m_\mu}{2M_i}\right) - \left(F_M(q_{nr}^2)\frac{m_\mu}{2m_p}\right)\frac{m_\mu}{2M_i},$$

$$G_A \equiv -F_A(q_{nr}^2) - F_V(q_{nr}^2)\frac{m_\mu}{2M_i} - F_M(q_{nr}^2)\frac{m_\mu}{2m_p},$$

$$G_P \equiv F_P(q_{nr}^2)\frac{m_\mu^2}{m_\pi^2} + [F_A(q_{nr}^2) - F_V(q_{nr}^2)]\frac{m_\mu}{2M_i} - F_M(q_{nr}^2)\frac{m_\mu}{2m_p},$$

$$q_{nr}^2 \equiv (p^{(f)} - p^{(i)})^2_{\substack{\text{nonrad.}\\ \text{muon capture}}} = (p^{(\mu)} - p^{(\nu)})^2_{\substack{\text{nonrad.}\\ \text{muon capture}}}$$
$$= (-m_\mu^2 + 2E^{(\nu)} m_\mu)_{\substack{\text{nonrad.}\\ \text{muon capture}}}$$
$$\cong m_\mu^2 - 2m_\mu \frac{m_\mu^2}{2M_f} = 0.88 m_\mu^2 : \mu^- p \to \nu_\mu n$$
$$= 0.96 m_\mu^2 : \mu^- {}^3\text{He} \to \nu_\mu {}^3\text{H}. \tag{46e}$$

Equations (45a)–(46e) demonstrate explicitly the "hyperfine effect" in the observable quantities

$$\mathcal{A}_{\hat{\nu},\hat{k}},\ \mathcal{A}_{\hat{k},\hat{z}},$$

$$\frac{d\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)}{dx},$$

$$\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma),$$

and

$$\Gamma(\mu^- N_i \to \nu_\mu N_f)$$

arising from the dependence on the $P(S,S_z)$ of the $\langle|\mathcal{T}|^2\rangle$ [Eq. (42h)] and of the corresponding $\langle|\mathcal{T}_{\text{nonrad.}}|^2\rangle$ (Ref. 16).

We proceed to give numerical values for

$$\mathcal{A}_{\hat{\nu},\hat{k}},$$

$$\mathcal{A}_{\hat{k},\hat{z}},$$

$$\frac{d\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma)}{dx},$$

$$\Gamma(\mu^- N_i \to \nu_\mu N_f \gamma),$$

$$\Gamma(\mu^- N_i \to \nu_\mu N_f),$$

and $R$ for the various cases of experimental interest.

*(I)*: $\mu^- p \to \nu_\mu n \gamma$

Here, using Ref. 16, we have

$$F_V(0.88m_\mu^2) = F_V(0) \times 0.977 = 0.977,$$

$$F_M(0.88m_\mu^2) = F_M(0) \times 0.971 = 3.598,$$

$$F_A(0.88m_\mu^2) = F_A(0) \times 0.978 = 1.213,$$

$$F_P(0.88m_\mu^2) \cong -\frac{F_A(0.88m_\mu^2)}{1+0.88m_\mu^2/m_\pi^2}(1+0.02) = -0.822$$

[see also Eq. (8f)]. From these values $F_{V,M,A}(\frac{3}{5}m_\mu^2)$ can be obtained by linear interpolation and

$$F_P(\tfrac{3}{5}m_\mu^2) \cong -\frac{F_A(\frac{3}{5}m_\mu^2)}{1+\frac{3}{5}m_\mu^2/m_\pi^2}(1+0.02) = -0.927$$

so that the $F_{V,M,A,P}(\frac{3}{5}m_\mu^2)$ entering into the $W_n^{(i)}(\frac{3}{5}m_\mu^2, x) \equiv W_n^{(i)}(x)$ are all numerically determined and the $W_n^{(i)}(x)$ are immediately calculable from Eqs. (B1)–(B58). In this way we obtain

$$W_1^{(0)}(x) = 8.46 - 10.55x + 6.94x^2, \quad (47a)$$

$$W_1^{(1)}(x) = -5.49 + 7.70x + 1.44x^2, \quad (47b)$$

$$W_1^{(2)}(x) = 0.78 - 1.90x + 1.13x^2, \quad (47c)$$

$$W_2^{(0)}(x) = 0.38 - 1.08x - 1.19x^2, \quad (47d)$$

$$W_2^{(1)}(x) = -0.31 - 0.89x + 2.38x^2, \quad (47e)$$

$$W_2^{(2)}(x) = -0.78 + 1.97x - 1.19x^2, \quad (47f)$$

$$W_3^{(0)}(x) = 4.83 - 3.48x - 0.57x^2, \quad (47g)$$

$$W_3^{(1)}(x) = -6.26 + 12.09x - 4.50x^2, \quad (47h)$$

$$W_3^{(2)}(x) = 3.54 - 8.61x + 5.07x^2, \quad (47i)$$

$$W_7^{(0)}(x) = -6.49 + 9.97x - 3.14x^2, \quad (47j)$$

$$W_7^{(1)}(x) = 2.71 - 5.86x + 3.14x^2, \quad (47k)$$

$$W_8^{(0)}(x) = 13.59 - 15.55x + 3.99x^2, \quad (47l)$$

$$W_8^{(1)}(x) = -5.64 + 10.93x - 3.76x^2, \quad (47m)$$

$$W_8^{(2)}(x) = 0 - 0.68x + 0.68x^2 \quad (47n)$$

and treat the individual cases of particular sets of values for the $P(S,S_z)$ separately.

*Case I(a): No $S=1 \to S=0$ conversion: $P(S,S_z)$ as in Eq. (42g).* Using Eqs. (45a)–(45c), (46a)–(46e), and Eqs. (47a)–(47n), we have

$$\mathcal{A}_{\hat{\nu},\hat{k}} = \frac{N_{\hat{\nu},\hat{k}}}{D_{\hat{\nu},\hat{k}}}, \quad (48a)$$

with

$$N_{\hat{\nu},\hat{k}} = (-5.49 + 7.70x + 1.44x^2) + [(-0.31 - 0.89x + 2.38x^2) + (-6.26 + 12.09x - 4.50x^2)(\hat{k}\cdot\hat{z})^2]P_\mu P_p, + (-12.13 + 20.90x - 6.90x^2)\hat{k}\cdot\hat{z}\,\tfrac{1}{2}(P_\mu + P_p),$$

$$D_{\hat{\nu},\hat{k}} = (9.24 - 12.45x + 8.06x^2) + [(-0.40 + 0.89x - 2.38x^2) + (8.37 - 12.09x + 4.50x^2)(\hat{k}\cdot\hat{z})^2]P_\mu P_p + (16.30 - 22.09x + 7.81x^2)\hat{k}\cdot\hat{z}\,\tfrac{1}{2}(P_\mu + P_p),$$

$$\mathcal{A}_{\hat{k},\hat{z}} = \frac{N_{\hat{k},\hat{z}}}{D_{\hat{k},\hat{z}}}, \quad (48b)$$

with

$$N_{\hat{\nu},\hat{k}} = (14.49 - 17.73x + 5.26x^2)\,\tfrac{1}{2}(P_\mu + P_p),$$

$$D_{\hat{\nu},\hat{k}} = (8.72 - 11.18x + 7.31x^2) + (6.13 - 6.77x + 0.47x^2)P_\mu P_p,$$

$$\frac{\dfrac{d\Gamma(\mu^- p \to \nu_\mu n\gamma)}{dx}}{\dfrac{\alpha}{12\pi}\Gamma_0(12x(1-x)^2)} = (8.72 - 11.18x + 7.31x^2) + (2.12 - 2.54x - 1.21x^2)P_\mu P_p, \quad (48c)$$

$$\frac{\Gamma(\mu^- p\to\nu_\mu n\gamma)}{(\alpha/12\pi)\Gamma_0}=5.71+0.86P_\mu P_P\,, \tag{48d}$$

$$\frac{\Gamma(\mu^- p\to\nu_\mu n)}{\Gamma_0}=5.99-5.58P_\mu P_P\,, \tag{48e}$$

$$R=\frac{\alpha}{12\pi}\,\frac{5.71+0.86P_\mu P_P}{5.99-5.58P_\mu P_P}=(1.85\times10^{-4})\,\frac{1+0.15P_\mu P_P}{1-0.93P_\mu P_P}. \tag{48f}$$

Case I(a) occurs when the $\mu^-$ beam stops in low density gaseous hydrogen.[16] We note that the values of $\Gamma(\mu^- p\to\nu_\mu n)$ and $R$ are very sensitive to the value of $P_\mu P_P$, the value $P_\mu P_P=1$ corresponding to $P(0,0)=0$ [and $P(1,1)=1, P(1,0)=P(1,-1)=0$] and so to muon capture from the triplet ($S=1$, $S_z=1$) total-spin configuration of ($\mu^- p$). Unfortunately, however, attainment in practice of anything but very small values of $P_\mu P_P$ appears to be extremely difficult.

*Case I(b): Complete $S=1\to S=0$ conversion: $P(S,S_z)$ as in Eq. (42g).* Using Eqs. (45d)–(45f), (46a)–(46e) and Eqs. (47a)–(47n), we have

$$\mathcal{A}_{\hat\nu,\hat k}=\frac{1.70-1.72x-1.20x^2}{2.07-3.07x+10.70x^2}\,, \tag{48g}$$

$$\mathcal{A}_{\hat k,\hat z}=0, \tag{48h}$$

$$\frac{\dfrac{d\Gamma(\mu^- p\to\nu_\mu n\gamma)}{dx}}{\dfrac{\alpha}{12\pi}\Gamma_0(12x(1-x)^2)}=(8.72-11.18x+7.31x^2)+(2.12-2.54x-1.21x^2)\cdot(-3)=2.36-3.56x+10.94x^2, \tag{48i}$$

$$\frac{\Gamma(\mu^- p\to\nu_\mu n\gamma)}{(\alpha/12\pi)\Gamma_0}=5.71+0.86\cdot(-3)=3.13\,, \tag{48j}$$

$$\frac{\Gamma(\mu^- p\to\nu_\mu n)}{\Gamma_0}=5.99-5.58\cdot(-3)=22.73\,, \tag{48k}$$

$$R=\frac{\alpha}{12\pi}\,\frac{3.13}{22.73}=2.67\times10^{-5}. \tag{48l}$$

Case I(b) occurs when the $\mu^-$ beam stops in medium density gaseous hydrogen.[16]

*Case I(c): $\mu^- p\to\nu_\mu n\gamma$ from the total-spin $\frac12$ ortho-($p\mu^- p$) molecule: $P(1,\pm1)=P(1,0)=\frac{1}{12}$, $P(0,0)=\frac34$.* Using Eqs. (45g)–(45i), (46a)–(46e), and Eqs. (47a)–(47n), we have

$$\mathcal{A}_{\hat\nu,\hat k}=\frac{-0.70+1.42x-0.32x^2}{4.46-6.17x+9.82x^2}\,, \tag{48m}$$

$$\mathcal{A}_{\hat k,\hat z}=0, \tag{48n}$$

$$\frac{\dfrac{d\Gamma(\mu^- p\to\nu_\mu n\gamma)}{dx}}{\dfrac{\alpha}{12\pi}\Gamma_0(12x(1-x)^2)}=(8.72-11.18x+7.31x^2)+(2.12-2.54x-1.21x^2)\cdot(-2)=4.48-6.10x+9.73x^2, \tag{48o}$$

$$\frac{\Gamma(\mu^- p\to\nu_\mu n\gamma)}{(\alpha/12\pi)\Gamma_0}=5.71+0.86\cdot(-2)=3.99\,, \tag{48p}$$

$$\frac{\Gamma(\mu^- p\to\nu_\mu n)}{\Gamma_0}=5.99-5.58\cdot(-2)=17.15\,, \tag{48q}$$

$$R=\frac{\alpha}{12\pi}\,\frac{3.99}{17.15}=4.50\times10^{-5}. \tag{48r}$$

Case I(c) occurs when the $\mu^-$ beam stops in high-density gaseous hydrogen or in liquid hydrogen.[16]

*(II):* $\mu^-\,{}^3\mathrm{He}\to\nu_\mu\,{}^3\mathrm{H}\gamma$

Here, using Ref. 17, we have

$$F_V(0.96\,m_\mu^2)=F_V(0)\times0.82=0.82\,,$$
$$F_M(0.96\,m_\mu^2)=F_M(0)\times0.87=-4.73,$$
$$F_A(0.96\,m_\mu^2)=F_A(0)\times0.87=-1.06\,,$$
$$F_P(0.96\,m_\mu^2)\cong-\frac{F_A(0.96\,m_\mu^2)}{1+0.96\,m_\mu^2/m_\pi^2}=0.68$$

[see also Eq. (8f)]. Note that $F_A$ in the ${}^3\mathrm{He}\to{}^3\mathrm{H}$ case is negative while $F_A$ in the $p\to n$ case is positive. From these values, $F_{V,M,A}(\frac35\,m_\mu^2)$ can be obtained by linear interpolation and

$$F_P(\tfrac35\,m_\mu^2)\cong-\frac{F_A(\frac35\,m_\mu^2)}{1+\frac35\,m_\mu^2/m_\pi^2}=0.83$$

so that the $F_{V,M,A,P}(\frac35\,m_\mu^2)$ entering into the $W_i^{(n)}(\frac35\,m_\mu^2,x)\equiv W_i^{(n)}(x)$ are all numerically determined and the $W_i^{(n)}(x)$ are immediately calculable from Eqs. (B1)–(B58). In this way, we obtain

$$W_1^{(0)}(x)=8.73-10.31x+6.36x^2\,, \tag{49a}$$
$$W_1^{(1)}(x)=-5.61+8.28x+0.02x^2\,, \tag{49b}$$
$$W_1^{(2)}(x)=0.57-1.75x+1.18x^2\,, \tag{49c}$$
$$W_2^{(0)}(x)=1.23-1.19x-0.94x^2\,, \tag{49d}$$
$$W_2^{(1)}(x)=0.63-0.32x+1.89x^2\,, \tag{49e}$$
$$W_2^{(2)}(x)=-0.57+1.52x-0.94x^2\,, \tag{49f}$$
$$W_3^{(0)}(x)=-1.87+0.37x-0.08x^2, \tag{49g}$$
$$W_3^{(1)}(x)=-8.80+12.44x-4.71x^2\,, \tag{49h}$$
$$W_3^{(2)}(x)=8.02-12.81x+4.79x^2\,, \tag{49i}$$
$$W_7^{(0)}(x)=-9.07+11.17x-3.45x^2\,, \tag{49j}$$

$$W_7^{(1)}(x)=7.46-10.91x+3.45x^2\,, \tag{49k}$$

$$W_8^{(0)}(x)=7.92-11.69x+4.39x^2\,, \tag{49l}$$

$$W_8^{(1)}(x)=-4.87+11.11x-4.08x^2\,, \tag{49m}$$

$$W_8^{(2)}(x)=0-0.63x+0.63x^2. \tag{49n}$$

In $\mu^- {}^3\text{He}\to\nu_\mu {}^3\text{H}\gamma$, $S=0\to S=1$ conversion and $({}^3\text{He}\mu^- {}^3\text{He})$ molecule formation do not take place so that the only situation of practical interest is

*Case II(a): No $S=0\to S=1$ conversion: $P(S,S_z)$ as in Eq. (42g).* Using Eqs. (45a)–(45c), (46a)–(46e), and Eqs. (49a)–(49n), we have

$$\mathcal{A}_{\hat{\nu},\hat{k}}=\frac{N_{\hat{\nu},\hat{k}}}{D_{\hat{\nu},\hat{k}}}\,, \tag{49o}$$

with

$$N_{\hat{\nu},\hat{k}}=(-5.61+8.28x+0.02x^2)+[(0.63-0.32x+1.89x^2)+(-8.80+12.44x-4.71x^2)(\hat{k}\cdot\hat{z})^2]P_\mu P_{^3\text{He}}$$
$$+(-13.94+22.28x-7.53x^2)\hat{k}\cdot\hat{z}\tfrac{1}{2}(P_\mu+P_{^3\text{He}}),$$

$$D_{\hat{\nu},\hat{k}}=(9.30-12.06x+7.54x^2)+[(0.66+0.33x-1.88x^2)+(6.15-12.44x+4.71x^2)(\hat{k}\cdot\hat{z})^2]P_\mu P_{^3\text{He}}$$
$$+(15.38-23.23x+8.47x^2)\hat{k}\cdot\hat{z}\tfrac{1}{2}(P_\mu+P_{^3\text{He}})\,,$$

$$\mathcal{A}_{\hat{k},\hat{z}}=\frac{N_{\hat{k},\hat{z}}}{D_{\hat{k},\hat{z}}}\,, \tag{49p}$$

with

$$N_{\hat{\nu},\hat{k}}=(10.41-15.54x+5.75x^2)\tfrac{1}{2}(P_\mu+P_{^3\text{He}})\,,$$

$$D_{\hat{\nu},\hat{k}}\doteq(8.92-10.89x+6.75x^2)+(1.84-4.58x-0.27x^2)P_\mu P_{^3\text{He}}\,,$$

$$\frac{d\Gamma(\mu^- {}^3\text{He}\to\nu_\mu {}^3\text{H}\gamma)/dx}{\frac{\alpha}{12\pi}\Gamma_0(12x(1-x)^2)}=(8.92-10.89x+6.75x^2)+(1.31-1.98x-0.74x^2)P_\mu P_{^3\text{He}}\,, \tag{49q}$$

$$\frac{\Gamma(\mu^- {}^3\text{He}\to\nu_\mu {}^3H\gamma)}{(\alpha/12\pi)\Gamma_0}=5.91+0.37P_\mu P_{^3\text{He}}\,, \tag{49r}$$

$$\frac{\Gamma(\mu^- {}^3\text{He}\to\nu_\mu {}^3\text{H})}{\Gamma_0}=4.61-0.49P_\mu P_{^3\text{He}}\,, \tag{49s}$$

$$R=\frac{\alpha}{12\pi}\,\frac{5.91+0.3P_\mu P_{^3\text{He}}}{4.61-0.49P_\mu P_{^3\text{He}}}=2.48\times10^{-4}\,\frac{1+0.06P_\mu P_{^3\text{He}}}{1-0.11P_\mu P_{^3\text{He}}}\,. \tag{49t}$$

Here, though $P_\mu$ is expected to be small (experimentally, $P_\mu\simeq0.06$ in ${}^4$He and should be approximately the same in ${}^3$He) one could, hopefully, develop techniques for polarizing the ${}^3$He nuclei (in liquid ${}^3$He) and so attain $P_{^3\text{He}}\cong1$. In this situation, determination of $\mathcal{A}_{\hat{\nu},\hat{k}}$ [by observation in coincidence of the recoil ${}^3$H and the $\gamma$ at $\hat{p}_{^3\text{He}}\cdot\hat{k}=-1$, $(m_\mu-2k_0)/|m_\mu-2k_0|$] and of $\mathcal{A}_{\hat{k},\hat{z}}$ would be of particular interest.

We conclude this discussion of applications by appending several comments:

(1) We emphasize that all our expression for

$$\mathcal{A}_{\hat{\nu},\hat{k}},\quad \mathcal{A}_{\hat{k},\hat{z}}\,,$$

$$\frac{d\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)}{dx}\,,$$

and

$$\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)$$

receive comparable contributions from $\mathcal{T}^{(l)}$ and $\mathcal{T}^{(h)}$. This is most easily seen by examination of the explicit expressions for $W_i^{(n);(ll)}(x)$, $W_i^{(n);(lh)}(x)$, and $W_i^{(n);(hh)}(x)$ in Eqs. (42d), (42e), and Appendix B. Thus, for example, the quantity

$$W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)=8.72-11.18x+7.31x^2$$
$$=\frac{d\Gamma(\mu^- p\to\nu_\mu n\gamma)/dx}{(\alpha/12\pi)\Gamma_0(12x(1-x)^2)}$$

in case I(a) with $P_\mu P_P=0$ [see Eqs. (46a) and (48c)] is composed additively of contributions $[W_1^{(0);(ll)}(x)+\tfrac{1}{3}W_1^{(2);(ll)}]$, $[W_1^{(0);(lh)}(x)+\tfrac{1}{3}W_1^{(2);(lh)}(x)]$, $[W_1^{(0);(hh)}(x)+\tfrac{1}{3}W_1^{(2);(hh)}(x)]$ (see Eq. (42e)) which are, respectively, $5.48-2.68x+0.68x^2$, $2.49-6.42x+1.49x^2$, and $0.75-2.08x+5.15x^2$ [see Eqs.

(B1)–(B3) and (B7)–(B9)]. Along the same line, neglect of terms containing $F_M(\frac{3}{5}\,m_\mu{}^2)$ and $F_P(\frac{3}{5}\,m_\mu{}^2)$ corresponds to dropping $\mathcal{T}^{(h)}$, i.e., setting $W_i^{(n);(lh)}(x)=0$ and $W_i^{(n);(hh)}(x)=0$ [see Eqs. (40b) and (42c)], and in addition retaining only these terms in $W_i^{(n);(ll)}(x)$ which depend solely on $F_V(\frac{3}{5}\,m_\mu{}^2)$ and $F_A(\frac{3}{5}\,m_\mu{}^2)$. In view of the remark just made, this neglect would result in very considerable modification of the predictions for

$$\mathcal{A}_{\hat\nu,\hat k},\ \mathcal{A}_{\hat k,\hat z},$$

$$\frac{d\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)}{dx},$$

and

$$\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)$$

—as an example of a particularly large modification:

$$\left(\frac{\dfrac{d\Gamma(\mu^- p\to\nu_\mu n\gamma)}{dx}}{\dfrac{\alpha}{12\pi}\,\Gamma_0(12x(1-x)^2)}\right)_{\text{case I(b)}} = \left([W_1^{(0)}(x)+\tfrac{1}{3}W_1^{(2)}(x)]+\{[W_2^{(0)}(x)+\tfrac{1}{3}W_2^{(2)}(x)]+\tfrac{1}{3}[W_3^{(0)}(x)+\tfrac{1}{3}W_3^{(2)}(x)]\}(-3)\right)$$

$$=(8.72-11.18x+7.31x^2)+(2.12-2.54x-1.21x^2)(-3)$$

$$=2.36-3.56x+10.94x^2$$

[see Eqs. (46a) and (48h)] would become

$$\left([W_1^{(0);(ll)}(x)+\tfrac{1}{3}W_1^{(2);(ll)}(x)]+\{[W_2^{(0);(ll)}(x)+\tfrac{1}{3}W_2^{(2);(ll)}(x)]+\tfrac{1}{3}[W_3^{(0);(ll)}(x)+\tfrac{1}{3}W_3^{(2);(ll)}(x)]\}(-3)\right)_{F_M=F_P=0}$$

$$=\left(\{[F_V(\tfrac{3}{5}m_\mu{}^2)]^2+3[F_A(\tfrac{3}{5}m_\mu{}^2)]^2\}+\tfrac{1}{3}\{[2F_V(\tfrac{3}{5}m_\mu{}^2)F_A(\tfrac{3}{5}m_\mu{}^2)]+2[F_A(\tfrac{3}{5}m_\mu{}^2)]^2\}(-3)\right)$$

$$=[F_V(\tfrac{3}{5}m_\mu{}^2)-F_A(\tfrac{3}{5}m_\mu{}^2)]^2=0.058$$

[see Eqs. (B1), (B7), (B10), (B16), (B19), and (B25)].

(2) We note that the average value of $\hat\nu\cdot\hat k$ is, using Eqs. (42h) and (42e)

$$(\hat\nu\cdot\hat k)_{\text{av}}=\frac{\displaystyle\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\,\hat\nu\cdot\hat k\langle|\mathcal{T}|^2\rangle}{\displaystyle\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\,\langle|\mathcal{T}|^2\rangle}$$

$$=\frac{\frac{1}{3}W_1^{(1)}(x)+\frac{1}{3}(W_2^{(1)}(x)+\frac{1}{3}W_3^{(1)}(x))[P(1,1)+P(1,0)+P(1,-1)-3P(0,0)]}{[W_1^{(0)}(x)+\frac{1}{3}W_1^{(2)}(x)]+\{[W_2^{(0)}(x)+\frac{1}{3}W_2^{(2)}(x)]+\frac{1}{3}[W_3^{(0)}(x)+\frac{1}{3}W_3^{(2)}(x)]\}[P(1,1)+P(1,0)+P(1,-1)-3P(0,0)]}\,. \tag{50a}$$

$(\hat\nu\cdot\hat k)_{\text{av}}$ is always small, a fact which helps justify the approximation $W_i^{(n)}(q^2,x)\cong W_i^{(n)}(\frac{3}{5}\,m_\mu{}^2,x)\equiv W_i^{(n)}(x)$ [see Eqs. (43a)–(43d) *et seq.*]. Thus, for example, in case I(b) where $[P(1,1)+P(1,0)+P(1,-1)-3P(0,0)]=-3$ [Eq. (42g)], we have from Eqs. (50a) and Eqs. (47a)–(47n)

$$(\hat\nu\cdot\hat k)_{\text{av}}=\frac{1}{3}\,\frac{1.70-1.72x-1.20x^2}{2.36-3.56x+10.94x^2}=\begin{cases}0.24: & x=0\\ 0.05: & x=\frac{1}{2}\\ -0.04: & x=1\,.\end{cases} \tag{50b}$$

(3) Though the approximation of

$$\frac{F_P(q^2)}{-F_A(q^2)/(1+q^2/m_\pi{}^2)}=1+\frac{m_\pi{}^2}{q^2}\left(1-\frac{f_{\pi N_i N_f}(q^2)/f_{\pi N_i N_f}(0)}{F_A(q^2)/F_A(0)}\right)$$

by 1 in both the $\mu^- p$ and $\mu^-\,{}^3\text{He}$ cases [Eq. (8f)] appears reasonably well established on the basis of the analysis of $\mu^- p\to\nu_\mu n$ and $\mu^-\,{}^3\text{He}\to\nu_\mu\,{}^3\text{H}$,[16,17] it is nevertheless worth pointing out that the values of $\mathcal{A}_{\hat\nu,\hat k}$, $\mathcal{A}_{\hat k,\hat z}$, $d\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)/dx$, and $\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)$ are all rather sensitive to the value assumed for

$$\frac{F_P(q^2)}{-F_A(q^2)/(1+q^2/m_\pi{}^2)}=1+\frac{m_\pi{}^2}{q^2}\left(1-\frac{f_{\pi N_i N_f}(q^2)/f_{\pi N_i N_f}(0)}{F_A(q^2)/F_A(0)}\right)\equiv\xi$$

the sensitivity being greatest for $\mathcal{A}_{\hat\nu,\hat k}$ and $\mathcal{A}_{\hat k,\hat z}$. It is therefore interesting to calculate the values of $\mathcal{A}_{\hat\nu,\hat k}$, $\mathcal{A}_{\hat k,\hat z}$, $d\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)/dx$, and $\Gamma(\mu^- N_i\to\nu_\mu N_f\gamma)$, e.g., for $\xi=\frac{1}{2}$ and $\xi=2$, and compare these values with the values calculated for $\xi=1$ and already given in Eqs. (49o)–(49r). Considering, for example, case II(a) with $P_\mu\cong 0$, $P_{^3\text{He}}\cong 1$, we have, using Eqs. (45a)–(45c), (46a), (46b), and Eqs. (B1)–(B58)

$$\mathcal{a}_{\hat{\nu},\hat{k}} = \frac{-0.51 - 2.09\hat{k}\cdot\hat{z}}{5.36 + 2.66\hat{k}\cdot\hat{z}}: \quad \xi = \tfrac{1}{2}, \quad x = \tfrac{2}{3}, \tag{51a}$$

$$\mathcal{a}_{\hat{\nu},\hat{k}} = \frac{-0.08 - 1.22\hat{k}\cdot\hat{z}}{4.61 + 1.83\hat{k}\cdot\hat{z}}: \quad \xi = 1, \quad x = \tfrac{2}{3} \quad \text{[Eq. (49o)]}, \tag{51b}$$

$$\mathcal{a}_{\hat{\nu},\hat{k}} = \frac{2.78 + 0.61\hat{k}\cdot\hat{z}}{5.90 + 1.18\hat{k}\cdot\hat{z}}: \quad \xi = 2, \quad x = \tfrac{2}{3}, \tag{51c}$$

$$\mathcal{a}_{\hat{k},\hat{z}} = 0.43: \quad \xi = \tfrac{1}{2}, \quad x = \tfrac{2}{3}, \tag{51d}$$

$$\mathcal{a}_{\hat{k},\hat{z}} = 0.28: \quad \xi = 1, \quad x = \tfrac{2}{3} \quad \text{[Eq. (49p)]}, \tag{51e}$$

$$\mathcal{a}_{\hat{k},\hat{z}} = 0.06: \quad \xi = 2, \quad x = \tfrac{2}{3}, \tag{51f}$$

$$\frac{\dfrac{d\Gamma(\mu^- \, ^3\text{He} \to \nu_\mu \, ^3\text{H}\gamma)}{dx}}{\dfrac{\alpha}{12\pi}\Gamma_0(12x(1-x)^2)} = 5.42: \quad \xi = \tfrac{1}{2}, \quad x = \tfrac{2}{3}, \tag{51g}$$

$$\frac{\dfrac{d\Gamma(\mu^- \, ^3\text{He} \to \nu_\mu \, ^3\text{H}\gamma)}{dx}}{\dfrac{\alpha}{12\pi}\Gamma_0(12x(1)-x^2)} = 4.66: \quad \xi = 1, \quad x = \tfrac{2}{3} \quad \text{[Eq. (49q)]}, \tag{51h}$$

$$\frac{\dfrac{d\Gamma(\mu^- \, ^3\text{He} \to \nu_\mu \, ^3\text{H}\gamma)}{dx}}{\dfrac{\alpha}{12\pi}\Gamma_0(12x(1-x)^2)} = 5.80: \quad \xi = 2, \quad x = \tfrac{2}{3}, \tag{51i}$$

$$\Gamma(\mu^- \, ^3\text{He} \to \nu_\mu \, ^3\text{H}\gamma)/\Gamma_0 = 12.8 \times 10^{-4}: \quad \xi = \tfrac{1}{2}, \tag{51j}$$

$$\Gamma(\mu^- \, ^3\text{He} \to \nu_\mu \, ^3\text{H}\gamma)/\Gamma_0 = 11.4 \times 10^{-4}: \quad \xi = 1 \quad \text{[Eq. (49r)]}, \tag{51k}$$

$$\Gamma(\mu^- \, ^3\text{He} \to \nu_\mu \, ^3\text{H}\gamma)/\Gamma_0 = 12.4 \times 10^{-4}: \quad \xi = 2. \tag{51l}$$

(4) The photon emitted in $\mu^- N_i \to \nu_\mu N_f \gamma$ possesses a net circular polarization

$$P_{\text{circ.}} = \frac{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} (\langle |\mathcal{T}|^2\rangle_+ - \langle |\mathcal{T}|^2\rangle_-)}{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} (\langle |\mathcal{T}|^2\rangle_+ + \langle |\mathcal{T}|^2\rangle_-)} = 1 - 2\,\frac{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} \langle |\mathcal{T}|^2\rangle_-}{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} (\langle |\mathcal{T}|^2\rangle_+ + \langle |\mathcal{T}|^2\rangle_-)}$$

$$= 1 - 2\,\frac{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} \langle |\mathcal{T}|^2\rangle_-}{\int \frac{d\Omega^{(\gamma)}}{4\pi} \int \frac{d\Omega^{(\nu)}}{4\pi} \langle |\mathcal{T}|^2\rangle}, \tag{52a}$$

where

$$\langle |\mathcal{T}|^2\rangle_\pm \equiv \sum_{S,S_z} \Big( \sum_{s_z^{(\nu)}, s_z^{(f)}} |\mathcal{T}^{(l)}(\vec{\epsilon}_\pm^{\,*}, s_z^{(\nu)}, s_z^{(f)}; S, S_z) + \mathcal{T}^{(h)}(\vec{\epsilon}_\pm^{\,*}, s_z^{(\nu)}, s_z^{(f)}; S, S_z)|^2 \Big) P(S, S_z) \tag{52b}$$

with $\vec{\epsilon}_+^{\,*}$ and $\vec{\epsilon}_-^{\,*}$ right and left circular polarization vectors, respectively ($\vec{\epsilon}_\pm = (1/\sqrt{2})(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)$, $\hat{k} = \hat{\epsilon}_3$ with $\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3$ mutually orthogonal). Equation (52b) for $\langle |\mathcal{T}|^2\rangle_\pm$ is to be compared with Eq. (42f) for $\langle |\mathcal{T}|^2\rangle = \langle |\mathcal{T}|^2\rangle_+ + \langle |\mathcal{T}|^2\rangle_-$.

Evaluating the sums on the right side of Eq. (52b), we get

$$\langle|\mathcal{T}|^2\rangle_\pm=\left(\frac{Ge}{\sqrt{2}}\frac{1}{(2k_0)^{1/2}}\frac{1}{m_\mu}\right)^2\{U_{1;\pm}+U_{2;\pm}[P(1,1)+P(1,0)+P(1,-1)-3P(0,0)]+U_{3;\pm}[P(1,0)-P(0,0)]$$
$$+[U_{4;\pm}(\hat{\nu}\cdot\hat{z})^2+U_{5;\pm}(\hat{\nu}\cdot\hat{z})(\hat{k}\cdot\hat{z})+U_{6;\pm}(\hat{k}\cdot\hat{z})^2][P(1,1)+P(1,-1)-2P(1,0)]$$
$$+[U_{7;\pm}\hat{\nu}\cdot\hat{z}+U_{8;\pm}\hat{k}\cdot\hat{z}][P(1,1)-P(1,-1)]\}, \qquad (52c)$$

$$U_{i;\pm}=\sum_{n=0}^{2}U_{i;\pm}^{(n)}(\hat{\nu}\cdot\hat{k})^n=\sum_{n=0}^{2}(U_{i;\pm}^{(n);(ll)}+U_{i;\pm}^{(n);(lh)}+U_{i;\pm}^{(n);(hh)})(\hat{\nu}\cdot\hat{k})^n,$$

$$W_i^{(n);(\xi)}=U_{i;+}^{(n);(\xi)}+U_{i;-}^{(n);(\xi)}:\ \ \xi=ll,lh,hh,$$

where the $U_{i;\pm}^{(n)}$ are structure functions which are homogeneous and quadratic in the $F_{V,M,A,P}(q^2)$ and which also depend on $k_0$—Eq. (52c) for $\langle|\mathcal{T}|^2\rangle_\pm$ is to be compared with Eq. (42h) for $\langle|\mathcal{T}|^2\rangle$. We note that $\mathcal{T}^{(l)}(\vec{\epsilon}_-^{\,*},s_z^{(\nu)},s_z^{(f)};S,S_z)=0$ [this follows from Eq. (40a) since $(1-\vec{\sigma}^{(L)}\cdot\hat{k})\vec{\sigma}^{(L)}\cdot\vec{\epsilon}_-^{\,*}=0$]; thus only $\mathcal{T}^{(h)}(\vec{\epsilon}_-^{\,*},s_z^{(\nu)},s_z^{(f)};S,S_z)$ contributes to $\langle|\mathcal{T}|^2\rangle_-$ [i.e., $U_{i;-}^{(n)}=U_{i;-}^{(n);(hh)}$] so that $\langle|\mathcal{T}|^2\rangle_-$ depends only on $F_M(q^2)$ and $F_P(q^2)$ [see Eq. (40b)] and $P_{\text{circ.}}=1$ only if the terms involving $F_M(q^2)$ and $F_P(q^2)$ are neglected.

If we now approximate the structure functions $U_{i;-}^{(n);(\xi)}(q^2,k_0/m_\mu)$ by $U_{i;-}^{(n);(\xi)}(\frac{3}{5}m_\mu^{\,2},k_0/m_\mu)$ [using the arguments of Eqs. (43a)–(43d) *et seq.*], calculate the $U_{i;-}^{(n);(\xi)}(\frac{3}{5}m_\mu^{\,2},k_0/m_\mu)$ from the $F_{V,M,A,P}(\frac{3}{5}m_\mu^{\,2})$ listed above, insert the values obtained into Eq. (52c) to get $\langle|\mathcal{T}|^2\rangle_-$, and integrate over all $\hat{\nu}$ and $\hat{k}$, we have

$$\int\frac{d\Omega^{(\gamma)}}{4\pi}\int\frac{d\Omega^{(\nu)}}{4\pi}\langle|\mathcal{T}|^2\rangle_-=\left(\frac{Ge}{\sqrt{2}}\frac{1}{(2k_0)^{1/2}}\frac{1}{m_\mu}\right)^2$$
$$\times\left\{\left[\tfrac{1}{3}\left(F_P(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu^{\,2}}{m_\pi^{\,2}}\right)^2(4-18x+32x^2)+\tfrac{8}{3}\left(F_P(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu^{\,2}}{m_\pi^{\,2}}\right)\left(F_M(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu}{2m_p}\right)(1-4x)\right.\right.$$
$$\left.+3\left(F_M(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu}{2m_p}\right)^2\right]+\left[\tfrac{1}{9}\left(F_P(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu^{\,2}}{m_\pi^{\,2}}\right)^2(-4+30x-56x^2)-\frac{16}{9}\left(F_P(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu^{\,2}}{m_\pi^{\,2}}\right)\right.$$
$$\left.\times\left(F_M(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu}{2m_p}\right)(1-4x)-2\left(F_M(\tfrac{3}{5}m_\mu^{\,2})\frac{m_\mu}{2m_p}\right)^2\right]$$
$$\left.\times[P(1,1)+P(1,0)+P(1,-1)-3P(0,0)]\right\} \qquad (52d)$$

which, together with Eqs. (44f) and (47a)–(47i) for $\int d\Omega^{(\gamma)}/4\pi\int d\Omega^{(\nu)}/4\pi\langle|\mathcal{T}|^2\rangle$, yields upon substitution in Eq. (52a)

*case I(a)*

$$P_{\text{circ.}}=1-2\frac{(0.22-0.59x+3.17x^2)-(0.02-0.20x+1.85x^2)P_\mu P_P}{(8.72-11.18x+7.31x^2)+(2.12-2.54x-1.21x^2)P_\mu P_P}=0.45\ \ \text{for } x=\tfrac{2}{3}\ \text{ and }\ P_\mu P_P=0, \qquad (52e)$$

*case I(b)*

$$P_{\text{circ.}}=1-2\frac{(0.28-1.19x+8.72x^2)}{(2.36-3.56x+10.94x^2)}=-0.39\ \ \text{for } x=\tfrac{2}{3}, \qquad (52f)$$

*case I(c)*

$$P_{\text{circ.}}=1-2\frac{(0.26-0.99x+6.87x^2)}{(4.48-6.10x+9.73x^2)}=-0.12\ \ \text{for } x=\tfrac{2}{3}, \qquad (52g)$$

*case II(a)*

$$P_{\text{circ.}}=1-2\frac{(0.19+0.04x+2.52x^2)-(0.06+0.18x+1.47x^2)P_\mu P_{^3\text{He}}}{(8.92-10.89x+6.75x^2)+(1.31-1.98x-0.74x^2)P_\mu P_{^3\text{He}}}$$
$$=0.43\ \ \text{for } x=\tfrac{2}{3}\ \text{ and } P_\mu P_{^3\text{He}}=0. \qquad (52h)$$

As can be seen directly from Eqs. (52d) and (52a) the value of $P_{\text{circ.}}$ is also sensitive to the value assumed for

$$\frac{F_P(\frac{3}{5}m_\mu^{\,2})}{-F_A(\frac{3}{5}m_\mu^{\,2})/[1+(\frac{3}{5}m_\mu^{\,2})/m_\pi^{\,2}]}=\left[1+\left(m_\pi^{\,2}/q^2\right)\left(1-\frac{f_{\pi N_i N_f}(q^2)/f_{\pi N_i N_f}(0)}{F_A(q^2)/F_A(0)}\right)\right]_{q^2=(3/5)m_\mu^{\,2}}\equiv\xi.$$

Unfortunately, the experimental determination of $P_{\text{circ.}}$ appears to be extremely difficult.

## SUMMARY

In conclusion, we summarize our results by emphasizing that we have derived an explicit expression for the transition amplitude of radiative muon capture by a nucleus in terms of the charge on the muon and the form factors characterizing the corresponding nonradiative muon capture [Eqs. (40a) and (40b)]. Moreover, this transition amplitude, albeit rather approximate, satisfies exactly the constraint equation of CEC, CVC, and PCAC and can be used to give a detailed numerical prediction of the various observable quantities associated with $\mu^- p \to \nu_\mu n\gamma$ and $\mu^- {}^3\text{He} \to \nu_\mu {}^3\text{H}\gamma$ [Eqs. (48a)–(48r), (49o)–(49t), (51a)–(51k), (52e)–(52h)].

## APPENDIX A

In this appendix, we begin with a description of the procedure for the determination of all the relevant $F_{ij}^{(a),(b),(c),(d)}$ by means of the CEC and CVC constraints of Eqs. (15a)–(15h) and (17a)–(17h) and the LH of Eq. (20), starting from an appropriate set of $F_{ij}^{(a),(b),(c),(d)}$. The appropriate set used is that given in the text [see Eqs. (24a)–(24c)]:

$$F_{00}^{(b)} = -m_p^2\left[\frac{F_M(q^2)}{2m_p}\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right) + \left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right) \times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right)\right], \quad \text{(A1)}$$

$$F_{11}^{(a)} = m_p^2\left[\frac{F_M(q^2)}{2m_p}\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right) - 2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\frac{e_f+\mu_f}{(Q+q)\cdot k}\right], \quad \text{(A2)}$$

$$F_{23}^{(a)} = 0 . \quad \text{(A3)}$$

We first note that Eq. (15b) together with Eq. (20) yields a unique solution for $F_{12}^{(b)}$ and $F_{13}^{(b)}$ [ see Eqs. (21a) and (21b)]:

$$F_{12}^{(b)} = m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right) \times\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \quad \text{(A4)}$$

$$F_{13}^{(b)} = -m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right) \times\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right). \quad \text{(A5)}$$

Analogously, Eqs. (15g) and (15h), together with Eq. (20), determine $F_{22}^{(c)}$, $F_{32}^{(c)}$, $F_{23}^{(c)}$, and $F_{33}^{(c)}$:

$$F_{22}^{(c)} = \tfrac{1}{2} i m_p^2 F_M(q^2)\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \quad \text{(A6)}$$

$$F_{32}^{(c)} = -\tfrac{1}{2} i m_p^2 F_M(q^2)\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right), \quad \text{(A7)}$$

$$F_{23}^{(c)} = 0 , \quad \text{(A8)}$$

$$F_{33}^{(c)} = 0 . \quad \text{(A9)}$$

Further, Eq. (15a) and Eq. (20) show that $F_{00}^{(a)}$ depends only on $q^2$. Similarly, Eq. (15e) and Eq. (20) show that $F_{13}^{(a)}$ depends only on $q^2$ so that Eq. (17b) becomes

$$(F_{11}^{(a)}+F_{00}^{(b)}+F_{13}^{(a)})q\cdot k + (F_{12}^{(a)}-F_{00}^{(b)})Q\cdot k = -q^2F_{13}^{(a)} + 2Mm_pF_{00}^{(a)} + m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right) = \text{function of } q^2 \text{ only} . \quad \text{(A10)}$$

Equations (A1) and (A2) yield

$$F_{11}^{(a)}+F_{00}^{(b)} = -m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right) \times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}+\frac{e_f+\mu_f}{(Q+q)\cdot k}\right) \quad \text{(A11)}$$

so that, combining Eqs. (A10) and (A11) with Eq. (20), we obtain

$$F_{13}^{(a)} = 0 , \quad \text{(A12)}$$

$$F_{12}^{(a)} - F_{00}^{(b)} = m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right) \times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right), \quad \text{(A13)}$$

$$-q^2F_{13}^{(a)} + 2Mm_pF_{00}^{(a)} = m_p^2\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)(\mu_i-\mu_f) \quad \text{(A14)}$$

which, in turn, yield [using also Eq. (A1)]

$$F_{12}^{(a)} = -\tfrac{1}{2} m_p F_M(q^2)\left(\frac{\mu_i}{2m_i}+\frac{\mu_f}{2M_f}\right), \quad \text{(A15)}$$

$$F_{00}^{(a)} = m_p\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right). \quad \text{(A16)}$$

With $F_{00}^{(a)}$, $F_{12}^{(a)}$ and $F_{13}^{(a)}$ given by Eqs. (A16), (A15), and (A12), Eqs. (15a), (15d), (15e), and Eq. (20) give

$$F_{22}^{(b)} = m_p^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\left(\frac{1}{(Q-q)\cdot k}\frac{\mu_i}{2M_i} - \frac{1}{(Q+q)\cdot k}\frac{\mu_f}{2M_f}\right), \quad \text{(A17)}$$

$$F_{23}^{(b)}=-m_p{}^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A18}$$

$$F_{22}^{(d)}=-\tfrac{1}{2}im_p{}^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A19}$$

$$F_{32}^{(d)}=\tfrac{1}{2}im_p{}^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A20}$$

$$F_{23}^{(d)}=0\,, \tag{A21}$$

$$F_{33}^{(d)}=0\,, \tag{A22}$$

while Eq. (17g), with $F_{22}^{(b)}$, $F_{12}^{(b)}$, $F_{00}^{(b)}$, and $F_{23}^{(d)}$ given by Eqs. (A17), (A4), (A1), and (A21), yields

$$F_{22}^{(d)}Q\cdot k+F_{21}^{(d)}q\cdot k=-\tfrac{1}{2}im_p{}^3F_M(q^2)\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right). \tag{A23}$$

Equation (A23) and Eq. (20) imply

$$F_{22}^{(d)}=-\tfrac{1}{2}im_p{}^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A24}$$

$$F_{21}^{(d)}=\tfrac{1}{2}im_p{}^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2m_f}\right). \tag{A25}$$

We note that Eq. (A19) and Eq. (A24) agree, which provides a consistency check. With $F_{21}^{(d)}$, $F_{00}^{(b)}$, and $F_{11}^{(a)}$ given by Eqs. (A25), (A1), and (A11) we obtain, using Eq. (15c) and Eq. (20),

$$F_{31}^{(d)}=-\tfrac{1}{2}im_p{}^3F_M(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A26}$$

$$F_{00}^{(d)}=-im_p{}^2\left[\frac{F_M(q^2)}{2m_p}\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}+\frac{e_f+\mu_f}{(Q+q)\cdot k}\right)\right]. \tag{A27}$$

Further, with $F_{22}^{(c)}$, $F_{22}^{(b)}$, $F_{23}^{(c)}$, and $F_{00}^{(a)}$ specified by Eqs. (A6), (A17), (A8), and (A16), Eq. (17d) leads to

$$F_{21}^{(c)}=-im_p{}^3\left[\frac{F_M(q^2)}{2m_p}\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right)\right] \tag{A28}$$

which is seen to satisfy Eq. (20). Equation (A28), together with Eqs. (15f) and (20), then determines $F_{31}^{(c)}$ and $F_{00}^{(c)}$:

$$F_{31}^{(c)}=im_p{}^3\left[\frac{F_M(q^2)}{2m_p}\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+1)\cdot k}\,\frac{\mu_f}{2M_f}\right)\right], \tag{A29}$$

$$F_{00}^{(c)}=im_p\left[\frac{F_M(q^2)}{2m_p}(e_i+e_f)+\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right)\right]. \tag{A30}$$

Finally, with $F_{23}^{(a)}$, and $F_{00}^{(a)}$ given by Eqs. (A3) and (A16), Eq. (17a) and Eq. (20) yield

$$F_{21}^{(a)}=m_p{}^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A31}$$

$$F_{22}^{(a)}=-m_p{}^3\left(F_V(q^2)+\frac{M}{m_p}F_M(q^2)\right)\times\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right) \tag{A32}$$

which completes the determination of all the relevant $F_{ij}^{(a),(b),(c),(d)}$ in Eqs. (25a)–(25z).

We note that the CVC constraint equations (17e) and (17h) provide consistency checks of the above results. Further, the CVC constraint equations (17c) and (17f) which relate $F_{11}^{(b)}$, $F_{21}^{(b)}$, $F_{1j}^{(c)}$, and $F_{1j}^{(d)}$ to $F_{00}^{(b)}$, $F_{00}^{(c)}$, and $F_{00}^{(d)}$ need not be used since, as noted in the text [before Eq. (23)], $F_{11}^{(b)}$, $F_{21}^{(b)}$, $F_{1j}^{(c)}$, and $F_{1j}^{(d)}$ do not contribute to $\mathcal{T}^{(h)}$.

We proceed next to determine all the relevant $G_{ij}^{(a),(b),(c),(d)}$ by means of the CEC and PCAC constraints of Eqs. (16a)–(16h) and (18a)–(18j) and the LH of Eq. (20), starting from an appropriate set of $G_{ij}^{(a),(b),(c),(d)}$. The appropriate set used is that given in the text [see Eqs. (27)–(31)]:

$$G_{00}^{(a)}=-m_pF_A(q^2)\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right), \tag{A33}$$

$$G_{00}^{(b)}=m_p^{\,2}F_A(q^2)\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right), \tag{A34}$$

$$G_{13}^{(a)}=-m_p^{\,2}F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right), \tag{A35}$$

$$G_{22}^{(a)}=0, \tag{A36}$$

$$G_{23}^{(a)}=-m_p^{\,3}F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right). \tag{A37}$$

We first note that Eqs. (16b), (16g), (16h), and (18j) together with Eq. (20) lead to

$$G_{12}^{(b)}=-m_p^{\,2}F_A(q^2)\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \tag{A38}$$

$$G_{13}^{(b)}=m_p^{\,2}F_A(q^2)\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right), \tag{A39}$$

$$G_{22}^{(c)}=0, \tag{A40}$$

$$G_{32}^{(c)}=0, \tag{A41}$$

$$G_{23}^{(c)}=im_p^{\,3}F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \tag{A42}$$

$$G_{33}^{(c)}=-im_p^{\,3}F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right), \tag{A43}$$

$$f_P=-m_\pi^{\,2}\left(F_A(q^2)+\frac{q^2}{m_\pi^{\,2}}F_P(q^2)\right)\times\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right), \tag{A44}$$

$$f_E=4Mm_p\left(F_A(q^2)+\frac{q^2}{m_\pi^{\,2}}F_P(q^2)\right)\times\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right) \tag{A45}$$

and, with $G_{00}^{(a)}$, $G_{12}^{(b)}$, $G_{22}^{(c)}$, $G_{23}^{(c)}$, and $f_E$ given by Eqs. (A33), (A38), (A40), (A42), and (A45), we obtain, from Eqs. (18d) and (20),

$$G_{21}^{(c)}=im_p^{\,3}\left[F_A(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right)-F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right)\right], \tag{A46}$$

$$G_{23}^{(c)}=im_p^{\,3}F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{e_i}{(Q-q)\cdot k}-\frac{e_f}{(Q+q)\cdot k}\right), \tag{A47}$$

while Eq. (A46), together with Eqs. (16f) and (20), yields

$$G_{31}^{(c)}=-im_p^{\,3}\left[F_A(q^2)\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right)-F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{e_i}{(Q-q)\cdot k}+\frac{e_f}{(Q+q)\cdot k}\right)\right], \tag{A48}$$

$$G_{00}^{(c)}=-im_p\left[F_A(q^2)\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right)-F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\right]. \tag{A49}$$

We note that Eq. (18e) provides a consistency check for the above results.

Using $G_{00}^{(b)}$ and $G_{13}^{(a)}$ as given by Eqs. (A34) and (A35), together with the fact that, by Eqs. (16d) and (20), $G_{12}^{(a)}$ depends only on $q^2$, we have from Eqs. (18b) and (20)

$$G_{12}^{(a)}=0, \tag{A50}$$

$$G_{11}^{(a)}=2m_p^{\,2}F_A(q^2)\,\frac{e_f+\mu_f}{(Q+q)\cdot k}+m_p^{\,2}F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{\mu_i}{2M_i}-\frac{\mu_f}{2M_f}\right), \tag{A51}$$

$$f_A=\left(F_A(q^2)+\frac{q^2}{m_\pi^{\,2}}F_P(q^2)\right)(\mu_i-\mu_f) \tag{A52}$$

whence, from Eqs. (18i) and (20)

$$f_2=-im_p^{\,2}\left(F_A(q^2)+\frac{q^2}{m_\pi^{\,2}}F_P(q^2)\right)\times\left(\frac{\mu_i}{(Q-q)\cdot k}-\frac{\mu_f}{(Q+q)\cdot k}\right), \tag{A53}$$

$$f_3=im_p^{\,2}\left(F_A(q^2)+\frac{q^2}{m_\pi^{\,2}}F_P(q^2)\right)\times\left(\frac{\mu_i}{(Q-q)\cdot k}+\frac{\mu_f}{(Q+q)\cdot k}\right). \tag{A54}$$

Also, with $G_{13}^{(a)}$ and $G_{12}^{(a)}$ as given by Eqs. (A35) and (A50), we get, using Eqs. (16d), (16e), and (20),

$$G_{22}^{(d)}=0, \tag{A55}$$

$$G_{32}^{(d)}=0, \tag{A56}$$

$$G_{23}^{(d)}=-im_p^{\,4}F_P(q^2)\,\frac{2M}{m_\pi^{\,2}}\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A57}$$

$$G_{33}^{(d)}=im_p{}^4F_P(q^2)\,\frac{2M}{m_\pi{}^2}\times\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A58}$$

while, with $G_{00}^{(b)}$, $G_{12}^{(b)}$, $G_{22}^{(d)}$, $G_{23}^{(d)}$, and $f_2$ given by Eqs. (A34), (A38), (A55), (A57), and (A53), we obtain $G_{21}^{(d)}$, in a form consistent with Eq. (20), from Eq. (18g):

$$G_{21}^{(d)}=im_p{}^4F_P(q^2)\,\frac{2M}{m_\pi{}^2}\times\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}-\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right). \tag{A59}$$

Equations (A59), (A51), and (A34), together with Eqs. (16c) and (20), yield

$$G_{31}^{(d)}=-im_p{}^4F_P(q^2)\,\frac{2M}{m_\pi{}^2}\times\left(\frac{1}{(Q-q)\cdot k}\,\frac{\mu_i}{2M_i}+\frac{1}{(Q+q)\cdot k}\,\frac{\mu_f}{2M_f}\right), \tag{A60}$$

$$G_{00}^{(d)}=im_p{}^2F_A(q^2)\times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}+\frac{e_f+\mu_f}{(Q+q)\cdot k}\right). \tag{A61}$$

We note that Eq. (18h) provides a consistency check for the above results.

Finally, with $G_{00}^{(a)}$, $G_{00}^{(b)}$, $G_{22}^{(a)}$, and $G_{23}^{(a)}$ given by Eqs. (A33), (A34), (A36), and (A37), we obtain, using Eqs. (18a) and (20),

$$G_{21}^{(a)}=m_p{}^3F_P(q^2)\,\frac{2M}{m_\pi{}^2}\times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right), \tag{A62}$$

$$\tilde{f}_E=4Mm_p\left(F_A(q^2)+\frac{q^2}{m_\pi{}^2}\,F_P(q^2)\right)\times\left(\frac{e_i+\mu_i}{(Q-q)\cdot k}-\frac{e_f+\mu_f}{(Q+q)\cdot k}\right)+2m_pF_A(q^2)\left(\frac{\mu_i}{2M_i}+\frac{\mu_f}{2M_f}\right). \tag{A63}$$

We thus complete the determination of all the relevant $G_{ij}^{(a),(b),(c),(d)}$ in Eqs. (32a)–(32z).

## APPENDIX B

In this appendix, we list the structure functions $W_i^{(n)}=W_i^{(n);(ll)}+W_i^{(n);(lh)}+W_i^{(n);(hh)}$ as derived from Eqs. (40a)–(42c). Using the notation $F_{V,M,A,P}\equiv F_{V,M,A,P}(q^2)$ and $x\equiv k_0/m_\mu$ (with $0\leqslant x\lesssim 1$), we have

$$W_1^{(0);(ll)}=F_V{}^2+3F_A{}^2+2F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}+\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-2x+2x^2)+4F_AF_M\frac{m_\mu}{2m_p}(1-2x)+2\left(F_M\frac{m_\mu}{2m_p}\right)^2(1-3x+3x^2), \tag{B1}$$

$$W_1^{(0);(lh)}=F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}8x+\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1+x)+8F_AF_M\frac{m_\mu}{2m_p}+F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1+7x-12x^2)+4\left(F_M\frac{m_\mu}{2m_p}\right)^2(1-2x), \tag{B2}$$

$$W_1^{(0);(hh)}=2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-3x+8x^2)+2F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-x)+6\left(F_M\frac{m_\mu}{2m_p}\right)^2, \tag{B3}$$

$$W_1^{(1);(ll)}=F_V{}^2-F_A{}^2+2F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}+\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2-4F_AF_M\frac{m_\mu}{2m_p}(1-2x)-2\left(F_M\frac{m_\mu}{2m_p}\right)^2(1-2x)^2, \tag{B4}$$

$$W_1^{(1);(lh)}=4F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}(1-2x)+2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2+4F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-3x+4x^2)-4\left(F_M\frac{m_\mu}{2m_p}\right)^2(1-2x), \tag{B5}$$

$$W_1^{(1);(hh)}=4\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2x(-1+3x)+F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}4x-4\left(F_M\frac{m_\mu}{2m_p}\right)^2, \tag{B6}$$

$$W_1^{(2);(ll)}=2\left[\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2-\left(F_M\frac{m_\mu}{2m_p}\right)^2\right]x(1-x), \tag{B7}$$

$$W_1^{(2);(lh)}=\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-x)-F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-x)(1-4x), \tag{B8}$$

$$W_1^{(2);(hh)}=2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-x)(1-2x)+2F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-x), \tag{B9}$$

$$W_2^{(0);(ll)}=0, \tag{B10}$$

$$W_2^{(0);(lh)} = (F_V - F_A)F_P \frac{m_\mu^2}{m_\pi^2} - \left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 x - F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-x), \tag{B11}$$

$$W_2^{(0);(hh)} = -4\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 x^2 + 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(-1+3x) - 2\left(F_M \frac{m_\mu}{2m_p}\right)^2, \tag{B12}$$

$$W_2^{(1);(ll)} = 0, \tag{B13}$$

$$W_2^{(1);(lh)} = (F_V - F_A)F_P \frac{m_\mu^2}{m_\pi^2} - \left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2, \tag{B14}$$

$$W_2^{(1);(hh)} = -2\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 x(3-4x) + 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-4x) + 2\left(F_M \frac{m_\mu}{2m_p}\right)^2, \tag{B15}$$

$$W_2^{(2);(ll)} = 0, \tag{B16}$$

$$W_2^{(2);(lh)} = F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-x) - \left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 (1-x), \tag{B17}$$

$$W_2^{(2);(hh)} = -2\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 (1-x)(1-2x) - 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-x), \tag{B18}$$

$$W_3^{(0);(ll)} = 2F_V F_A + 2F_A^2 + 2F_V F_P \frac{m_\mu^2}{m_\pi^2} x + 2F_A F_P \frac{m_\mu^2}{m_\pi^2}(1-x) + 2F_V F_M \frac{m_\mu}{2m_p}(1-x) + 2F_A F_M \frac{m_\mu}{2m_p}(1-3x) + 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-x)(1-2x) - 2\left(F_M \frac{m_\mu}{2m_p}\right)^2 x(1-2x), \tag{B19}$$

$$W_3^{(0);(lh)} = -F_V F_P \frac{m_\mu^2}{m_\pi^2}(1-2x) + F_A F_P \frac{m_\mu^2}{m_\pi^2}(1+6x) + \left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 (-1+6x-4x^2) + 2F_V F_M \frac{m_\mu}{2m_p} + 4F_A F_M \frac{m_\mu}{2m_p} + 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(2-x-4x^2) + 2\left(F_M \frac{m_\mu}{2m_p}\right)^2 (1-3x), \tag{B20}$$

$$W_3^{(0);(hh)} = 2\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 (-1+3x) + F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p} 8x + 2\left(F_M \frac{m_\mu}{2m_p}\right)^2. \tag{B21}$$

$$W_3^{(1);(ll)} = 2F_V F_A - 2F_A^2 + 2F_V F_P \frac{m_\mu^2}{m_\pi^2} - 4F_A F_M \frac{m_\mu}{2m_p}(1-2x) - 2\left(F_M \frac{m_\mu}{2m_p}\right)^2 (1-2x)^2, \tag{B22}$$

$$W_3^{(1);(lh)} = F_V F_P \frac{m_\mu^2}{m_\pi^2} + F_A F_P \frac{m_\mu^2}{m_\pi^2}(3-8x) + \left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 + (2F_V - 4F_A)F_M \frac{m_\mu}{2m_p} + 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-6x+8x^2) + \left(F_M \frac{m_\mu}{2m_p}\right)^2 (-4+10x), \tag{B23}$$

$$W_3^{(1);(hh)} = 2\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 x(3-4x) + 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-4x) - 2\left(F_M \frac{m_\mu}{2m_p}\right)^2, \tag{B24}$$

$$W_3^{(2);(ll)} = 2F_V F_P \frac{m_\mu^2}{m_\pi^2}(1-x) - 2F_A F_P \frac{m_\mu^2}{m_\pi^2}(1-x) + 2F_V F_M \frac{m_\mu}{2m_p}(1-x) + 2F_A F_M \frac{m_\mu}{2m_p}(1-x) - 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-x)(1-2x) + 2\left(F_M \frac{m_\mu}{2m_p}\right)^2 (1-x)(1-2x), \tag{B25}$$

$$W_3^{(2);(lh)} = 2(F_V - F_A)F_P \frac{m_\mu^2}{m_\pi^2}(1-x) + 2\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 (1-x)(1-2x) - 2F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-x)(3-4x) + 4\left(F_M \frac{m_\mu}{2m_p}\right)^2 (1-x), \tag{B26}$$

$$W_3^{(2);(hh)} = 4\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 (1-x)(1-2x), \tag{B27}$$

$$W_4^{(0);(ll)} = W_4^{(0);(lh)} = W_4^{(0);(hh)} = W_4^{(1);(ll)} = W_4^{(1);(lh)} = W_4^{(1);(hh)} = W_4^{(2);(ll)} = W_4^{(2);(lh)} = W_4^{(2);(hh)} = 0, \tag{B28}$$

$$W_5^{(0);(ll)}=2F_VF_A-2F_A{}^2+2F_VF_P\frac{m_\mu{}^2}{m_\pi{}^2}(1-x)-2F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}x-2F_VF_M\frac{m_\mu}{2m_p}x-2F_AF_M\frac{m_\mu}{2m_p}(2-3x)-2F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}x(1-2x)-2\left(F_M\frac{m_\mu}{2m_p}\right)^2(1-x)(1-2x), \tag{B29}$$

$$W_5^{(0);(lh)}=F_VF_P\frac{m_\mu{}^2}{m_\pi{}^2}(1+2x)+F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}3(1-2x)+\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-4x^2)+(2F_V-4F_A)F_M\frac{m_\mu}{2m_p}+2F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-6x+4x^2)+\left(F_M\frac{m_\mu}{2m_p}\right)^2(-4+6x), \tag{B30}$$

$$W_5^{(0);(hh)}=2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2x(3-4x)+2F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-2x)-2\left(F_M\frac{m_\mu}{2m_p}\right)^2, \tag{B31}$$

$$W_5^{(1);(ll)}=W_3^{(2);(ll)}, \tag{B32}$$

$$W_5^{(1);(lh)}=W_3^{(2);(lh)}, \tag{B33}$$

$$W_5^{(1);(hh)}=W_3^{(2);(hh)}, \tag{B34}$$

$$W_5^{(2);(ll)}=W_5^{(2);(lh)}=W_5^{(2);(hh)}=0, \tag{B35}$$

$$W_6^{(0);(ll)}=W_3^{(0);(ll)}, \tag{B36}$$

$$W_6^{(0);(lh)}=W_3^{(0);(lh)}, \tag{B37}$$

$$W_6^{(0);(hh)}=W_3^{(0);(hh)}, \tag{B38}$$

$$W_6^{(1);(ll)}=W_3^{(1);(ll)}-W_5^{(0);(ll)}, \tag{B39}$$

$$W_6^{(1);(lh)}=W_3^{(1);(lh)}-W_5^{(0);(lh)}, \tag{B40}$$

$$W_6^{(1);(hh)}=W_3^{(1);(hh)}-W_5^{(0);(hh)}, \tag{B41}$$

$$W_6^{(2);(ll)}=W_6^{(2);(lh)}=W_6^{(2);(hh)}=0, \tag{B42}$$

$$W_7^{(0);(ll)}=W_5^{(0);(ll)}, \tag{B43}$$

$$W_7^{(0);(lh)}=F_VF_P\frac{m_\mu{}^2}{m_\pi{}^2}(1+2x)+F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}3(1-2x)+\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-4x^2)+(2F_V-4F_A)F_M\frac{m_\mu}{2m_p}+2F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-6x+6x^2)+\left(F_M\frac{m_\mu}{2m_p}\right)^2(-4+6x), \tag{B44}$$

$$W_7^{(0);(hh)}=2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2x+4F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-2x)-2\left(F_M\frac{m_\mu}{2m_p}\right)^2, \tag{B45}$$

$$W_7^{(1);(ll)}=W_5^{(1);(ll)}, \tag{B46}$$

$$W_7^{(1);(lh)}=2(F_V-F_A)F_P\frac{m_\mu{}^2}{m_\pi{}^2}(1-x)+2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-x)(1-2x)-6F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-x)(1-2x)+2\left(F_M\frac{m_\mu}{2m_p}\right)^2(1-x), \tag{B47}$$

$$W_7^{(1);(hh)}=-4F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-x), \tag{B48}$$

$$W_7^{(2);(ll)}=W_7^{(2);(lh)}=W_7^{(2);(hh)}=0, \tag{B49}$$

$$W_8^{(0);(ll)}=W_1^{(0);(ll)}+W_6^{(0);(ll)}, \tag{B50}$$

$$W_8^{(0);(lh)}=2F_VF_P\frac{m_\mu{}^2}{m_\pi{}^2}x+14F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}x+2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2x(3-2x)+(2F_V+12F_A)F_M\frac{m_\mu}{2m_p}+2F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(2+3x-10x^2)+2\left(F_M\frac{m_\mu}{2m_p}\right)^2(3-7x), \tag{B51}$$

$$W_8^{(0);(hh)}=8\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2x^2+16F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}x+4\left(F_M\frac{m_\mu}{2m_p}\right)^2, \tag{B52}$$

$$W_8^{(1);(ll)}=W_1^{(1);(ll)}+W_6^{(1);(ll)}, \tag{B53}$$

$$W_8^{(1);(lh)}=F_VF_P\frac{m_\mu{}^2}{m_\pi{}^2}(1-2x)+F_AF_P\frac{m_\mu{}^2}{m_\pi{}^2}(3-10x)+2\left(F_P\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2(1-2x+4x^2)+4F_P\frac{m_\mu{}^2}{m_\pi{}^2}F_M\frac{m_\mu}{2m_p}(1-3x+5x^2)+2\left(F_M\frac{m_\mu}{2m_p}\right)^2(-2+5x), \tag{B54}$$

$$W_8^{(1);(hh)} = 2\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 x(1-6x) + 4F_P \frac{m_\mu^2}{m_\pi^2} F_M \frac{m_\mu}{2m_p}(1-3x) \quad \text{(B55)}$$

$$W_8^{(2);(ll)} = W_1^{(2);(ll)} + W_6^{(2);(ll)}, \quad \text{(B56)}$$

$$W_8^{(2);(lh)} = 0, \quad \text{(B57)}$$

$$W_8^{(2);(hh)} = -4\left(F_P \frac{m_\mu^2}{m_\pi^2}\right)^2 x(1-x), \quad \text{(B58)}$$

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