Collisions between composite particles at medium and high energies

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Collisions between nuclei are studied by means of a simple extension of the Glauber high energy approximation. An expression for the optical phase shift function, exact within the framework of the Glauber approximation, is expanded in an infinite series and includes the effects of nuclear correlations. The first term corresponds to the standard optical limit result of the Glauber theory, and the higher order corrections arise from the processes in which one or more nucleons of either nucleus can undergo multiple collisions. The center-of-mass correlation is treated consistently so that our results do not exhibit the large-q divergence which characterizes the usual optical limit. It is shown that with realistic constraints on nucleon-nucleon total cross sections, the optical phase shift function *does not* approach the usual optical limit result when the mass numbers of the colliding nuclei become very large. With a proper treatment of center-of-mass correlations, the optical phase shift series converges rapidly for light nuclei and allows one to perform realistic calculations. The effects of higher order corrections on total and inelastic cross sections and on elastic scattering intensities are examined. The effects of the Coulomb field are included in an average phase approximation and results are compared with measurements.

[NUCLEUS REACTIONS Nucleus-nucleus scattering. Corrections to the optical limit in high energy approximation. Center-of-mass correlation effects. Calculated $\sigma(E, \theta)$, total cross sections, inelastic cross sections, and slope parameters.]

I. INTRODUCTION

Theories of scattering are perhaps best tested by applying them to a wide variety of collisions. Indeed, it would be satisfying if one could describe both nucleon-nucleus and nucleus-nucleus collisions by means of the same underlying theoretical framework. At high energies the multiple scattering theory due to Glauber[1,2] has provided a basis for calculations which have been quite successful in describing the collisions of hadrons from various nuclear targets ranging from deuterium to uranium.[3-6] One of the early applications of the theory was to hadron-deuteron collisions and it was predicted that double collisions would become the dominant mechanism for scattering away from the forward direction.[2] For heavier nuclei many additional multiple scattering processes can occur, and the contributions to the scattering amplitudes from successively higher order multiple scattering terms fall off more slowly as the momentum transfer increases. Consequently, higher order collisions were predicted to compete with and eventually become more important than the lower order collisions as the momentum transfer increased. This has now been well established in the extensive applications of the theory to nucleon-nucleus elastic scattering data near 1 GeV and to the elastic plus quasielastic data at 19.3 GeV.[3-5] For pion-nucleus scattering the theory has been successful, surprisingly, down to rather low energies.[6]

If one has confidence in the theory, one can use it to probe nuclear structure. Attempts to determine the radii of neutron distributions in this way have recently been made[7] for various nuclei. Alternatively, using simple nuclear targets, one can use the theory to extract information regarding hadron-nucleon amplitudes. In fact, proton-deuteron measurements together with proton-proton measurements have been widely used to extract proton-neutron amplitudes.[8-10]

There exists an extensive literature on the subject of corrections to the Glauber approximation. Evaluation of these corrections is a delicate matter because of the cancellations[11] between off-shell and higher order multiple scattering effects. These cancellations also occur in the more general case of heavy ion collisions.[12] Recently, corrections to the Glauber theory, which retain the previously noted cancellations, have been obtained.[13] In fact, even more subtle cancellations may be involved in the Glauber approximation. There is some evidence[13] that the dominant corrections arising from noneikonal, Fermi motion, and kinematic effects exhibit substantial cancellations among themselves. Therefore, theoretical analyses which remove only one of the above approximations may be less accurate than those based upon the conventional Glauber theory. At incident energies of a few GeV/nucleon (GeV/n), the combined effects of these corrections become small, as expected. At intermediate energies, they can be added on to the full Glauber scattering amplitude.

In the case of nucleus-nucleus collisions, there have been a number of theoretical analyses at high energies based upon the Glauber approximation or some variants of it.[9,14-28] There have also been a few attempts[29] to extend the optical potential formalism of Kerman, McManus, and Thaler[30] to nucleus-nucleus collisions. However, these calculations have been restricted to first order. Within the framework of the Glauber approximation, a solution to deuteron-deuteron scattering was presented in Ref. 14, and subsequently applied with success[31] to the high energy data. The case of deuteron-nucleus scattering has also been studied recently.[32] The full Glauber multiple scattering series for general nucleus-nucleus collisions was considered in detail by Czyż and Maximon.[16] However, the number of multiple interactions in this case grows so rapidly that the evaluation of the full multiple scattering series becomes intractable. (Full calculations have been carried out for α-α collisions using Gaussian wave functions.[16,20]) For heavier nuclei, an optical limit approximation[16,18] is generally used. This optical limit result has been employed to study total cross sections and fragmentation cross sections,[25] and for testing concepts such as factorization.[21-23] It has also been applied to medium energy heavy-ion elastic scattering data.[24]

It has been realized[12,26,28] that a serious attempt to analyze heavy-ion scattering, in the optical limit, is plagued with difficulties. The cross sections diverge at large momentum transfers when the center-of-mass correlation function is retained (as it should be[16]). Furthermore, recent total cross section measurements at the Berkeley Bevalac[33] indicate serious disagreement with optical limit predictions for ^{12}C-^{12}C collisions. In view of the recent rapid increase of interest in the field, it is appropriate to examine heavy-ion scattering and the validity of the optical limit.

In the present analysis we study the problem of nucleus-nucleus collisions in a theoretical framework where the full multiple scattering character of the Glauber series is retained. Rather than work with the full series which becomes intractable, we attempt to sum it by introducing an optical phase-shift function (which is related to a unique optical potential). A series for the phase shift function is obtained. The procedure is similar in many respects to that of Glauber[1] for the case of nucleon-nucleus scattering. However, the character of the phase shift series is quite different in the case of nucleus-nucleus scattering, due to the occurrence of additional new types of multiple interactions. In fact the series, which is usually considered to reduce to the optical limit in the case $A_1, A_2 \to \infty$, appears to diverge in that case when realistic limits are taken for nucleon-nucleon amplitudes. It is, nevertheless, quite useful for light and medium weight nuclei and for studying processes which do not probe small impact parameter collisions. In Sec. II we derive the results for the optical phase shift function $\chi_{\rm opt}$, where the first-order term corresponds to the usual optical limit. Correction terms up to the fourth order are obtained. In Sec. III, center-of-mass correlations are treated consistently by introducing them in the series for $\chi_{\rm opt}$, so that the resulting cross sections do not suffer from the divergence at large momentum transfers. In Sec. IV the phase-shift series is evaluated for the case of Gaussian wave functions and the higher order terms are estimated in limiting cases. The influence of higher order corrections on total cross sections, inelastic cross sections, and nucleus-nucleus slope parameters, is discussed in Sec. V and comparison is made with the recent 0.87 and 2.1 GeV/n experiments. The effects of the Coulomb interaction are treated in Sec. VI. In Sec. VII we study the elastic scattering intensities and the results are applied to the 1.37 GeV α-^{12}C data from Saclay. In Sec. VIII a short range approximation for the nucleon-nucleon interaction is considered, and finally we summarize our results in Sec. IX.

II. NUCLEUS-NUCLEUS SCATTERING AMPLITUDE

The scattering amplitude operator for collisions between nuclei with mass numbers A_1 and A_2 can be written as[1,14]

$$F(\vec{q};\{\vec{r}_i\},\{\vec{r}_j'\}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left(1 - e^{i\chi_{A_1A_2}(\vec{b};\{\vec{r}_i\},\{\vec{r}_j'\})}\right), \quad (2.1)$$

where k is the wave number of incident nucleus, $\hbar\vec{q}$ is the momentum transferred from the projectile to the target nucleus, $\{\vec{r}_i\}$ and $\{\vec{r}_j'\}$ represent the coordinates of the bound nucleons in the incident and target nuclei, respectively, and $\vec{b}$ is the impact parameter vector. The nucleus-nucleus phase-shift function $\chi_{A_1A_2}$ can, in turn, be expressed in terms of nucleon-nucleon phase-shift functions χ_{NN} by

$$\chi_{A_1A_2}(\vec{b};\{\vec{r}_i\},\{\vec{r}'_j\}) = \sum_{i=1}^{A_1}\sum_{j=1}^{A_2}\chi_{ij}(k_N;\vec{b}-\vec{s}_i+\vec{s}'_j), \quad (2.2)$$

where $\vec{s}_i$ and $\vec{s}'_j$ are the projections of the nucleon coordinates onto the impact parameter plane, and where k_N denotes the wave number of the nucleons in the projectile. With the definition

$$\Gamma_{NN}(k_N;\vec{b}) = 1 - e^{i\chi_{NN}(k_N;\vec{b})} \quad (2.3)$$

for the nucleon-nucleon (NN) profile functions, we obtain

$$e^{i\chi_{A_1A_2}(\vec{b};\{\vec{r}_i\},\{\vec{r}'_j\})} = \prod_{i=1}^{A_1}\prod_{j=1}^{A_2}[1-\Gamma_{ij}(k_N;\vec{b}-\vec{s}_i+\vec{s}'_j)]. \quad (2.4)$$

The profile functions Γ_{NN} are, in general, operators which do not commute with each other, and therefore Eq. (2.4) would be valid only if completely antisymmetrized wave functions were used to describe the (A_1+A_2) particle system. However, since we do not deal with the effects of spin and isospin in this paper, the profile functions do commute with each other and therefore Eq. (2.4) is valid for more general cases.

The nucleus-nucleus elastic scattering amplitude is given by the expectation value of the scattering amplitude operator in the ground states of the two nuclei, i.e.,

$$F_{\mathrm{el}}(\vec{q}) = \frac{ik}{2\pi}K(q)\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left\langle\Psi_{A_1A_2}\left|\left(1-\prod_{i=1}^{A_1}\prod_{j=1}^{A_2}[1-\Gamma_{ij}(k_N;\vec{b}-\vec{s}_i+\vec{s}'_j)]\right)\right|\Psi_{A_1A_2}\right\rangle, \quad (2.5)$$

where Ψ_{A_i} are wave functions which depend upon $3A_i$ coordinates and where a center-of-mass correlation function $K(q)$ has been introduced in order to remove the constraint due to the center of mass. This is possible if the wave functions Ψ_{A_i} factorize into center-of-mass and internal wave functions. This case will be discussed in detail in Sec. III. The expression (2.5) for the elastic scattering amplitude is quite difficult to evaluate for general forms of nuclear densities and NN scattering amplitudes. We shall, therefore, define an optical phase-shift function[12] by

$$F_{\mathrm{el}}(\vec{q}) = \frac{ik}{2\pi}K(q)\int d^2b\, e^{i\vec{q}\cdot\vec{b}}(1-e^{i\chi_{\mathrm{opt}}(\vec{b})}). \quad (2.6)$$

If we define

$$f(\lambda) = \left\langle\Psi_{A_1A_2}\left|\prod_{i=1}^{A_1}\prod_{j=1}^{A_2}[1-\lambda\Gamma_{ij}(\vec{b}-\vec{s}_i+\vec{s}'_j)]\right|\Psi_{A_1}\Psi_{A_2}\right\rangle, \quad (2.7)$$

then, from Eqs. (2.5) and (2.6),

$$i\chi_{\mathrm{opt}}(\vec{b}) = \ln f(\lambda)\big|_{\lambda=1} = i\sum_{j=1}^{\infty}\chi_j(\vec{b}). \quad (2.8)$$

Using the notation

$$f^{(n)} \equiv \frac{\partial^n}{\partial\lambda^n}f(\lambda)\bigg|_{\lambda=0}, \quad (2.9)$$

we obtain

$$\begin{aligned} i\chi_1(\vec{b}) &= f^{(1)},\\ i\chi_2(\vec{b}) &= \frac{1}{2!}(-f^{(1)2}+f^{(2)}),\\ i\chi_3(\vec{b}) &= \frac{1}{3!}(2f^{(1)3}-3f^{(1)}f^{(2)}+f^{(3)}),\\ i\chi_4(\vec{b}) &= \frac{1}{4!}(-6f^{(1)4}+2f^{(1)2}f^{(2)}-3f^{(2)2}\\ &\quad -4f^{(1)}f^{(3)}+f^{(4)}), \end{aligned} \quad (2.10)$$

and so on, where

$$f^{(1)} = -\sum_{i=1}^{A_1}\sum_{j=1}^{A_2}\langle\Psi_{A_1}\Psi_{A_2}|\Gamma_{ij}(\vec{b}-\vec{s}_i+\vec{s}'_j)|\Psi_{A_1}\Psi_{A_2}\rangle$$

and, in general,

$$f^{(n)} = (-1)^n\sum_{i_1\cdots i_n}^{A_1}{}'\sum_{j_1\cdots j_n}^{A_2}{}'\langle\Psi_{A_1}\Psi_{A_2}|\Gamma_{i_1j_1}(\vec{b}-\vec{s}_{i_1}+\vec{s}_{j_1})\cdots\Gamma_{i_nj_n}(\vec{b}-\vec{s}_{i_n}+\vec{s}'_{j_n})|\Psi_{A_1}\Psi_{A_2}\rangle. \quad (2.11)$$

The primes on the summation signs indicate the restriction that two pairs of indices cannot be equal at the same time (for example, if $i_1=i_2$ then $j_1\neq j_2$ and vice versa). In order to evaluate $f^{(n)}$, it is convenient to isolate the terms in which none of the indices are equal and the terms in which some of the indices may be equal. (These later terms will correspond to the cases in which nucleons of either nucleus may undergo multiple collisions.) The terms $f^{(2)}$ and $f^{(3)}$ can be written as

$$f^{(2)}=\sum_{i,k}^{A_1}{}'\sum_{j,l}^{A_2}{}'\langle\Gamma_{ij}\Gamma_{kl}\rangle$$

$$=\sum_{i\neq k}^{A_1}\sum_{j\neq l}^{A_2}\langle\Gamma_{ij}\Gamma_{kl}\rangle+\sum_{i}^{A_1}\sum_{j\neq l}^{A_2}\langle\Gamma_{ij}\Gamma_{il}\rangle+\sum_{i\neq k}^{A_1}\sum_{j}^{A_2}\langle\Gamma_{ij}\Gamma_{kj}\rangle, \tag{2.12}$$

$$f^{(3)}=-\sum_{i,k,m}^{A_1}{}'\sum_{j,l,n}^{A_2}{}'\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\rangle$$

$$=\sum_{\substack{i\neq k\neq m\\ i\neq m}}\sum_{\substack{j\neq l\neq n\\ j\neq n}}\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\rangle+3\sum_{i=k\neq m}\sum_{\substack{j\neq l\neq n\\ j\neq n}}\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\rangle+3\sum_{\substack{i\neq k\neq m\\ i\neq m}}\sum_{j=l\neq n}\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\rangle$$

$$+6\sum_{i=k\neq m}\sum_{\substack{j=n\\ j\neq l\neq n}}\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\rangle+\sum_{i=k=m}\sum_{\substack{j\neq l\neq n\\ j\neq n}}\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\rangle+\sum_{\substack{i\neq k\neq m\\ i\neq m}}\sum_{j=l=n}\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\rangle\,. \tag{2.13}$$

Similarly,

$$f^{(4)}=\sum_{i,k,m,p}^{A_1}{}'\sum_{j,l,n,q}^{A_2}{}'\langle\Gamma_{ij}\Gamma_{kl}\Gamma_{mn}\Gamma_{pq}\rangle. \tag{2.14}$$

$f^{(4)}$ and higher order terms can also be broken up, but their evaluation becomes increasingly lengthy and tedious as the number of possible types of multiple interactions grows rapidly. We shall restrict ourselves to terms up to fourth order. A fourth order calculation, as we shall see, is adequate for describing collisions between light nuclei for momentum transfers which are not too large.

In order to relate the optical phase-shift function to the experimentally measured NN amplitudes, we have from Eq. (2.1) for the case of NN collisions

$$f_{NN}(k_N;\vec{q})=\frac{ik_N}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\Gamma_{NN}(k_N;\vec{b}), \tag{2.15}$$

which upon Fourier inversion leads to

$$\Gamma_{NN}(k_N;\vec{b})=(2\pi ik_N)^{-1}\int d^2q\, e^{-i\vec{q}\cdot\vec{b}}f_{NN}(k_N;\vec{q}). \tag{2.16}$$

We now assume, for simplicity, that all NN amplitudes are equal, which is approximately true at high energies. (This, however, is a matter of convenience and not of necessity. The generalization of our results to the case $f_{pp}\neq f_{np}$ is tedious but straightforward.) With Eq. (2.16) we obtain

$$f^{(1)}(b)=-A_1A_2C_1(b),$$

$$f^{(2)}(b)=A_1A_2[(A_1-1)(A_2-1)D_1(b)+(A_2-1)D_2(b)+(A_1-1)D_3(b)],$$

$$f^{(3)}(b)=-A_1A_2[(A_1-1)(A_1-2)(A_2-1)(A_2-2)E_1(b)+3(A_1-1)(A_2-1)[(A_2-2)E_2(b)+(A_1-2)E_3(b)]$$
$$+6(A_1-1)(A_2-1)E_4(b)+(A_2-1)(A_2-2)E_5(b)+(A_1-1)(A_1-2)E_6(b)],$$

$$f^{(4)}(b)=A_1A_2\{(A_1-1)(A_1-2)(A_1-3)(A_2-1)(A_2-2)(A_2-3)G_1(b)$$
$$+6(A_1-1)(A_1-2)(A_2-1)(A_2-2)[(A_2-3)G_2(b)+(A_1-3)G_3(b)]$$
$$+4(A_1-1)(A_2-1)[(A_2-2)(A_2-3)G_4(b)+(A_1-2)(A_1-3)G_5(b)]$$
$$+(A_1-1)(A_1-2)(A_2-1)(A_2-2)[24G_6(b)+6G_7(b)]$$
$$+3(A_1-1)(A_2-1)[(A_2-2)(A_2-3)G_8(b)+(A_1-2)(A_1-3)G_9(b)$$
$$+4(A_2-2)G_{10}(b)+4(A_1-2)G_{11}(b)+2G_{12}(b)]$$
$$+(A_2-1)(A_2-2)(A_2-3)G_{13}(b)+(A_1-1)(A_1-2)(A_1-3)G_{14}(b)$$
$$+12(A_1-1)(A_2-1)[(A_2-2)G_{15}(b)+(A_1-2)G_{16}(b)]\}, \tag{2.17}$$

where the functions C_1, D_i, E_i, and G_i are obtained as follows. Let us define the generalized (four-body) nuclear form factor S_{A_i} by

$$S_{A_i}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)=\int d^3r_1\cdots d^3r_{A_i}\exp\left(i\sum_{j=1}^{4}\vec{q}_j\cdot\vec{r}_j\right)|\Psi_{A_i}(\vec{r}_1,\ldots,\vec{r}_{A_i})|^2 \tag{2.18}$$

$$=\int d^3r_1\cdots d^3r_4\exp\left(i\sum_{j=1}^{4}\vec{q}_j\cdot\vec{r}_j\right)\rho_{A_i}^{(4)}(\vec{r}_1,\ldots,\vec{r}_4), \tag{2.19}$$

where $\rho_{A_i}^{(4)}$ is the four-particle density. (If one of the q_i is zero, for example, then S_{A_i} would be the form factor of the three-particle density and so on.) Here we have defined the n-particle density as

$$\rho_{A_i}^{(n)}=\int d^3r_{n+1}\cdots d^3r_{A_i}|\Psi_{A_i}(\vec{r}_1,\ldots,\vec{r}_{A_i})|^2. \tag{2.20}$$

Let us also define, for convenience, the integral *operators*

$$I_n=(2\pi ik_N)^{-n}\int d^2q_1e^{-i\vec{q}_1\cdot\vec{b}}f(\vec{q}_1)\cdots\int d^2q_ne^{-i\vec{q}_n\cdot\vec{b}}f(\vec{q}_n). \tag{2.21}$$

The quantities C_1, D_i, E_i, and G_i can be written as

$$
\begin{aligned}
C_1(b)&=(2\pi ik_N)^{-1}\int d^2q_1e^{-i\vec{q}_1\cdot\vec{b}}f(\vec{q}_1)S_{A_1}(\vec{q}_1,0,0,0)S_{A_2}(-\vec{q}_1,0,0,0)\\
&=I_1[S_{A_1}(\vec{q}_1,0,0,0)S_{A_2}(-\vec{q}_1,0,0,0)],\\
D_1(b)&=I_2[S_{A_1}(\vec{q}_1,\vec{q}_2,0,0)S_{A_2}(-\vec{q}_1,-\vec{q}_2,0,0)],\\
D_2(b)&=I_2[S_{A_1}(0,\vec{q}_1+\vec{q}_2,0,0)S_{A_2}(-\vec{q}_1,-\vec{q}_2,0,0)],\\
D_3(b)&=D_2(S_{A_1}\leftrightarrow S_{A_2}),\\
E_1(b)&=I_3[S_{A_1}(\vec{q}_1,\vec{q}_2,\vec{q}_3,0)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,0)],\\
E_2(b)&=I_3[S_{A_1}(\vec{q}_1+\vec{q}_2,0,\vec{q}_3,0)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,0)],\\
E_3(b)&=E_2(S_{A_1}\leftrightarrow S_{A_2}),\\
E_4(b)&=I_3[S_{A_1}(0,\vec{q}_1+\vec{q}_2,\vec{q}_3,0)S_{A_2}(0,-\vec{q}_2,-\vec{q}_1-\vec{q}_3,0)],\\
E_5(b)&=I_3[S_{A_1}(0,0,\vec{q}_1+\vec{q}_2+\vec{q}_3,0)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,0)],\\
E_6(b)&=E_5(S_{A_1}\leftrightarrow S_{A_2}),\\
G_1(b)&=I_4[S_{A_1}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,-\vec{q}_4)],\\
G_2(b)&=I_4[S_{A_1}(\vec{q}_1+\vec{q}_2,0,\vec{q}_3,\vec{q}_4)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,-\vec{q}_4)],\\
G_3(b)&=G_2(S_{A_1}\leftrightarrow S_{A_2}),\\
G_4(b)&=I_4[S_{A_1}(\vec{q}_1+\vec{q}_2+\vec{q}_3,0,0,\vec{q}_4)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,-\vec{q}_4)],\\
G_5(b)&=G_4(S_{A_1}\leftrightarrow S_{A_2}),\\
G_6(b)&=I_4[S_{A_1}(\vec{q}_1+\vec{q}_2,0,\vec{q}_3,\vec{q}_4)S_{A_2}(-\vec{q}_1-\vec{q}_3,-\vec{q}_2,0,-\vec{q}_4)],\\
G_7(b)&=I_4[S_{A_1}(\vec{q}_1+\vec{q}_2,0,\vec{q}_3,\vec{q}_4)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3-\vec{q}_4,0)],\\
G_8(b)&=I_4[S_{A_1}(\vec{q}_1+\vec{q}_2,0,\vec{q}_3+\vec{q}_4,0)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,-\vec{q}_4)],\\
G_9(b)&=G_8(S_{A_1}\leftrightarrow S_{A_2})\\
G_{10}(b)&=I_4[S_{A_1}(0,0,\vec{q}_1+\vec{q}_2+\vec{q}_3,\vec{q}_4)S_{A_2}(0,-\vec{q}_2,-\vec{q}_3,-\vec{q}_1,-\vec{q}_4)],\\
G_{11}(b)&=G_{10}(S_{A_1}\leftrightarrow S_{A_2}),\\
G_{12}(b)&=I_4[S_{A_1}(0,\vec{q}_1+\vec{q}_2,0,\vec{q}_3+\vec{q}_4)S_{A_2}(0,0,-\vec{q}_1-\vec{q}_3,-\vec{q}_2-\vec{q}_4)],\\
G_{13}(b)&=I_4[S_{A_1}(0,0,0,\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,-\vec{q}_4)],\\
G_{14}(b)&=G_{13}(S_{A_1}\leftrightarrow S_{A_2})\\
G_{15}(b)&=I_4[S_{A_1}(0,\vec{q}_1+\vec{q}_2,0,\vec{q}_3+\vec{q}_4)S_{A_2}(0,-\vec{q}_2,-\vec{q}_1-\vec{q}_3,-\vec{q}_4)]\\
G_{16}(b)&=G_{15}(S_{A_1}\leftrightarrow S_{A_2}),
\end{aligned} \tag{2.22}
$$

where the notation ($S_{A_1} \leftrightarrow S_{A_2}$) implies the interchange of nuclear form factors.

Equation (2.8), together with Eqs. (2.10), (2,17), and (2.22), provides the expression for the nucleus-nucleus optical phase shift function which includes all the effects of nuclear correlations through fourth order. At this stage if one assumes an independent particle model for the nuclei, the number of terms in Eq. (2.22) is significantly reduced. This case will be discussed in Sec. IV and explicit analytic formulas for the special case of Gaussian densities will be obtained in Appendix A.

III. CENTER-OF-MASS CORRELATIONS

The internal nuclear wave functions $\Phi_{A_i}(\{\vec{r}_j^{\,\rm int}\})$ depend upon $3(A_i-1)$ coordinates. However, it is often more convenient to use the wave functions $\Psi_{A_i}(\{\vec{r}_j\})$ which contain the center-of-mass dependence. This can be done in the special case in which the center-of-mass and relative wave functions factorize,[16] i.e.,

$$\Psi_{A_i}(\{\vec{r}_j\}) = R_{A_i}(\vec{R}_i)\Phi_{A_i}(\{\vec{r}_j^{\,\rm int}\}), \tag{3.1}$$

where

$$\vec{r}_j^{\,\rm int} = \vec{r}_j - \vec{R}_i . \tag{3.2}$$

Since we have, due to translational invariance,

$$F(\vec{q};\{\vec{r}_i\},\{\vec{r}_j'\}) = F(\vec{q},\{\vec{r}_i^{\,\rm int}\},\{\vec{r}_j'^{\,\rm int}\})e^{i\vec{q}\cdot(\vec{R}_1-\vec{R}_2)}, \tag{3.3}$$

we can write

$$\langle\Psi_{A_1}\Psi_{A_2}|\,F(\vec{q};\{\vec{r}_i\},\{\vec{r}_j'\})|\Psi_{A_1}\Psi_{A_2}\rangle = \langle\Phi_{A_1}\Phi_{A_2}|F(\vec{q};\{\vec{r}_i^{\,\rm int}\},\{\vec{r}_j'^{\,\rm int}\})|\Phi_{A_1}\Phi_{A_2}\rangle \times\langle R_{A_1}R_{A_2}|e^{i\vec{q}\cdot(\vec{R}_1-\vec{R}_2)}|R_{A_1}R_{A_2}\rangle . \tag{3.4}$$

This relation, with the definition

$$K(q) = (\langle R_{A_1}R_{A_2}|e^{i\vec{q}\cdot(\vec{R}_1-\vec{R}_2)}|R_{A_1}R_{A_2}\rangle)^{-1}, \tag{3.5}$$

leads immediately to Eq. (2.5). This is the conventional way[16] of treating the center-of-mass correlation in the Glauber approximation. We have retained it in the previous section in order to make contact with the earlier calculations. This procedure is exact if Eq. (3.1) holds (which is the case with Gaussian and harmonic oscillator wave functions) and if the scattering amplitude is translationally invariant (which is true for the full Glauber amplitude). However, this procedure can lead to serious difficulties when approximations are made to the scattering amplitude. In fact, in the usual Glauber optical limit, the factor $K(q)$ leads to an unphysical *divergence* in the cross sections at large q.[16,26,28] The problem would disappear if all the higher order corrections to the optical limit were calculated. In actual practice, as we shall see, the convergence of series (2.8) is often slow and the divergence due to this inconsistent treatment of center-of-mass motion persists.

One obvious way to avoid this problem is to include the c.m. correlation to the same order as the optical phase-shift function. This can be done by noting that the elastic scattering amplitude may be written as

$$F_{\rm el}(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left\langle\Phi_{A_1}\Phi_{A_2}\right|\left(1-\prod_{i=1}^{A_1}\prod_{j=1}^{A_2}[1-\Gamma_{ij}(k_N;\vec{b}-\vec{s}_i^{\,\rm int}+\vec{s}_j'^{\,\rm int})]\right)\left|\Phi_{A_1}\Phi_{A_2}\right\rangle. \tag{3.6}$$

One can define a modified optical phase shift $\bar{\chi}_{\rm opt}$ by[28]

$$F_{\rm el}(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}(1-e^{i\bar{\chi}_{\rm opt}(b)}) \tag{3.7}$$

and, proceeding in a manner similar to the previous section, obtain

$$i\bar{\chi}_{\rm opt}(b) = i\sum_{j=1}^{\infty}\bar{\chi}_j(b), \tag{3.8}$$

where the $\bar{\chi}_j$ are again given by Eqs. (2.10) and (2.17) with the quantities C_1, D_i, E_i, and G_i replaced by the new quantities $\bar{C}_1$, $\bar{D}_i$, $\bar{E}_i$, and $\bar{G}_i$. From these equations we have, for example,[34]

$$\bar{C}_1(b) = \langle\Phi_{A_1}\Phi_{A_2}|\Gamma_{ij}(\vec{b}-\vec{s}_i^{\,\rm int}+\vec{s}_j'^{\,\rm int})|\Phi_{A_1}\Phi_{A_2}\rangle = (2\pi ik_N)^{-1}\int d^2q_1 e^{-i\vec{q}_1\cdot\vec{b}}f(\vec{q}_1)\langle\Phi_{A_1}\Phi_{A_2}|\,e^{-i\vec{q}_1\cdot(\vec{s}_i^{\,\rm int}-\vec{s}_j'^{\,\rm int})}|\Phi_{A_1}\Phi_{A_2}\rangle . \tag{3.9}$$

Now using Eqs. (3.1), (3.2), and (3.5), this can be replaced by

$$\overline{C}_1(b)=(2\pi i k_N)^{-1}\int d^2q_1\, e^{-i\vec{q}_1\cdot\vec{b}}\, f(\vec{q}_1)\langle\Psi_{A_1}\Psi_{A_2}|e^{-i\vec{q}_1\cdot(\vec{s}_i-\vec{s}_j)}|\Psi_{A_1}\Psi_{A_2}\rangle K(\vec{q}_1)$$
$$=I_1[S_{A_1}(\vec{q}_1,0,0,0)S_{A_2}(-\vec{q}_1,0,0,0)K(\vec{q}_1)], \tag{3.10}$$

which is the expression for C_1 in Eq. (2.22) except for the extra factor of $K(\vec{q}_1)$. Similarly, in the higher order terms, $\overline{D}_i$ has the same form as D_i except for an extra factor of $K(\vec{q}_1+\vec{q}_2)$ in the integrand. The expressions for E_i and G_i with extra factors of $K(\vec{q}_1+\vec{q}_2+\vec{q}_3)$ and $K(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)$, respectively, in the integrand yield the new $\overline{E}_i$ and $\overline{G}_i$. Thus, referring to Eq. (2.22) we obtain, for example,

$$\overline{D}_1(b)=I_2[S_{A_1}(\vec{q}_1,\vec{q}_2,0,0)\times S_{A_2}(-\vec{q}_1,-\vec{q}_2,0,0)K(\vec{q}_1+\vec{q}_2)],$$
$$\overline{E}_1(b)=I_3[S_{A_1}(\vec{q}_1,\vec{q}_2,\vec{q}_3,0)\,S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,0)\times K(\vec{q}_1+\vec{q}_2+\vec{q}_3)],$$
$$\overline{G}_1(b)=I_4[S_{A_1}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)\times S_{A_2}(-\vec{q}_1,-\vec{q}_2,-\vec{q}_3,-\vec{q}_4)\times K(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)]\,. \tag{3.11}$$

The rest of the functions $\overline{D}_i$, $\overline{E}_i$, and $\overline{G}_i$ can be obtained from Eqs. (2.22) by similar modifications.

A second (and equivalent) way of obtaining the new phase-shift functions $\overline{\chi}_j$ is by simply equating Eqs. (2.6) and (3.7) which yields, immediately, the relation[28]

$$e^{i\overline{\chi}_{\rm opt}(\vec{b})}=(2\pi)^{-2}\int d^2q\,d^2b'\,e^{-i\vec{q}\cdot(\vec{b}-\vec{b}')}\,e^{i\chi_{\rm opt}(\vec{b})}K(q)$$
$$=\int_0^\infty J_0(qb)K(q)q\,dq\int_0^\infty J_0(qb')\times e^{i\chi_{\rm opt}(b')}b'db'\,. \tag{3.12}$$

This is equivalent to the relation

$$\overline{Q}_i(b)=\int_0^\infty J_0(qb)K(q)q\,dq\times\int_0^\infty J_0(qb')Q_i(b')b'db'\,, \tag{3.13}$$

where Q_i stands for the quantities C_1, D_i, E_i, G_i, and $\overline{Q}_i$ stands for the similar barred quantities. This second procedure [Eq. (3.13)] is particularly useful for obtaining $\overline{\chi}_j(b)$ if one already has analytic expressions for $\chi_j(b)$.

Once the χ_j's have been calculated the $\overline{\chi}_j$'s may also be obtained directly from Eq. (3.12). We obtain

$$i\overline{\chi}_1(b)=(2\pi)^{-2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})}i\chi_1(b')K(q)d^2b'd^2q\,, \tag{3.14}$$

$$i\overline{\chi}_2(b)=(2\pi)^{-2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})}[i\chi_2(b')-\tfrac{1}{2}\chi_1^{\,2}(b')]K(q)d^2b'd^2q+\tfrac{1}{2}\overline{\chi}_1^{\,2}(b)\,, \tag{3.15}$$

$$i\overline{\chi}_3(b)=(2\pi)^{-2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})}[i\chi_3(b')-\chi_1(b')\chi_2(b')-\tfrac{1}{6}i\chi_1^{\,3}(b')]K(q)d^2b'd^2q+[\overline{\chi}_1(b)\overline{\chi}_2(b)+\tfrac{1}{6}i\overline{\chi}_1^{\,3}(b)]\,, \tag{3.16}$$

$$i\overline{\chi}_4(b)=(2\pi)^{-2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})}[i\chi_4(b')-\tfrac{1}{2}\chi_2^{\,2}(b')-\chi_1(b')\chi_3(b')-\tfrac{1}{2}i\chi_1^{\,2}(b')\chi_2(b')-\tfrac{1}{24}\chi_1^{\,4}(b')]K(q)d^2b'd^2q$$
$$+[\tfrac{1}{2}\overline{\chi}_2^{\,2}(b)+\overline{\chi}_1(b)\overline{\chi}_3(b)+\tfrac{1}{2}i\overline{\chi}_1^{\,2}(b)\overline{\chi}_2(b)+\tfrac{1}{24}\overline{\chi}_1^{\,4}(b)]\,. \tag{3.17}$$

For more complicated nuclear densities and *NN* amplitudes, where integrations have to be performed numerically, it may be more convenient to use the first procedure [i.e., Eqs. (3.10), (3.11), etc.].

IV. GAUSSIAN WAVE FUNCTIONS AND LARGE *A* LIMITS

As we have pointed out, the optical limit result of the Glauber approximation corresponds to the first term of the series (2.8) for $\chi_{\rm opt}$. In order to illustrate the effects of the higher order corrections, let us assume that the nuclei can be described by an independent particle model, i.e.,

$$|\Psi_{A_i}(\{\vec{r}_i\})|^2=\prod_{j=1}^{A_i}|\phi(\vec{r}_j)|^2\,, \tag{4.1}$$

which leads to the simplification

$$S_{A_i}(\vec{q}_1,\ldots,\vec{q}_4)=S_{A_i}(\vec{q}_1)\cdots S_{A_i}(\vec{q}_4)\,. \tag{4.2}$$

(This is equivalent to neglecting all correlations except for those due to the center-of-mass constraints.) We shall further assume that

$$S_{A_i}(q) = e^{-q^2 R_i^2/4}. \tag{4.3}$$

Although such a form factor is realistic only for light nuclei, it leads to analytic results which are quite useful for studying the convergence properties of the optical phase-shift series. Furthermore, if the parameters R_i are chosen to correspond to the experimental rms radii, Gaussian form factors can yield reliable results for scattering at small momentum transfers (and hence for total cross sections, for example).

For NN scattering amplitudes, we shall take the usual high energy parametrization

$$f(k_N;q) = \frac{k_N \sigma (i+\rho)}{4\pi} e^{-aq^2/2}. \tag{4.4}$$

The quantities C_1, D_i, E_i, G_i, $\overline{C}_1$, $\overline{D}_i$, $\overline{E}_i$, and $\overline{G}_i$ can now be evaluated analytically and explicit formulas are given in Appendix A.

A. Particle-nucleus scattering

Before studying nucleus-nucleus collisions, let us first consider the case of a single particle incident on a nucleus, i.e., $A_1 = 1$, $A_2 = A$, and $R_1 = 0$. [Since we are interested in the limit $A \to \infty$, in which case $K(q) \to 1$, we shall ignore the c.m. correlations in the rest of this section.] We obtain from Eqs. (2.10) and (2.11)

$$\begin{aligned} i\chi_1(b) &= -A \int d^3r_1 \Gamma_1(\vec{b} - \vec{s}_1)\rho^{(1)}(\vec{r}_1), \\ i\chi_2(b) &= -\frac{A}{2} \int d^3r_1 d^3r_2 \Gamma_1(\vec{b}-\vec{s}_1)\Gamma_2(\vec{b}-\vec{s}_2) \\ &\quad \times [A\rho^{(1)}(\vec{r}_1)\rho^{(1)}(\vec{r}_2) - (A-1)\rho^{(2)}(\vec{r}_1,\vec{r}_2)] \\ &= -\frac{A}{2}\int d^3r_1 d^3r_2 \Gamma_1(\vec{b}-\vec{s}_1)\Gamma_2(\vec{b}-\vec{s}_2) \\ &\quad \times [\rho^{(2)}(\vec{r}_1,\vec{r}_2) - AC_A(\vec{r}_1,\vec{r}_2)], \end{aligned} \tag{4.5}$$

where we have defined the pair correlation function

$$C_A(\vec{r}_1,\vec{r}_2) = \rho^{(2)}(\vec{r}_1,\vec{r}_2) - \rho^{(1)}(\vec{r}_1)\rho^{(1)}(\vec{r}_2). \tag{4.6}$$

Now for an uncorrelated wave function of form (4.1), $C_A = 0$ and the coefficient of the surviving term in $i\chi_2$ is A (which is also the coefficient in $i\chi_1$). More explicitly, for Gaussian wave functions and NN amplitudes described earlier, we have

$$\begin{aligned} i\chi_1(b) &= -Aye^{-b^2/R^2}; \\ y &= \frac{\sigma(1-i\rho)}{2\pi R^2}, \quad R^2 = R_2^2 + 2a, \\ i\chi_2(b) &= (i\chi_1)\tfrac{1}{2}ye^{-b^2/R^2}, \end{aligned} \tag{4.7}$$

and, in general, from Eqs. (2.10) and (2.11) we have

$$i\chi_n(b) = (i\chi_1)\frac{y^{n-1}}{n} e^{-(n-1)b^2/R^2}. \tag{4.8}$$

The expansion parameter for the phase-shift is proportional to y. Since $R \propto A^{1/3}$, we have $y \propto A^{-2/3}$ and the optical limit result $i\chi_{\rm opt} = i\chi_1$ becomes a good approximation for large A in the case of particle-nucleus scattering.[12]

B. Nucleus-nucleus scattering

Let us now turn to the case where both colliding objects are nuclei. We have now

$$i\chi_1(b) = -A_1A_2\int d^3r_1 d^3r_1' \Gamma_{11}\rho_{A_1}^{(1)}(\vec{r}_1)\rho_{A_2}^{(1)}(\vec{r}_1'), \tag{4.9}$$

where $\Gamma_{ij} \equiv \Gamma_{ij}(\vec{b} - \vec{s}_i + \vec{s}_j')$ and

$$\begin{aligned} i\chi_2(b) = \tfrac{1}{2}A_1A_2\Big[&\int d^3r_1 d^3r_1' d^3r_2 d^3r_2' \Gamma_{11}\Gamma_{22}[(1-A_1-A_2)\rho_{A_1}^{(2)}(\vec{r}_1,\vec{r}_2)\rho_{A_2}^{(2)}(\vec{r}_1',\vec{r}_2') + A_1A_2C_{A_1A_2}(\vec{r}_1,\vec{r}_2;\vec{r}_1',\vec{r}_2')] \\ &+ \int d^3r_1 d^3r_1' \Gamma_{11}\Big((A_2-1)\int d^3r_2' \Gamma_{12}\rho_{A_1}^{(1)}(\vec{r}_1)\rho_{A_2}^{(2)}(\vec{r}_1',\vec{r}_2') + (A_1-1)\int d^3r_2\Gamma_{21}\rho_{A_1}^{(2)}(\vec{r}_1,\vec{r}_2)\rho_{A_2}^{(1)}(\vec{r}_1')\Big)\Big], \end{aligned} \tag{4.10}$$

where

$$C_{A_1A_2}(\vec{r}_1,\vec{r}_2;\vec{r}_1',\vec{r}_2') = \rho_{A_1}^{(2)}(\vec{r}_1,\vec{r}_2)\rho_{A_2}^{(2)}(\vec{r}_1',\vec{r}_2') - \rho_{A_1}^{(1)}(\vec{r}_1)\rho_{A_1}^{(1)}(\vec{r}_2)\rho_{A_2}^{(1)}(\vec{r}_1')\rho_{A_2}^{(1)}(\vec{r}_2'). \tag{4.11}$$

Again, for an independent particle model, $C_{A_1A_2} = 0$. The leading terms which survive in $i\chi_2$ (let us assume $A_1 = A_2 = A$ for simplicity) are proportional to A^3, whereas $i\chi_1$ is proportional to A^2. Therefore, the term $i\chi_2$ is *not necessarily* small compared with $i\chi_1$. In order to estimate the higher order corrections let us again take the case of Gaussian wave functions and NN amplitudes. The detailed expressions for $i\chi_2, \ldots, i\chi_4$ are quite lengthy. But for the special case $A_1 = A_2 \gg 1$, R_1^2

$= R_2^2 \gg a$, they simplify to

$$i\chi_1(b) \simeq -A^2 y e^{-b^2/R^2}, \quad y = \frac{\sigma(1-i\rho)}{2\pi R^2},$$

$$R^2 \simeq 2R_1^2,$$

$$i\chi_2(b) \simeq i\chi_1(b)(Ay)(e^{-b^2/R^2} - 1.33e^{-b^2/3R^2}),$$

$$i\chi_3(b) \simeq i\chi_1(b)(Ay)^2(1.67e^{-2b^2/R^2} - 4e^{-4b^2/3R^2} + 2e^{-b^2/R^2} + 0.67e^{-b^2/2R^2}),$$

$$i\chi_4(b) \simeq i\chi_1(b)(Ay)^3(3.5e^{-3b^2/R^2} - 12e^{-7b^2/3R^2} + 8e^{-2b^2/R^2} + 4e^{-5b^2/3R^2} + 2.67e^{-3b^2/2R^2} - 6.4e^{-7b^2/5R^2} - 0.27e^{-3b^2/5R^2}). \quad (4.12)$$

The second term in $i\chi_2$ comes from two new kinds of multiple scattering in which one nucleon of nucleus 1 interacts with two nucleons of nucleus 2 and vice versa. The first term in $i\chi_2$ corresponds to two different nucleons of nucleus 1 interacting with two different nucleons of nucleus 2. This term is the counterpart of the $i\chi_2$ term in the particle-nucleus case. We note that even this term in $i\chi_2$ is proportional to $(Ay)i\chi_1$ and *does not* vanish in the limit $A_1, A_2 \to \infty$. In order to look more closely at the expansion parameter, let us evaluate the χ_j at $b=0$, $b=R$, and $b=2R$. We obtain

$$\frac{\chi_2(0)}{\chi_1(0)} \sim -0.33Ay, \quad \frac{\chi_2(R)}{\chi_1(R)} \sim -0.59Ay,$$

$$\frac{\chi_2(2R)}{\chi_1(2R)} \sim -0.333Ay, \quad \frac{\chi_3(0)}{\chi_2(0)} \sim -Ay,$$

$$\frac{\chi_3(R)}{\chi_2(R)} \sim -0.53Ay, \quad \frac{\chi_3(2R)}{\chi_2(2R)} \sim -0.329Ay,$$

$$\frac{\chi_4(0)}{\chi_3(0)} \sim -1.5Ay, \quad \frac{\chi_4(R)}{\chi_3(R)} \sim -0.9Ay,$$

$$\frac{\chi_4(2R)}{\chi_3(2R)} \sim -0.321Ay. \quad (4.13)$$

We notice that, unlike the case of particle-nucleus scattering, here the series is oscillatory. This, again, is due to the occurrence of new types of multiple scattering in the nucleus-nucleus case. Furthermore, the expansion parameter now is Ay, which is proportional to $A^{1/3}$ and which may become greater than unity for large A. In order to make qualitative estimates, let us take $R_i = (\frac{2}{3})^{1/2}\langle r_i^2\rangle^{1/2}$ where $\langle r^2\rangle^{1/2}$ is the nuclear rms radius. This would insure that the resulting nuclear form factor would be accurate at small momentum transfers. At an incident energy of 2.1 GeV/n, $\sigma = 42.7$ mb and therefore, for the case of Pb-Pb collisions, $Ay \sim 3.3$. In fact, we find that in the limit $A_1, A_2 \to \infty$ (with σ = constant), the series (2.8) for the phase-shift function *appears to diverge*.

We should point out that these estimates are obtained from Eq. (4.12) which is valid in the limiting case $A_1, A_2 \to \infty$. We have considered this case because it has been argued in the past that $i\chi_{opt} \to i\chi_1$ as A_1, A_2 become large. That this is not true is evident from these estimates.

For finite nuclei, one can evaluate $\chi_1, \ldots, \chi_4$ by using the formulas of Appendix A. In Figs. 1–3 we present the real parts of $i\bar{\chi}_{opt}(b)$ and $i\chi_{opt}(b)$. In the figures we have used the notation

$$\chi(N) = \sum_{i=1}^{N} \chi_i$$

and a similar notation for the modified phase shifts $\bar{\chi}(N)$. The results for $i\chi_{opt}$ in ^{16}O-^{16}O collisions are shown in Fig. 1(a). The higher order corrections to $i\chi_1$ are most important at small b and their effects tend to become smaller at large

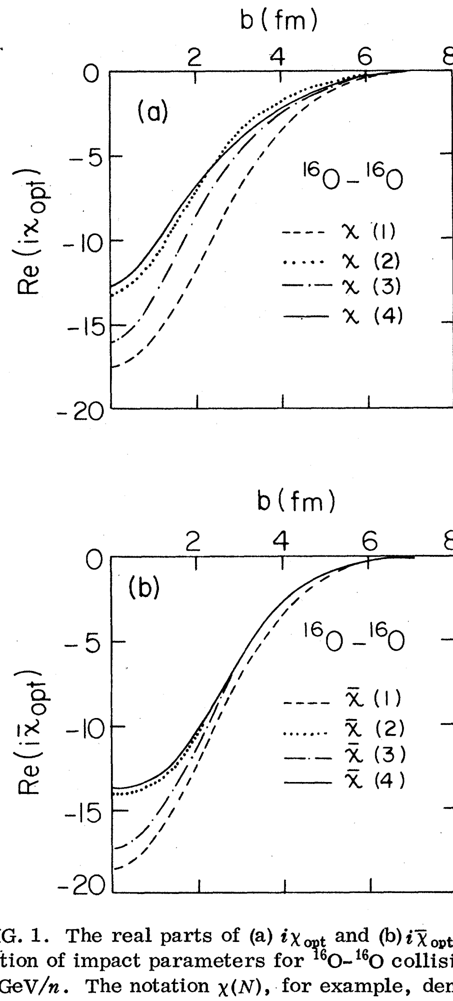

FIG. 1. The real parts of (a) $i\chi_{opt}$ and (b) $i\bar{\chi}_{opt}$ as a function of impact parameters for ^{16}O-^{16}O collisions at 2.1 GeV/n. The notation $\chi(N)$, for example, denotes $\chi_1 + \cdots + \chi_N$.

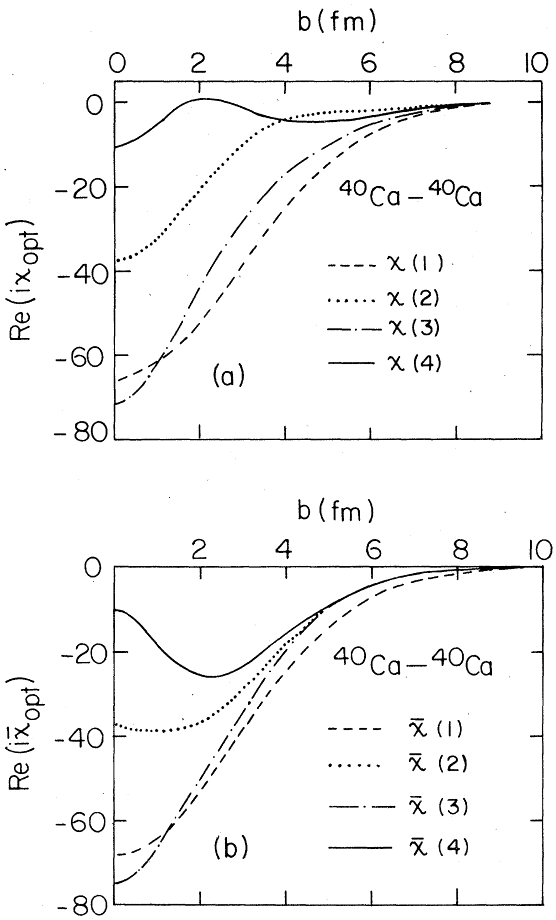

FIG. 2. The real parts of (a) $i\chi_{\rm opt}$ and (b) $i\bar{\chi}_{\rm opt}$ for ^{40}Ca-^{40}Ca collisions at 2.1 GeV/n.

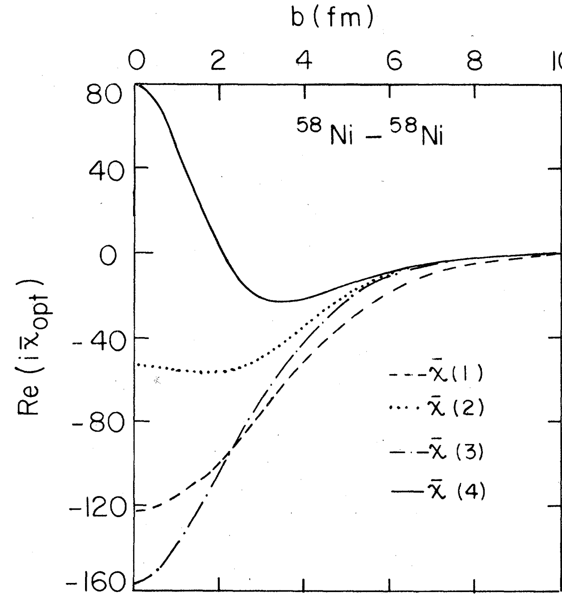

FIG. 3. The real parts of $i\bar{\chi}_{\rm opt}$ for ^{58}Ni-^{58}Ni collisions at 2.1 GeV/n.

b. In Fig. 1(b) we show the results for the modified phase-shift functions $\bar{\chi}_1, \ldots, \bar{\chi}_4$. We see that the proper treatment of the center-of-mass correlation significantly improves the convergence of the series for the optical phase-shift function. We also show the results for $\chi_{\rm opt}$ and $\bar{\chi}_{\rm opt}$ for the case of ^{40}Ca-^{40}Ca in Fig. 2 and for $\bar{\chi}_{\rm opt}$ for ^{58}Ni-^{58}Ni collisions in Fig. 3. We notice that even in the case of ^{40}Ca-^{40}Ca collisions the series for $\chi_{\rm opt}$ and $\bar{\chi}_{\rm opt}$ are diverging near $b=0$. For ^{58}Ni-^{58}Ni, the results shown in Fig. 3 illustrate the divergence of $\bar{\chi}_{\rm opt}$ for large nuclei, as expected from our previous qualitative estimates. As these figures illustrate, the series does converge at larger impact parameters and it may be possible, therefore, to use it even for large nuclei for calculating cross sections which depend mostly upon peripheral processes. However, one has to be cautious in such calculations because for large nuclei $i\bar{\chi}_{\rm opt}$ can (and does, as shown in Fig. 3) take on unphysical values at small impact parameter (e.g., $|e^{i\chi}|>1$) when the series for $\bar{\chi}_{\rm opt}=\sum_j \chi_j$ is truncated at even j.

In order to investigate what region of impact parameters the cross sections are most sensitive to, it is useful to plot the nucleus-nucleus profile function

$$\Gamma(b)=1-e^{i\bar{\chi}_{\rm opt}(b)}\,. \tag{4.14}$$

In Fig. 4 we show the results for $\mathrm{Re}\Gamma(b)$ and $\mathrm{Re}[b\Gamma(b)]$ for ^{40}Ca-^{40}Ca collisions. We notice that the effects of the higher order phase-shift corrections $\bar{\chi}_2, \ldots, \bar{\chi}_4$ on $\mathrm{Re}\Gamma(b)$ are noticeable only at large impact parameters. At small b, $\mathrm{Re}\Gamma(b)\simeq 1$. For heavy nuclei, if one assumes that there is strong absorption in collisions at small impact parameter, $\mathrm{Re}\Gamma(b)$ can be replaced by unity inside some critical impact parameter b_c. For b_c one can take the maximum value of the impact parameter for which the Γ's corresponding to $\bar{\chi}(1)$, $\bar{\chi}(2)$, and $\bar{\chi}(3)$ are all approximately equal to unity. We find, a range of values for b_c below the particular value which yields roughly the same final results for cross sections. This procedure, though reasonable for total cross section calculations (which are proportional to integrals over $\mathrm{Re}[b\Gamma(b)]$ and pick up most of the contributions from large b, as shown in Fig. 4), could lead to large inaccuracies in results for large angle elastic scattering angular distributions (which may probe smaller impact parameter collisions). In that case the effects on cross sections due to small variations in b_c should be carefully investigated.

We should point out that if the colliding objects

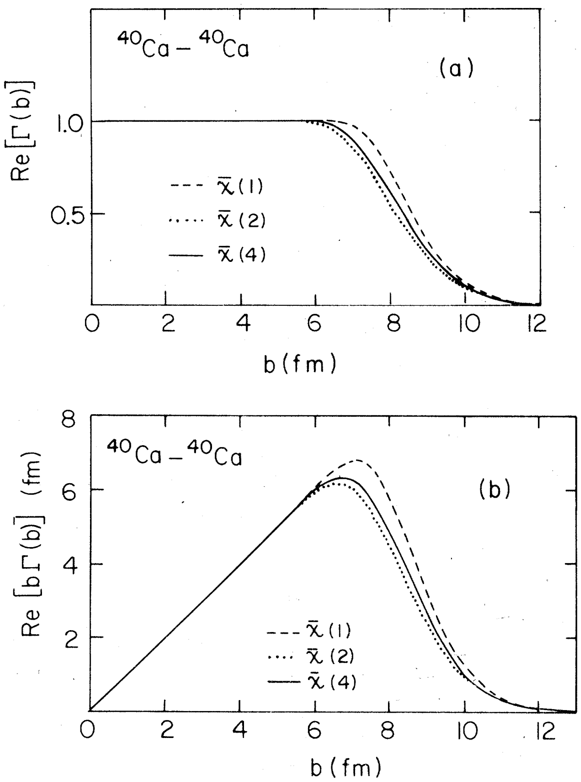

FIG. 4. The real parts of (a) $\Gamma(b)$ and (b) $b\Gamma(b)$ as a function of impact parameter for ^{40}Ca-^{40}Ca collisions. The results for $\bar{\chi}(3)$ (not shown) lie between those for $\bar{\chi}(2)$ and $\bar{\chi}(4)$.

are hadrons, which are considered as being composed of an infinite number of constituents x, then the appropriate limit is $\sigma_{xx} \to 0$ as $A_1, A_2 \to \infty$ with $\sigma_{xx} A_1 A_2 \to$ constant. Under these conditions, the parameter $Ay \sim A^{-1}$ and χ_{opt} approaches the optical limit χ_1 for $A_1, A_2 \to \infty$. If one further assumes that the constituent interactions have zero range,[12] then the optical phase-shift function becomes equivalent to the Chou-Yang model,[35] which has been quite successful in describing hadron-hadron cross sections at high energies.

V. TOTAL CROSS SECTIONS, INELASTIC CROSS SECTIONS, AND SLOPE PARAMETERS

In this section we investigate the effects of higher order phase-shift corrections and the consistent treatment of c.m. correlations on the calculations of total cross sections, total inelastic cross sections, and the slope parameters for nucleus-nucleus forward elastic scattering amplitudes. These quantities have recently been measured at the Berkeley Bevalac at incident energies of 0.87 and 2.1 GeV/n for collisions between light nuclei.[33]

Total cross sections may be obtained from the elastic scattering amplitudes by means of the optical theorem

$$\sigma_{tot} = (4\pi/k)\,\mathrm{Im}F_{el}(0) = 4\pi\,\mathrm{Re}\int_0^\infty (1 - e^{i\bar{\chi}_{opt}(b)})b\,db\,, \tag{5.1}$$

and the inelastic cross section can be obtained from the relation

$$\sigma_{inel} = \sigma_{tot} - \sigma_{el} = 2\pi\int_0^\infty (1 - |e^{i\bar{\chi}_{opt}(b)}|^2)b\,db\,. \tag{5.2}$$

Very near the forward direction the nucleus-nucleus elastic scattering amplitudes may be parametrized by the form

$$F_{el}(q) = F(0)e^{-(B_R + iB_I)q^2/2}\,, \tag{5.3}$$

and the differential cross section by

$$\frac{d\sigma}{d\Omega} = |F(0)|^2 e^{-B_R q^2}\,. \tag{5.4}$$

The slope parameter B_R is then given by

$$B_R = \lim_{q\to 0}\left[\mathrm{Re}\left(\frac{-2(\partial F_{el}/\partial q^2)}{F_{el}}\right)\right] \tag{5.5}$$

$$= \mathrm{Re}\left(\frac{\int_0^\infty (1 - e^{i\bar{\chi}_{opt}(b)})b^3 db}{2\int_0^\infty (1 - e^{i\bar{\chi}_{opt}(b)})b\,db}\right)\,. \tag{5.6}$$

Although Eq. (5.1) is valid with either $\bar{\chi}_{opt}$ or the unmodified phase shift function χ_{opt}, we should point out that Eqs. (5.2) and (5.6) are strictly valid only for the modified phase-shift function $\bar{\chi}_{opt}$. These formulas can also be used for $\chi_{opt}(b)$ in the approximation where the center-of-mass correlation function $K(q)$ is neglected. This approximation (which has been used in the past) underestimates σ_{el} and hence overestimates σ_{inel}.

In our calculations the parameters for NN scattering amplitudes are taken to be[36] $\sigma = 42.4$ mb, $a = 5$ (GeV/c)$^{-2}$, $\rho = -0.2$ at 0.87 GeV/n and $\sigma = 42.7$ mb, $a = 6.2$ (GeV/c)$^{-2}$, $\rho = -0.28$ at 2.1 GeV/n. The rms radii which we use are taken from electron scattering measurements[37] and are listed in Table I. Upon correcting for the finite proton size and the c.m. recoil, the parameter R_i is given by

$$R_i^2 = \tfrac{2}{3}(\langle r_{A_i}^2\rangle - \langle r_p^2\rangle)/(1 - 1/A_i)\,. \tag{5.7}$$

In Table II we show the results for total cross sections for A_1 and $A_2 \le 40$, at 0.87 and 2.1 GeV/n together with the recent experimental measurements. The errors are statistical only. For α-^{12}C collisions, the two measurements correspond to α-^{12}C and to ^{12}C-α collisions. Since they should be equal, their difference gives a good indication of the probable systematic errors.[33] For comparison, the results for both $\chi_{opt}(b)$ and $\bar{\chi}_{opt}(b)$ are

TABLE I. Nuclear rms radii.

A	1	2	4	12	16	24	40	58	116	208
$\langle r^2\rangle^{1/2}$ (fm)	0.81	2.17	1.71	2.453	2.71	2.98	3.50	3.80	4.55	5.49

shown. It is interesting to note that even for total cross sections, which depend on F_{el} at $q=0$ where the c.m. correlation function $K(q)$ is unity, the proper inclusion of the c.m. constraint in each $\bar{\chi}_i$ improves the convergence of the cross sections quite significantly. For example, for the ^{4}He-^{4}He collisions at 2.1 GeV/n the exact Glauber multiple scattering result yields 386 mb. [This is obtained by computing F_{el} explicitly from Eq. (2.5), which can be done for very light nuclei, $A_1, A_2 \lesssim 5$.] The usual first order optical limit result yields 429 mb. The first optical limit result obtained with a consistent treatment of the center of mass correlation is 386 mb.[28] (This result is the same as the fourth-order result and is equal to the exact Glauber result.) For ^{40}Ca-^{40}Ca collisions, the usual first order optical limit result is 4941 mb, and the corresponding fourth order result is 4512 mb. The first order optical limit result obtained with a consistent treatment of the center of mass correlation is 4845 mb,[28] which is considerably closer to the corresponding fourth-order result of 4535 mb. Thus simply treating the center-of-mass correlations consistently significantly improves the first order optical limit result. Similar improvements are seen to occur for the higher order results. In passing, we might point out that the fourth-order results [last column, $\bar{\chi}(4)$] are generally in better agreement with the measurements (sixth column) than are the usual first order optical limit results [second column, $\chi(1)$]. The large discrepancy[33] between ^{12}C-^{12}C total cross section measurements and earlier optical limit predictions is quite significantly reduced by the inclusion of c.m. correlations and higher order phase-shift correction. In addition, our first-order results [seventh column, $\bar{\chi}(1)$] are also generally in better agreement with the measurements than are the usual first order results (second column). Except for the very heavy nuclei, calculations for σ_{tot} can be performed to third order without any difficulty and the results are shown in Table III. In fourth order, however, $i\bar{\chi}_{opt}$ takes on unphysical values near $b \simeq 0$, as pointed out in the preceding section. However, as shown in Fig. 4, $\mathrm{Re}\Gamma(b)=1$ up to fairly large

TABLE II. Nucleus-nucleus total cross sections. The numbers listed in parentheses are for 0.87 GeV/n, whereas all the other values are for 2.1 GeV/n. The second through fifth columns show the theoretical predictions obtained from Eq. (2.6), and the seventh through the tenth columns show those obtained from Eq. (3.7). The sixth column lists the experimental measurements (Ref. 33). The two experimental values in the second row correspond to ^{4}He-^{12}C and ^{12}C-^{4}He collisions, respectively.

Nuclei	σ_{tot} (mb) with χ_{opt} equal to				σ_{tot} (mb)	σ_{tot} (mb) with $\bar{\chi}_{opt}$ equal to			
A_1-A_2	$\chi(1)$	$\chi(2)$	$\chi(3)$	$\chi(4)$	experiment	$\bar{\chi}(1)$	$\bar{\chi}(2)$	$\bar{\chi}(3)$	$\bar{\chi}(4)$
4-4	429	384	387	386	408±2.5	386	388	387	386
	(420)	(373)	(377)	(375)	(390±4.2)	(377)	(377)	(376)	(376)
4-12	902	788	810	802	835±5, 826±5.9	834	809	806	805
	(885)	(767)	(792)	(781)	(820±13, 790±7)	(817)	(790)	(788)	(785)
4-16	1097	961	989	979		1023	987	985	983
4-24	1387	1217	1260	1244		1307	1254	1252	1249
4-40	1939	1720	1778	1757		1851	1768	1767	1764
12-12	1605	1365	1453	1420	1347±25	1518	1431	1433	1431
	(1580)	(1329)	(1430)	(1384)	(1256±31)	(1493)	(1403)	(1406)	(1402)
12-16	1880	1599	1712	1669		1789	1679	1685	1682
12-24	2272	1931	2086	2026		2180	2034	2047	2043
12-40	3010	2584	2786	2709		2914	2709	2736	2730
16-16	2180	1855	1996	1942		2087	1951	1962	1958
16-24	2607	2212	2406	2332		2512	2335	2356	2352
16-40	3402	2916	3165	3071		3305	3061	3101	3094
24-24	3077	2607	2861	2765		2983	2754	2793	2787
24-40	3949	3385	3698	3577		3854	3542	3613	3601
40-40	4941	4307	4662	4512		4845	4440	4557	4535

TABLE III. Nucleus-nucleus total cross sections. The notation is the same as in Table II.

Nuclei A_1-A_2	σ_{tot} (mb) $\chi_{opt}=\chi_1$	σ_{tot} (mb) with $\bar{\chi}_{opt}$ equal to $\bar{\chi}(1)$	$\bar{\chi}(2)$	$\bar{\chi}(3)$	$\bar{\chi}(4)$
4-58	2 385	2 292	2 187	2 187	2 182
4-116	3 612	3 510	3 346	3 350	3 344
4-208	5 383	5 275	5 026	5 038	5 028
12-58	3 584	3 488	3 236	3 276	3 268
12-116	5 145	5 046	4 674	4 750	4 734
12-208	7 381	7 278	6 730	6 860	6 829
24-58	4 616	4 522	4 144	4 246	4 227
24-116	6 407	6 314	5 769	5 950	5 909
24-208	8 949	8 856	8 057	8 360	8 280
40-58	5 692	5 600	5 113	5 279	5 243
40-116	7 682	7 591	6 915	7 193	7 121
40-208	10 470	10 379	9 423	9 863	9 731
58-58	6 503	6 414	5 830	6 065	6 007
58-116	8 633	8 548	7 743	8 135	8 015
58-208	11 598	11 514	10 381	11 002	10 780
116-116	11 077	10 997	9 959	10 559	10 322
116-208	14 409	14 332	12 987	13 841	13 445

b even when the phase-shift series is diverging near $b \approx 0$. In fact, near $b \simeq 0$ deviations in $\mathrm{Re}\Gamma(b)$ from unity occur only when the optical phase-shift function begins to approach unphysical values ($|e^{i\bar{\chi}_{opt}}| > 1$). In these cases we simply replace $\Gamma(b)$ by unity as discussed. The results are shown in the sixth column of Table III. We also show the total cross sections obtained from the usual first-order optical limit χ_1 (column 2). We note, by comparing columns 2 and 7 of Table II and by comparing columns 2 and 3 of Table III, that the center-of-mass effects decrease with increading A_1, A_2, as expected. For α-α scattering, the difference in the first order cross sections is 11%, whereas for ^{116}Sn-^{208}Pb scattering it is only 0.5% On the other hand, the effect of the higher order phase-shift functions becomes important for medium and large A_1, A_2. For α-α collisions, the fourth order calculation for σ is 386 mb and the first-order calculation is 386 mb, so that there is negligibly small effect. For ^{116}Sn-^{208}Pb collisions, the fourth-order calculation for σ is 13.445 b and the first-order calculations is 14.332 b, so that there is an effect of ~6%.

Since it is possible to calculate deuteron-nucleus (d-A) total cross sections *exactly* in Glauber theory,[32] it would be interesting to see how rapidly the results for the d-A total cross sections converge when using the series Eq. (3.8) for the optical phase shift-functions $\bar{\chi}_{opt}(b)$. In Table IV we show deuteron-nucleus total cross sections for target nuclei denoted by their mass numbers A. In column 2 we show the results obtained by using $\chi(1)$, the standard first-order optical limit result, for the phase-shift function. In columns 3–6 we show the results obtained by using the new and higher order results of this paper. In column 7 we present the exact Glauber theory results and in column 8 we give the measured values.

We see that the new first-order results of $\bar{\chi}(1)$ are significantly closer to the exact results than those of the standard first-order result of $\chi(1)$ in all cases except for d-d collisions ($A_1 = A_2 = 2$). Furthermore, the fourth-order results of $\bar{\chi}(4)$ are all within 0.1% of the exact Glauber values. Since, as we see from Eqs. (4.7) and (4.8), the series for $\bar{\chi}_{opt}$ in particle-nucleus scattering is rapidly converging for large A, it is expected that the series would also converge rapidly in nucleus-nucleus collisions when one of the nuclei is very light, as is the case for d-A collisions. This is clearly confirmed by Table IV.

TABLE IV. Deuteron-nucleus total cross sections at 2.1 GeV/n. Column seven gives the exact Glauber theory results. The remaining notation is as in Table II.

Target nucleus A	σ_{tot} (mb) $\chi_{opt}=\chi(1)$	σ_{tot} (mb) with $\bar{\chi}_{opt}$ equal to $\bar{\chi}(1)$	$\bar{\chi}(2)$	$\bar{\chi}(3)$	$\bar{\chi}(4)$	σ_{tot} (mb) exact	σ_{tot} (mb) exp.
2	161.7	154.1	159.4	159.7	159.8	159.8	158 ± 0.8
4	293.7	267.2	264.7	263.6	263.3	263.3	{271 ± 1.5 262 ± 1.8
12	729.0	642.5	622.2	621.3	620.4	620.5	{644 ± 3.5 617 ± 3.0
16	910.4	801.4	776.9	776.6	775.5	775.5	
24	1208	1057	1021	1022	1020	1021	
40	1724	1523	1476	1481	1478	1478	
58	2162	1918	1858	1866	1862	1862	
116	3300	2983	2904	2917	2912	2911	
208	4857	4493	4407	4417	4412	4410	

TABLE V. Nucleus-nucleus inelastic cross sections. The notation is the same as in Table II.

Nuclei A_1-A_2	$\sigma_{\rm inel}$ (mb) with $\bar{\chi}_{\rm opt}$ equal to $\bar{\chi}(1)$	$\bar{\chi}(2)$	$\bar{\chi}(3)$	$\bar{\chi}(4)$	$\sigma_{\rm inel}$ (mb) experiment
4-4	265	263	263	263	276 ± 3.7
	(260)	(258)	(257)	(257)	(262 ± 13)
4-12	532	518	517	517	547 ± 3, 523 ± 4.6
	(524)	(509)	(508)	(508)	(542 ± 16, 516 ± 5.3)
4-16	644	625	625	625	
12-12	916	876	879	879	888 ± 19
	(906)	(863)	(866)	(866)	(939 ± 17)
12-16	1070	1021	1025	1025	
16-16	1237	1178	1184	1184	

In Table V we present theoretical predictions for the total inelastic cross sections for the lighter nuclei together with recent measurements.[33] The errors are statistical only. Again, for α-^{12}C collisions, the two measurements at each energy correspond to α-^{12}C and to ^{12}C-α collisions, and should be equal. Their difference (24 mb at 2.1 GeV/n and 26 mb at 0.87 GeV/n) gives an indication of the probable systematic errors. Our results show that the higher order corrections vary from ~1% for α-α collisions to ~4.5% for ^{16}O-^{16}O collisions. The fourth-order predictions are in reasonably good agreement with the data.

In Table VI we show the theoretical predictions for the slope parameters B_R in nucleus-nucleus forward elastic scattering, together with the values extracted[33] from the Berkeley measurements. Some theoretical analysis was used to extract the slope parameters from the data, and the errors assigned to the measured values include only the uncertainty in the fits used.[33] Again, for α-^{12}C collisions, the two measured values at each energy correspond to α-^{12}C and ^{12}C-α collisions, and should be equal. Their difference [12 and 3 (GeV/c)$^{-2}$] gives an indication of the probable systematic errors. Our results show that the higher order corrections vary from ~1% for α-α collisions to ~4% for ^{16}O-^{16}O collisions.

VI. EFFECTS OF THE COULOMB FIELD

It has been pointed out[9] that the Coulomb interaction has a significant effect on the elastic scattering angular distributions. While the Coulomb contributions are crucial at the minima, they can alter the cross sections substantially even at the subsidiary maxima. Here we shall obtain a simple formula for the inclusion of the Coulomb interaction which employs an average phase approximation and includes the effects due to center-of-mass correlations.

When the Coulomb interaction is included, the full scattering amplitude can be written as[9]

$$F_{\rm el}(q)=f_c^{\rm pt}(q)+\frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left[e^{i\chi_c^{\rm pt}(b)}-e^{i\bar{\chi}_c(b)+i\bar{\chi}_{\rm opt}(b)}\right], \tag{6.1}$$

where the point charge Coulomb phase shift is given by $\chi_c^{\rm pt}(b)=2n\ln(kb)$, where $n=Z_1Z_2e^2/\hbar v$ is

TABLE VI. Nucleus-nucleus slope parameters. The notation is the same as in Table II.

Nuclei A_1-A_2	Slope (GeV/c)$^{-2}$ with $\bar{\chi}_{\rm opt}$ equal to $\bar{\chi}(1)$	$\bar{\chi}(2)$	$\bar{\chi}(3)$	$\bar{\chi}(4)$	Slope (GeV/c)$^{-2}$ experiment
4-4	59.9	58.9	58.9	58.9	70 ± 4
	(58.4)	(57.5)	(57.4)	(57.4)	(63 ± 10)
4-12	109.6	107.1	107.0	107.1	129 ± 4, 117 ± 2.4
	(107.7)	(105.0)	(105.0)	(105.1)	(120 ± 13, 117 ± 3)
4-16	131.2	128.0	128.0	128.0	
12-12	180.4	173.5	174.1	174.1	204 ± 11
	(177.8)	(170.6)	(171.3)	(171.3)	(254 ± 18)
12-16	209.6	201.0	201.9	201.9	
16-16	241.2	230.7	232.0	231.9	

the usual Coulomb parameter and the point charge Coulomb amplitude is

$$f_c^{pt}(q) = -\frac{2nk}{q^2} e^{-2i[n \ln(q/2k) - \arg\Gamma(1+in)]} . \qquad (6.2)$$

The extended charge Coulomb phase shift function $\overline{\chi}_c(b)$ can be obtained in an average phase approximation which consists of averaging the point charge phase-shift function over the ground state of the colliding nuclei.[9,12] This yields

$$\overline{\chi}_c(b) = \left\langle \Phi_{A_1}\Phi_{A_2} \left| \sum_{i=1}^{Z_1} \sum_{j=1}^{Z_2} \frac{2e^2}{\hbar v} \ln(k|\vec{b} - \vec{s}_i^{\,\text{int}} + \vec{s}\,'^{\,\text{int}}_j|) \right| \Phi_{A_1}\Phi_{A_2} \right\rangle . \qquad (6.3)$$

If the colliding nuclei are described by Gaussian wave functions

$$|\Phi_{A_i}|^2 = (\pi R_i^2/A_i)^{3/2} \prod_{j=1}^{A_i} (\pi R_i^2)^{-3/2} e^{-r_j^2/R_i^2} \times \delta^3\Big(A_i^{-1} \sum_j \vec{r}_j\Big), \qquad (6.4)$$

where the δ function describes the center-of-mass constraint. Equation (6.3) can be evaluated in closed form with the result

$$\overline{\chi}_c(b) = \chi_c^{pt}(b) + nE_1(b^2/\overline{R}^2), \qquad (6.5)$$

where $E_1(z)$ is the exponential integral[38] and

$$\overline{R}^2 = R_1^2(1 - A_1^{-1}) + R_2^2(1 - A_2^{-1}). \qquad (6.6)$$

The full nucleus-nucleus elastic scattering amplitude may finally be written as

$$F_{el}(q) = f_c^{pt}(q) + ik \int_0^\infty J_0(qb)(kb)^{2in} \times (1 - e^{i[nE_1(b^2/\overline{R}^2) + \overline{\chi}_{opt}(b)]}) b\, db . \qquad (6.7)$$

We notice that for Gaussian wave functions, the effect of center-of-mass correlation on the Coulomb phase shift function $\overline{\chi}_c(b)$ is merely a shift in the nuclear radius $[R_i^2 \to \overline{R}_i^2 \equiv R_i^2(1 - 1/A_i)]$. This is the consequence of the fact that only single particle densities are required in the evaluation of Eq. (6.3). The effect of integrating over $3(A-1)$ coordinates in Eq. (6.4) with the δ function is simply to replace the parameter R_i^2 (in the single particle density when the c.m. correlation is neglected) by $\overline{R}_i^2$. Since the first-order modified phase-shift function $\overline{\chi}_1$ (for strong interactions) also requires only single particle densities, it too can be obtained from χ_1 by merely letting $R_i^2 \to \overline{R}_i^2$ (for Gaussian wave functions). (This is also true in case of harmonic oscillator wave functions[28].)

VII. ELASTIC SCATTERING ANGULAR DISTRIBUTIONS

The differential cross section for nucleus-nucleus elastic scattering may be calculated by means of Eq. (2.6) or, alternatively, by means of Eq. (3.7). In either case the optical phase-shift function, $\chi_{opt}(b)$ or $\overline{\chi}_{opt}(b)$, is given by means of the expansion (2.8) or (3.8). The calculation of $\chi_j(b)$ or $\overline{\chi}_j(b)$ in these series rapidly becomes exceedingly tedious as j increases. We have calculated the results for $j \le 4$.

As we have pointed out, unless the full (infinite) series (2.8) is retained for $\chi_{opt}(b)$, the elastic scattering amplitude given by Eq. (2.6) increases without bound for large increasing q. [We again note that Eq. (3.7) does not suffer from this problem.] This unphysical result arises from the factor $K(q)$ in Eq. (2.6) which increases rapidly with increasing q and the fact that the integral in that equation does not decrease rapidly enough if the series (2.8) is approximated by a truncation. Despite this divergence for large q, at any *fixed* q Eq. (2.6) can provide reliable results provided a sufficient number of terms $\chi_j(b)$ are retained in the truncated series for $\chi_{opt}(b)$. What constitutes a sufficient number of terms depends upon the

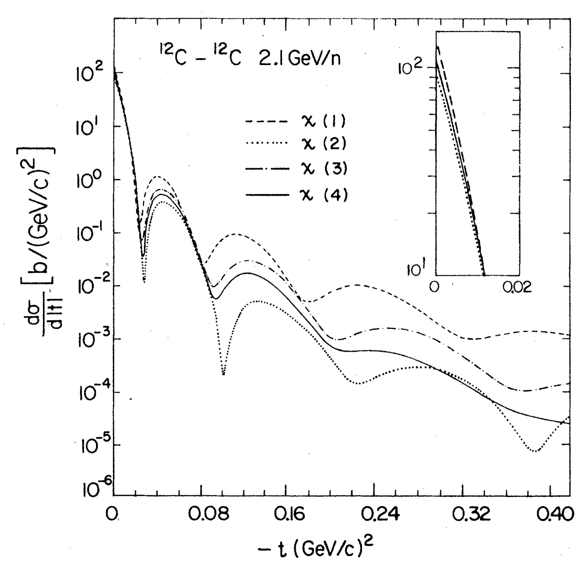

FIG. 5. ^{12}C-^{12}C elastic scattering at 2.1 GeV/n, including corrections up to fourth order for the unmodified optical phase-shift function.

nuclei involved, the momentum transfer of interest, and the accuracy desired.

In Fig. 5 we present the differential cross section $d\sigma/d|t|$ as a function of the squared four-momentum transfer t, for ^{12}C-^{12}C elastic scattering at 2.1 GeV/n, using Eq. (2.6) and the series $\chi_{opt} = \Sigma\chi_j$ for the unmodified phase-shift function as given by Eqs. (2.8), (2.10), (2.17), and (2.22). Since we are interested in the effects of the higher order phase-shift corrections on the elastic scattering intensities and are comparing theoretical expressions, we ignore Coulomb effects at this point. The four angular distributions shown correspond to truncations of the series (2.8) after one, two, three, or four terms. It is clear that beyond the forward peak the dashed curve, corresponding to the usual optical limit approximation $\chi_{opt}(b) \approx \chi_1(b)$, is quite inaccurate. Furthermore, beyond $-t \approx 0.2$ (GeV/c)^2 even the fourth-order result can only be described as perhaps being at best a qualitatively accurate result.

In Fig. 6 we again present $d\sigma/d|t|$ for ^{12}C-^{12}C elastic scattering at 2.1 GeV/n, but this time calculated from Eq. (3.7) and the series $\bar{\chi}_{opt}(b) = \sum \bar{\chi}_j(b)$ for the modified phase-shift function. It is apparent that the angular distributions are converging much more rapidly than in Fig. 5, and that the fourth-order result is quantitatively accurate even beyond the fourth maximum.

In Fig. 7, we show $d\sigma/d|t|$ for ^{12}C-^{12}C at 2.1 GeV/n for the cases $\chi_{opt} \approx \chi(1)$, $\chi_{opt} \approx \chi(4)$, $\bar{\chi}_{opt} \approx \bar{\chi}(1)$, and $\bar{\chi}_{opt} \approx \bar{\chi}(4)$. For the first two cases, Eq. (2.6) is used to calculate $d\sigma/d|t|$, and for the second two cases Eq. (3.7) is used. We also show the result obtained by using $\chi_{opt} \approx \chi_1$ in Eq. (2.6)

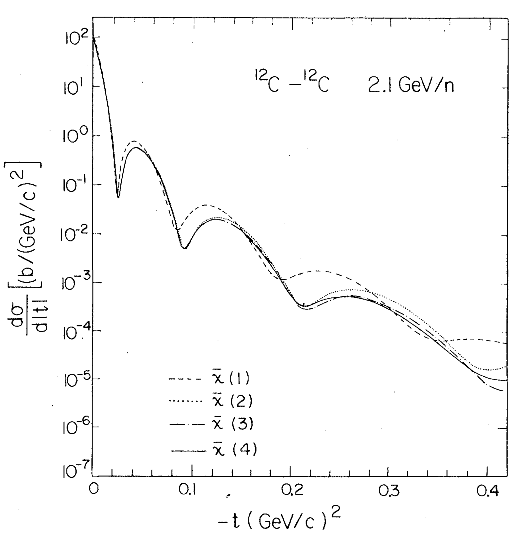

FIG. 6. Same as Fig. 5 for the modified optical phase-shift function.

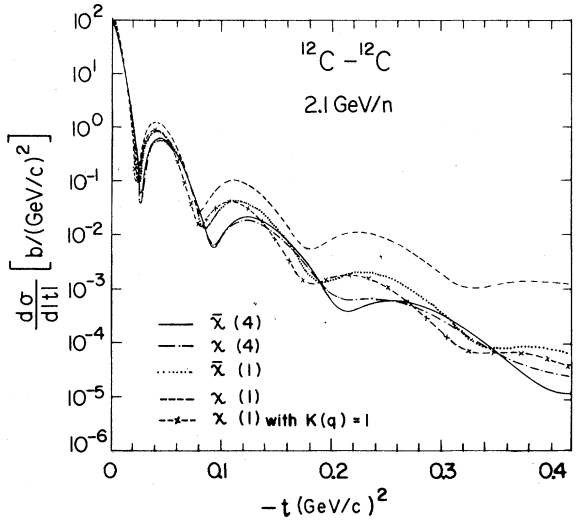

FIG. 7. ^{12}C-^{12}C elastic scattering at 2.1 GeV/n. Cross sections obtained from the unmodified and modified phase-shift functions are compared.

and arbitrarily setting $K(q)$ equal to unity. (This has on occasion been done in the past.) We see that the usual optical limit, $\chi_{opt} \approx \chi(1)$, does not lead to a reliable angular distribution [with or without the factor $K(q)$ in Eq. (2.6)]. The results obtained from the unmodified fourth-order optical phase-shift function are in qualitative agreement with the results obtained from the modified fourth-order optical phase-shift function $\bar{\chi}(4)$.

In Fig. 8 we present $d\sigma/d|t|$ for ^4He-^4He elastic scattering using Eq. (2.6) and the series for the unmodified phase-shift function χ_{opt}. We see that the first-order result is quite inaccurate beyond the forward peak, and that the fourth-order result is unreliable beyond the second minimum.

In Fig. 9 we show $d\sigma/d|t|$ for ^4He-^4He elastic scattering using Eq. (3.7) and the series for the modified phase-shift function $\bar{\chi}_{opt}$. We see that the first-order result is very accurate up till the second maximum, and the fourth-order result is quantitatively accurate beyond the third maximum and qualitatively reliable up till the third minimum.

In Fig. 10 we show $d\sigma/d|t|$ for ^4He-^4He collisions for the cases $\chi_{opt} \approx \chi(1)$, $\chi_{opt} \approx \chi(4)$, $\bar{\chi}_{opt} \approx \bar{\chi}(1)$, $\bar{\chi}_{opt} \approx \bar{\chi}(4)$. We also show the result obtained by exactly summing the Glauber multiple scattering series implicit in Eq. (2.5), which can be done when the nuclei are very light (in practice, when A_1, $A_2 \lesssim 5$). We see that the usual optical limit $\chi_{opt} \approx \chi(1)$ leads to an unreliable angular distribution before reaching the first minimum [with or without the factor $K(q)$ in Eq. (2.6)]. The results obtained from the unmodified fourth-order optical

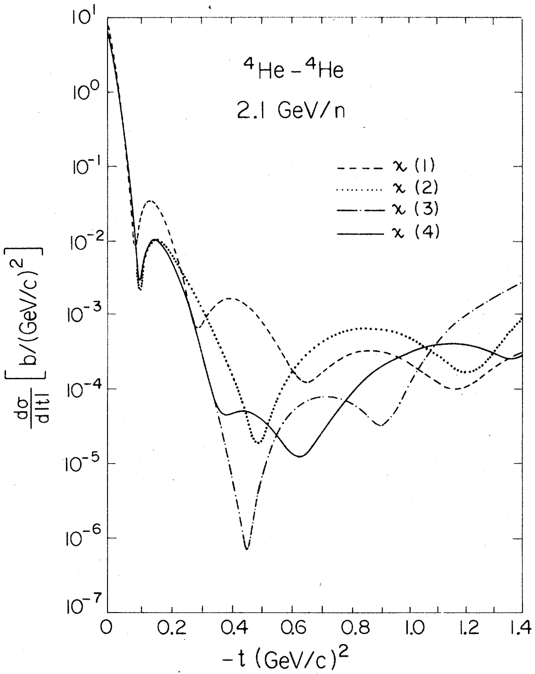

FIG. 8. Same as Fig. 5 for ^{4}He-^{4}He collisions.

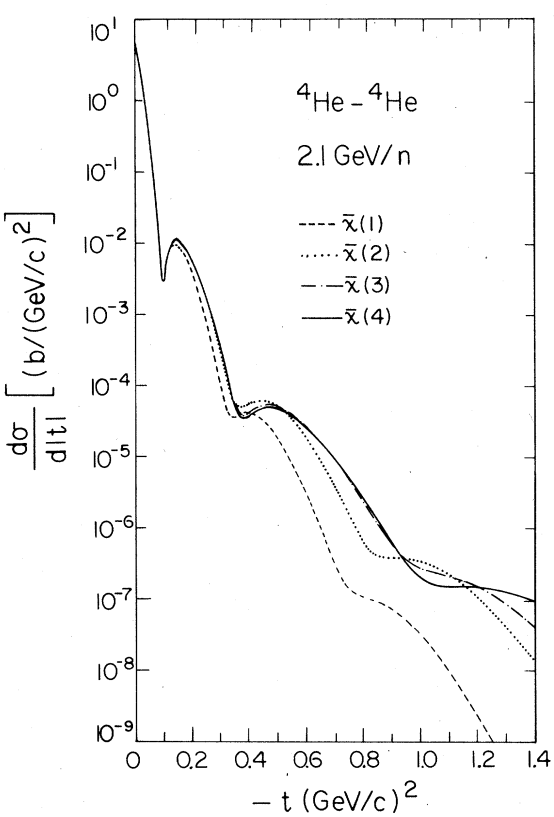

FIG. 9. Same as Fig. 6 for ^{4}He-^{4}He collisions.

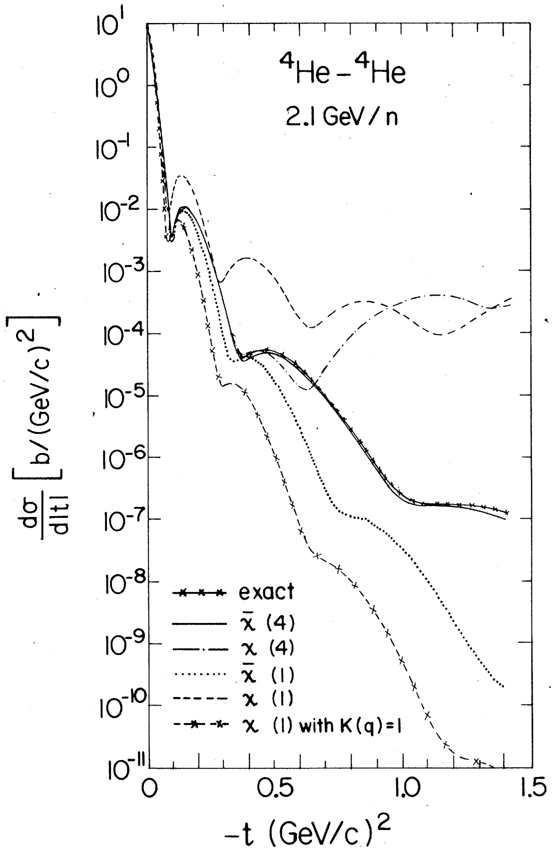

FIG. 10. Same as Fig. 7 for ^{4}He-^{4}He collisions. Also shown, for comparison, is the exact Glauber result.

phase-shift function becomes unreliable beyond $-t \approx 0.5$ (GeV/c)^2. The onset of the divergence in $d\sigma/d|t|$ for this case is apparent in the figure. On the other hand, the angular distributions obtained from the modified fourth-order optical phase-shift function compares exceedingly well with the exact Glauber calculation [Eq. (2.5)] for the *entire range* $0 < -t \lesssim 1.4$ (GeV/c)^2.

We see from Figs. 5–10 that in solving the center-of-mass problem in high-energy nucleus-nucleus collisions, we have at the same time obtained a much more rapidly converging series for the optical phase-shift function and hence for the elastic scattering angular distribution.

At present, high-energy elastic scattering data for collisions between nuclei heavier then deuterium do not exist. The data available at the highest energy is from Saclay[39] and involves the scattering of 1.37 GeV α particles from ^{12}C. This experiment corresponds to an incident energy of 343 MeV/n which is not high enough to test the validity of the high energy approximation. Nevertheless, in the absence of any other measurements at higher energies we shall apply our results to this case.

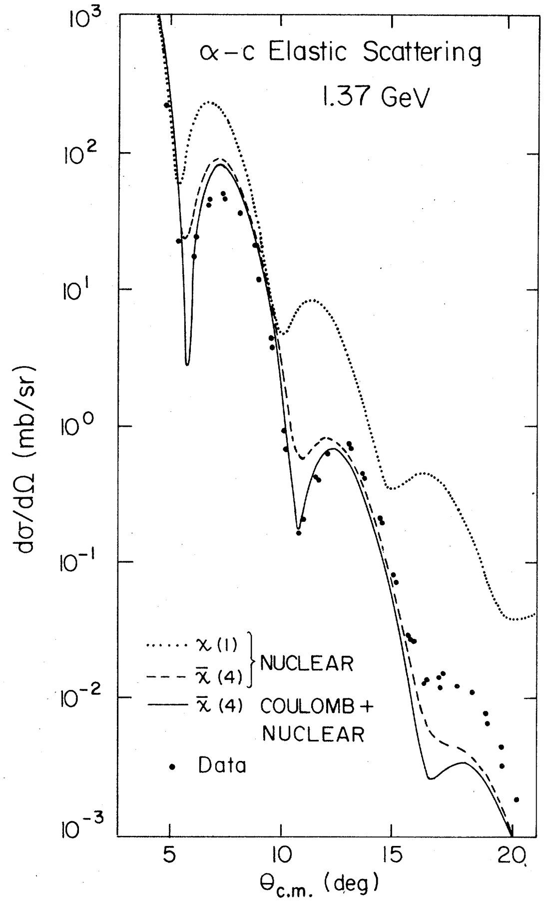

FIG. 11. Theoretical predictions, together with the data of Ref. 39, for α-^{12}C elastic scattering at 1.37 GeV.

Furthermore, since the attempts to describe the 1.37 GeV α-^{12}C data in terms of the "double folding" model (which is closely related to the usual Glauber optical limit) have met with little success,[39] it is worthwhile to investigate the effects of the higher order phase shift corrections and to see if they improve the agreement with the measurements, at least qualitatively. We should emphasize, however, that the Gaussian parametrization of Eq. (4.4) for NN amplitudes is unrealistic at 343 MeV/n, and that both the α and ^{12}C charge form factors are known to exhibit minima, a feature which is not described by the simple Gaussian form factors which we have assumed.

Good agreement between Glauber theory predictions and measurements for 344 MeV/n deuteron-deuteron elastic scattering has been obtained by Alberi, Bertocchi, and Białkowski.[31] Therefore, for NN amplitudes we have taken the parameters from Ref. 31. They are $\sigma_{pp} = 34$ mb, $\sigma_{np} = 27$ mb, $a_{pp} = 0.44$ (GeV/c)$^{-2}$, $a_{np} = 2.0$ (GeV/c)$^{-2}$, $\rho_{pp} = 0.6$, and $\rho_{np} = 0$. In Fig. 11 we show the results of our fourth-order calculations for $\bar{\chi}_{\text{opt}}$, with and without the Coulomb interactions, together with the data. Also shown, for comparison, is the usual optical limit prediction (dotted curve) which yields cross sections that are much too large. The effects of the Coulomb interaction were neglected in the analysis of Ref. 39. It is clear from Fig. 11 that these effects are quite important and they significantly improve the agreement with the data near the second minimum. The Coulomb effects are not negligible even at the maxima. Coulomb effects, however, make the agreement with data worse for $\theta > 15°$. But $\theta \sim 15°$ corresponds to $q^2 \sim 10.5$ fm^{-2}, where the Gaussian form factors for the nuclei are quite unrealistic. The high theoretical predictions occurring at the maximum near $\theta \sim 7°$ are not entirely surprising. A similar discrepancy is observed in the case of p-^{12}C scattering and can be removed by introducing a deformation in the ground state of carbon.[40]

VIII. ASYMPTOTIC APPROXIMATIONS FOR THE OPTICAL PHASE-SHIFT FUNCTION

As we have seen, the expressions for the higher order corrections to the optical phase shift function become increasingly complicated and require evaluation of multiple integrals. While the integrals can be evaluated in closed form for nuclear wave functions described by Gaussians (or sums of Gaussians), these are realistic only for collisions between light nuclei. It is therefore worthwhile to investigate some approximation schemes which may allow one to estimate the higher order corrections for general forms of nuclear densities in a relatively easy manner. A zero range approximation for the nucleon-nucleon interaction has been used in the literature for both nucleon-nucleus and nucleus-nucleus collisions in the usual optical limit calculations. Although this approximation, by itself, can lead to errors,[9] its accuracy improves significantly if one does not correct the measured nuclear charge form factors for the finite size of the proton (since the two approximations tend to partially cancel each other). For example, the first-order optical phase-shift function involves the product $P = S_{A_1}(\vec{q})S_{A_2}(-\vec{q})f(\vec{q})$ of nuclear form factors and the nucleon-nucleon amplitudes. Using the charge form factors $S_{A_i}^{\text{ch}}$ and correcting for proton size, we have

$$P = S_{A_1}^{\text{ch}}(\vec{q})S_{A_2}^{\text{ch}}(-\vec{q})[S_p(q)]^{-2}f(\vec{q}), \tag{8.1}$$

where S_p is the proton form factor. For small momentum transfers $S_p(q)$ is well described by the form $e^{-Cq^2/4}$. Using Eq. (4.4) for the NN amplitude, we have

$$P = S^{ch}_{A_1}(\vec{q}) S^{ch}_{A_2}(-\vec{q}) f(0) e^{-(a-c)/2} . \quad (8.2)$$

A proton rms radius of 0.81 fm corresponds to $C = (0.66\text{ fm})^2 \simeq 9.5\ (\text{GeV}/c)^{-2}$. At high energies the value of a varies typically between 10–11 $(\text{GeV}/c)^{-2}$. Thus there is almost complete cancellation due to the two aforementioned approximations. Furthermore, the first-order phase-shift function χ_1 or $\bar{\chi}_1$ is proportional to the integral over P of Eq. (8.2). Since both the nuclear charge form factors are sharply peaked in the forward direction, the small remaining q dependence due to $f(q)$ has little effect on the integral.

One should keep in mind, however, that the use of an NN amplitude which is not peaked in the forward direction is inconsistent with the Glauber approximation.[15] This shows up clearly in the fourth-order phase-shift function where the term $G_{12}(b)$ (which has a structure similar to the quadruple scattering term in deuteron-deuteron collisions) diverges in the limit $a \to 0$. This term, which is $O(1/A_i)$ relative to the leading term in χ_4, is nevertheless not negligible for light nuclei and is also significant when one is interested in details of angular distributions. However, as long as this term is evaluated carefully, the rest of the quantities C_1, D_i, E_i, and G_i can be greatly simplified in the zero range approximation. We obtain, in the independent particle model [Eq. (4.1)]

$$\begin{aligned} C_1(b) &= (2\pi i k_N)^{-1} \int d^2q\, e^{-i\vec{q}\cdot\vec{b}} S_{A_1}(\vec{q}) f(\vec{q}) S_{A_2}(-\vec{q}) \\ &\simeq (2\pi i k_N)^{-1} f(0) \int d^2q\, e^{-i\vec{q}\cdot\vec{b}} S_{A_1}(\vec{q}) S_{A_2}(-\vec{q}) \\ &= g \int d^2s\, \rho_{A_1}(\vec{s}) \rho_{A_2}(\vec{s}-\vec{b}) , \end{aligned} \quad (8.3)$$

where

$$\rho_{A_i}(\vec{s}) = \int_{-\infty}^{\infty} dz\, \rho_{A_i}[(s^2+z^2)^{1/2}], \quad g = 2\pi f(0)/ik_N . \quad (8.4)$$

In higher order terms, the expressions look simpler in coordinate space where the approximation $f(q) \simeq f(0)$ is equivalent to

$$\Gamma_{ij}(\vec{b}-\vec{s}_i+\vec{s}'_j) \simeq g\delta^2(\vec{b}-\vec{s}_i+\vec{s}'_j) ,$$

which yields (upon straightforward integration),

$$\begin{aligned} &D_2(b) = g^2 \int d^2s\, \rho_{A_1}(\vec{s}) \rho_{A_2}{}^2(\vec{s}-\vec{b}) , \quad D_3(b) = D_2(1 \leftrightarrow 2) , \\ &E_4(b) = g^3 \int d^2s\, \rho_{A_1}{}^2(\vec{s}) \rho_{A_2}{}^2(\vec{s}-\vec{b}) , \quad E_5(b) = g^3 \int d^2s \rho_{A_1}(\vec{s}) \rho_{A_2}{}^3(\vec{s}-\vec{b}) , \quad E_6(b) = E_5(1 \leftrightarrow 2) , \\ &G_{10}(b) = g^4 \int d^2s \rho_{A_1}{}^2(\vec{s}) \rho_{A_2}{}^3(\vec{s}-\vec{b}) , \quad G_{11}(b) = G_{10}(1 \leftrightarrow 2) , \quad G_{13}(b) = g^4 \int d^2s \rho_{A_1}(\vec{s}) \rho_{A_2}{}^4(\vec{s}-\vec{b}) , \\ &G_{14}(b) = G_{13}(1 \leftrightarrow 2) , \quad G_{15}(b) = G_{10}(b) , \quad G_{16}(b) = G_{11}(b) . \end{aligned} \quad (8.5)$$

The other D_i, E_i, and G_i occurring in Eq. (2.11) are related to the above quantities by Eq. (A1) in Appendix A. The quantities $\bar{C}_1$, $\bar{D}_i$, $\bar{E}_i$, and $\bar{G}_i$ which are required for calculations of the modified optical phase-shift function $\bar{\chi}_{\text{opt}}$ can be obtained from Eqs. (8.5) and (A1) by means of the relation (A3).

It is worth pointing out that a unique optical potential can always be obtained from the optical phase-shift function by means of the relation[1]

$$V_{\text{opt}}(\vec{r}) = \frac{\hbar v}{\pi} \frac{1}{r} \frac{d}{dr} \int_r^{\infty} \frac{\chi_{\text{opt}}(b)}{(b^2-r^2)^{1/2}}\, b\, db .$$

The first-order optical potential corresponding to χ_1 has a form similar to that of heavy-ion potentials at lower energies in the "double folding" model.[41] The potentials corresponding to higher order phase-shift corrections provide corrections to the double-folding model and arise from the processes in which one nucleon of the incident nucleus can interact with two or more nucleons of the target (and vice versa). The effects of the higher order corrections on the optical phase shift function (and hence on the optical potential) are to reduce its depth (this is true for real as well as imaginary parts). It is interesting to note that the theoretical heavy-ion potentials obtained from the double-folding model do give potentials which are too deep.[42,43] While it should be emphasized that the approximations inherent in the Glauber theory break down at low energies, the qualitative estimates of the size of higher order corrections (i.e., the reduction in the depth of the potential) may not be unrealistic.

XI. CONCLUSIONS

We have studied the case of nucleus-nucleus collisions in the Glauber approximation. The usual optical limit result for the phase-shift function is the first term of an infinite series for $\chi_{\text{opt}}(b)$. Its use leads to a nucleus-nucleus scattering amplitude which diverges at large momentum transfers. This divergence is avoided by introducing the center-of-mass correlations in each order of the optical phase-shift function. Use of the first term in this modified phase shift series leads to significant improvement in the results for total cross sections, inelastic cross sections, and nucleus-nucleus slope parameters. When higher order terms in the series are retained we obtain even greater improvement in these results. These higher order terms are also crucial for studying the elastic scattering angular distributions. For collisions between light and medium nuclei, the higher order phase shift corrections provide a basis for realistic calculations. For heavier nuclei, the phase-shift series diverges at small impact parameters. However, for cross sections which depend mostly upon peripheral collisions, approximate results can be obtained. Since the higher order phase-shift corrections involve complicated multiple integrals, we have also obtained simplified expressions for them in an approximation where the nucleon-nucleon interaction is treated as having zero range. A simple formula is also given for the inclusion of Coulomb interactions, which are shown to contribute significantly to these collisions even at angles away from the forward directions. Theoretical predictions agree well with the Bevalac measurements at 0.87 and 2.1 GeV/n involving collisions between light nuclei.

Note added in proof: The unphysical divergence of $\bar{\chi}_{\text{opt}}$ for small b, discussed in Sec. IV, is removed when Pauli or dynamical short range correlations are treated. See V. Franco and W. T. Nutt [Nucl. Phys. A (to be published)], G. K. Varma [Phys. Rev. C (to be published)], and V. Franco and W. T. Nutt (unpublished).

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APPENDIX A

When nuclei are described by independent particle models, the terms occurring in Eq. (2.17) simplify considerably. Equations (4.1) and (4.2), when used in Eq. (2.22), lead to

$$\begin{aligned}
D_1(b) &= C_1^{\,2}(b)\,,\\
E_1(b) &= C_1^{\,3}(b)\,,\\
E_2(b) &= C_1(b)D_2(b)\,,\\
E_3(b) &= C_1(b)D_3(b)\,,\\
G_1(b) &= C_1^{\,4}(b)\,,\\
G_2(b) &= C_1^{\,2}(b)D_2(b)\,,\\
G_3(b) &= C_1^{\,2}(b)D_3(b)\,,\\
G_4(b) &= C_1(b)E_5(b)\,,\\
G_5(b) &= C_1(b)E_6(b)\,,\\
G_6(b) &= C_1(b)E_4(b)\,,\\
G_7(b) &= D_2(b)D_3(b)\,,\\
G_8(b) &= D_2^{\,2}(b)\,,\\
G_9(b) &= D_3^{\,2}(b)\,.
\end{aligned} \tag{A1}$$

The other parameters in Eq. (2.17) can be evaluated analytically for nuclear form factors and NN amplitudes of the forms (4.3) and (4.4), respectively, with the results[12]

$$C_1(b) = (\sigma'/R^2)e^{-b^2/R^2};$$

$$\sigma' = \sigma(1-i\rho)/2\pi\,, \quad R^2 = R_1^{\,2} + R_2^{\,2} + 2a\,,$$

$$D_2(b) = (\sigma'^2/\alpha_1)e^{-2b^2/(R^2+R_1^2)}; \quad \alpha_1 = R^4 - R_1^{\,4}\,,$$

$$D_3(b) = D_2(1 \leftrightarrow 2)\,,$$

$$E_4(b) = (\sigma'^3/R^2\alpha_{12})\exp\left[-\frac{2b^2}{R^2}\left(1+\frac{2a^2}{\alpha_{12}}\right)\right];$$

$$\alpha_{12} = \alpha_1 - R_2^{\,4}\,,$$

$$E_5(b) = (\sigma'^3/\alpha_1\beta_1)\exp\left[-\frac{b^2}{R^2}\left(1+\frac{2\beta_1^{\,2}}{\alpha_1}\right)\right];$$

$$\beta_1 = R^2 - R_1^2\,, \quad E_6(b) = E_5(1 \leftrightarrow 2)\,, \quad G_{10}(b) = (\sigma'^4/R^2 P_1 \alpha_1)\exp\left[-b^2\left(\frac{2}{R^2} + \frac{\beta_1^2}{R^2\alpha_1} + \frac{Q_1^2}{P_1}\right)\right];$$

$$P_1 = R^{-2}[\alpha_{12} - \beta_1^2 R_1^4 \alpha_1^{-1}],$$

$$Q_1 = R^{-2}[2a - \beta_1^2 R_1^2 \alpha_1^{-1}],$$

$$G_{11}(b) = G_{10}(1 \leftrightarrow 2)\,, \quad G_{12}(b) = (\sigma'^4/R^2\alpha_{12}T)\exp\left\{-b^2\left[\frac{2}{R^2+R_1^2} + \frac{\alpha_1}{R^2\alpha_{12}}\left(\frac{\beta_2}{R^2} + \frac{\beta_1 R_1^2 R_2^2}{\alpha_1 R^2}\right)^2 + \frac{S^2}{T}\right]\right\};$$

$$S = 1 - R_2^2\beta_1\alpha_1^{-1} - R^{-4}\alpha_{12}^{-1}(\alpha_1\beta_2 + R_1^2R_2^2\beta_1)(R_1^2 + R_1^2R_2^4\alpha_1^{-1})\,, \quad T = R^2\alpha_{12}\alpha_1^{-1} - \alpha_1 R^{-2}\alpha_{12}^{-1}R_1^4(1 + R_2^4\alpha_1^{-1})^2\,,$$

$$G_{13}(b) = \frac{\sigma'^4 R^2}{V_1(\alpha_1^{*2} - R_1^4\beta_1^2)}\exp\left\{-b^2\left[\frac{2}{R^2+R_1^2} + \frac{U_1^2}{V_1} + \frac{\beta_1 U_1}{\alpha_1}\right]\right\};$$

$$U_1 = \beta_1(\alpha_1 - \beta_1 R_1^2)/[R^2(\alpha_1 + \beta_1 R_1^2)\,, \quad V_1 = R^{-2}[\alpha_1 - 2\beta_1^2 R_1^4/(\alpha_1 + \beta_1 R_1^2)]\,,$$

$$G_{14}(b) = G_{13}(1 \leftrightarrow 2)\,, \quad G_{15}(b) = (\sigma'^4/R^2\alpha_1\delta_1)\exp\left[-b^2\left(\frac{1}{R^2} + \frac{2}{R^2+R_1^2} + \frac{\gamma_1^2}{\delta_1}\right)\right];$$

$$\gamma_1 = \beta_1 R^{-2} - R_2^2(R^2 + R_1^2)^{-1}\,, \quad \delta_1 = \alpha_1 R^{-2} - R^2 R_2^4\alpha_1^{-1}\,, \quad G_{16}(b) = G_{15}(1 \leftrightarrow 2)\,. \qquad \text{(A2)}$$

Equations (A2) and (A1), when substituted in Eqs. (2.17) and (2.10), yield the series for the unmodified phase-shift function $\chi_{\text{opt}}(b)$. The series for the modified phase-shift function $\overline{\chi}_{\text{opt}}(b)$ can be obtained from the same equation by merely substituting the quantities $\overline{C}_1$, $\overline{D}_i$, $\overline{E}_i$, and $\overline{G}_i$ in place of C_1, D_i, E_i, and G_i. For nuclear wave functions corresponding to Eq. (4.3), the center-of-mass correlation $K(q)$ is given by[12,16]

$$K(q) = \exp[q^2(R_1^2/4A_1 + R_2^2/4A_2)]\,. \qquad \text{(A3)}$$

If we let Q_i denote C_1, D_i, E_i, and G_i than all the Q_i are of the form

$$Q_i(b) = \mu_i e^{-\nu_i b^2}\,, \qquad \text{(A4)}$$

and we obtain from Eq. (3.13)

$$\overline{Q}_i(b) = \frac{\mu_i}{\nu_i\nu_i'}e^{-b^2/\nu_i'}; \quad \nu_i' = \frac{1}{\nu_i} - \frac{R_1^2}{A_1} - \frac{R_2^2}{A_2}\,, \qquad \text{(A5)}$$

where $\overline{Q}_i$ stands for the new $\overline{C}_1$, $\overline{D}_i$, $\overline{E}_i$, and $\overline{G}_i$. As pointed out in Sec. III, the barred quantities can also be obtained directly starting from Eqs. (3.14)–(3.17), with results equivalent to (A5). [It is worth emphasizing that Eq. (A1) does not hold for the barred quantities, i.e., when c.m. correlations are contained in the optical phase-shift function. For example, the relation $\overline{D}_1 = \overline{C}_1^2$ *does not* hold. However, $\overline{D}_1$ can be obtained by using $D_1 = C_1^2$ in Eq. (3.13).]

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$$-\tfrac{1}{2}(c_p^2 e^{-a_p q^2/4} + c_n^2 e^{-a_n q^2/4})\sum_j \alpha_j/H_j$$

should be replaced by

$$-\frac{1}{4}\sum_j \alpha_j\left(\frac{c_p^2 e^{-a_p q^2/4}}{a_p + \beta_j} + \frac{c_n^2 e^{-a_n q^2/4}}{a_n + \beta_j}\right).$$

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