Radiative muon capture rates and the maximum photon energy*

R. S. Sloboda
Nuclear Research Centre, University of Alberta, Edmonton, Alberta, Canada and TRIUMF, Vancouver, British Columbia, Canada

Harold W. Fearing
TRIUMF, Vancouver, British Columbia, Canada
(Received 5 January 1978)

The differential photon spectrum for radiative muon capture is expanded about the average maximum photon energy k_m and the correction terms evaluated using for one a modified Thomas-Reiche-Kuhn sum rule, thus extending previous work for ordinary capture. The resulting rate is much less dependent on k_m than the usual closure result. The ratio k_m/ν appropriate for closure calculations, with ν the average neutrino energy, is determined and found to be approximately constant and, when correction terms are included, somewhat higher than values previously used. By similar techniques a consistency relation is derived which can be solved to explicitly estimate "physical" values of k_m and ν.

NUCLEAR REACTIONS Radiative muon capture. Differential photon spectrum, relative rate for ^{40}Ca. Dipole sum rules used to correct closure approximation, obtain estimates of mean excitation energy, average maximum photon energy.

I. INTRODUCTION

The calculation of radiative muon capture rates in nuclei has been pursued by a number of authors with the aim of extracting the magnitude of the weak induced pseudoscalar coupling g_P from measurements of the differential photon spectrum or the photon asymmetry. In essentially all of these calculations, e.g., Ref. 1,2, the closure approximation on the final nuclear states has been used to express the rate in terms of the ground state expectation value of a two-body operator. In this approach an average maximum photon energy k_m corresponding to an average excitation energy of the residual nucleus is introduced. The rate depends very strongly on k_m and hence, uncertainty in k_m poses a problem when one attempts to extract g_P from experiment. The situation is entirely analogous to that for ordinary muon capture, where the result of a closure approximation calculation is very sensitive to the choice of the average neutrino energy ν.

Recent calculations[3-6] of ordinary muon capture rates employing sum rules have succeeded in largely eliminating the ν dependence of the rate. In particular, Bernabeu[4] has shown that a first order expansion of the capture rate about ν gives a result which is essentially independent of the specific value of ν within a range of plausible values.

In the present paper we extend these ideas to radiative muon capture and show that a first order expansion of the expression for the photon spectrum about k_m gives a corrected expression for the spectrum which is much less dependent on the specific value of k_m used than the usual closure result.

In Sec. II we derive the corrected expression for the differential photon spectrum. As for ordinary capture, two additional terms arise besides the usual closure term. One can be calculated from the closure term and the other one is evaluated using a generalized Thomas-Reiche-Kuhn (TRK) sum rule.[7]

Several further corrections are considered which were not included in earlier calculations of ordinary capture. In particular, the effect of the Coulomb energy difference between initial and final nuclear states is included in a simple way. Exchange effects are also included, albeit in a very phenomenological way, by modifying the sum rule term by an overall factor which is obtained from a sum rule analysis of total photoabsorption cross sections.

In Sec. III we try to clarify the meaning of these average quantities k_m and ν and show that by a simple extension of the sum rule technique we can obtain a consistency equation which allows an explicit calculation of average quantities related to k_m and ν. Then, as an example, in Sec. IV our results are applied to the closed shell nucleus ^{40}Ca, using a very simple harmonic oscillator model, and we close with a brief discussion of these results and our conclusions.

II. THEORY

In an approximation which neglects the "velocity terms", the differential photon spectrum for radiative muon capture can be written

$$N(k)=(\alpha m_\mu G^2/4\pi^2)|\phi_\mu|^2\sum_{\lambda=\pm1} I^2(k,\lambda)\,. \tag{1}$$

Here ϕ_μ is the muon wave function, and the sum on λ is over the circular polarizations of the photon emitted with absolute value of momentum k. The function $I^2(k,\lambda)$ contains the nuclear matrix elements. If one assumes for radiative capture the relations given in Eqs. (6) and (7) of Ref. 8, which are the exact analogs of the relations $M_V^2=M_A^2=M_P^2$ usually assumed for ordinary capture, and which follow, for example, from SU(4) invariance of the internucleon forces,[9] then

$$\begin{aligned}\sum_\lambda I^2(k,\lambda)&=\int_{-1}^{1}dy\sum_{\overline{a}b} G_{ab}^2\left(\frac{k(k_m^{ab}-k)^2}{m_\mu^3}\theta(k_m^{ab}-k)|\langle b|\sum_j \tau^-(j)\,e^{-i\vec{s}_{ab}\cdot\vec{r}_j}|a\rangle|^2\right)\\&\equiv\int_{-1}^{1}dy\sum_{\overline{a}b}G_{ab}^2 I_{ab}^2(k_m^{ab},k)\,.\end{aligned} \tag{2}$$

In this expression, $y=\hat{k}\cdot\hat{\nu}$ and $\vec{s}_{ab}=(\vec{k}+\vec{\nu})_{ab}$, with $\vec{\nu}_{ab}$, $\vec{k}_{ab}$, and k_m^{ab}, respectively, the neutrino momentum, the photon momentum, and the maximum photon energy corresponding to the transition $a\to b$. G_{ab}^2 is a function of k_m^{ab}, y, and the weak coupling constants, and the sum on $\overline{a}b$ denotes an average over initial and a sum over final nuclear states. The maximum energy available to the photon is given by

$$\begin{aligned}k_m^{ab}&=m_\mu-(m_n-m_p)-E_{BE}-(E_b-E_a)\\&\equiv E-(E_b-E_a)\,,\end{aligned} \tag{3}$$

in which E_{BE} is the muon binding energy and E_a, E_b are the energies of the nuclear states.

Since G_{ab}^2 depends only weakly on k_m^{ab} through the combination $k_m^{ab}/2m\sim 0.05$, we replace k_m^{ab} in G_{ab}^2 by an appropriate average value k_m, and define

$$I_n(k)\equiv\int_{-1}^{1}dy\,y^n\sum_{\overline{a}b}I_{ab}^2(k_m^{ab},k)\,. \tag{4}$$

Thus the differential photon spectrum can be written

$$N(k)=(\alpha m_\mu G^2/4\pi^2)|\phi_\mu|^2\sum_n C_n I_n(k)\,. \tag{5}$$

The C_n, which are obtained by extracting the explicit y dependence from G_{ab}^2, are functions of k_m and the weak couplings. When the I_n are evaluated in the closure-harmonic oscillator model, Eq. (5) gives an expression for the spectrum in closure approximation $N(k_m,k)$ which exhibits strong dependence on the average maximum photon energy k_m, as we shall see in the next section.

In an attempt to remedy this situation we calculate correction terms to the closure approximation using techniques analogous to those used in Ref. 4 for ordinary capture. We thus expand $I_{ab}^2(k_m^{ab},k)$ to first order in k_m^{ab} about an average value k_m and obtain a corrected expression $\tilde{I}_{ab}^2$ for which the k_m^{ab} dependence is explicit and linear.

$$\begin{aligned}\tilde{I}_{ab}^2&=I_{ab}^2(k_m^{ab},k)\,|_{k_m^{ab}=k_m}\\&\quad+\frac{\partial}{\partial k_m^{ab}}I_{ab}^2(k_m^{ab},k)\,|_{k_m^{ab}=k_m}(k_m^{ab}-k_m)\,.\end{aligned}$$

Using Eq. (3),

$$\begin{aligned}\tilde{I}_{ab}^2&=\left(1+(E-k_m)\frac{\partial}{\partial k_m}\right)I_{ab}^2(k_m,k)\\&\quad-\frac{\partial}{\partial k_m}[(E_b-E_a)I_{ab}^2(k_m^{ab},k)]\,|_{k_m^{ab}=k_m}\,.\end{aligned} \tag{6}$$

Substituting Eq. (6) in Eq. (4), we obtain a new expression

$$\begin{aligned}\tilde{I}_n(k_m,k)&=\left(1+(E-k_m)\frac{\partial}{\partial k_m}\right)I_n(k_m,k)\\&\quad-\int_{-1}^{1}dy\,y^n\frac{\partial}{\partial k_m}\left(\sum_{\overline{a}b}\frac{k(k_m^{ab}-k)^2}{m_\mu^3}(E_b-E_a)\theta(k_m^{ab}-k)\,|\langle b|\sum_i\tau^-(i)\,e^{-i\vec{s}_{ab}\cdot\vec{r}_i}|a\rangle|^2\,|_{k_m^{ab}=k_m}\right),\end{aligned} \tag{7}$$

where $I_n(k_m,k)$ is the closure approximation to $I_n(k)$. When substituted in Eq. (5), this yields an expression for the differential photon spectrum which we shall call $\tilde{N}(k_m,k)$.

The final term in Eq. (7) can be calculated using a modified Thomas-Reiche-Kuhn sum rule.[7] For a many-particle Hamiltonian $H=T+V$, an operator $\sum_i O_i$ which satisfies the commutation relation

$[\sum_i O_i, V]=0$ then also satisfies

$$\sum_b (E_b' - E_a')\,|\langle b|\sum_i O_i|a\rangle|^2 = \frac{1}{2m}\langle a|\sum_i (\vec{\nabla}_i O_i^\dagger)\cdot(\vec{\nabla}_i O_i)|a\rangle, \quad (8)$$

with E_a', E_b' eigenstates of H, and m the nucleon mass.

There are two refinements which can now be made before applying this sum rule to the evaluation of Eq. (7). These have not generally been made in previous calculations for ordinary capture, but do seem to have numerical significance and so will be included here. The first deals with the Coulomb energy shift. Observe that Eq. (8) contains $E_b' - E_a'$, the difference in eigenvalues of the nuclear Hamiltonian, whereas Eq. (6) involves $E_b - E_a$, the difference in the actual nuclear energy levels, which of course includes the Coulomb energy. The two are related by $E_b - E_a = E_b' - E_a' - E_c$, where E_c is the Coulomb energy difference between the (A,Z) and $(A,Z-1)$ ground states. Substitution of this relation into Eq. (7) effectively replaces E by $E+E_c$ in the second term.

The second refinement has to do with the influence of exchange corrections. When the TRK sum rule is applied to photoabsorption processes it fails, predicting total photoabsorption cross sections which are too low. This is not entirely unexpected since it has been known for some time that nuclear exchange potentials not satisfying $[\sum_i O_i, V_{\text{exch}}]=0$ give rise to terms which make a significant contribution to the sum rule.[11,12] The inclusion of exchange forces would enhance the right hand side of Eq. (8) by an amount conventionally described by a phenomenological factor $(1+\alpha)$, which for specific nuclei may be determined from fits to experimental total photoabsorption cross sections. Moreover, since the processes of photoabsorption and muon capture are both dominated by dipole transitions and in that approximation involve similar operators, we may reasonably expect that values for α determined from photoabsorption data[13] can be used to estimate the α necessary to calculate muon capture rates. Thus we will make use of the modified TRK sum rule in the form

$$\sum_b (E_b' - E_a')\,|\langle b|\sum_i O_i|a\rangle|^2 = \frac{(1+\alpha)}{2m}\langle a|\sum_i (\vec{\nabla}_i O_i^\dagger)\cdot(\vec{\nabla}_i O_i)|a\rangle. \quad (9)$$

Applying Eq. (9) to the final term in Eq. (7), we obtain for the differential photon spectrum

$$\tilde{N}(k_m,k) = (\alpha m_\mu G^2/4\pi^2)\,|\phi_\mu|^2 \sum_n C_n \tilde{I}_n(k_m,k), \quad (10)$$

with

$$\tilde{I}_n(k_m,k) = \left[1 + (E+E_c-k_m)\frac{\partial}{\partial k_m}\right] I_n(k_m,k) - (1+\alpha)\frac{2Zk}{m m_\mu^3} \times \begin{cases} (2k_m^3 - 6kk_m^2 + 7k^2k_m - 3k^3)/(n+1) & n \text{ even} \\ (3kk_m^2 - 6k^2k_m + 3k^3)/(n+2) & n \text{ odd} \end{cases}. \quad (11)$$

To obtain the relative rate we need the ordinary capture rate calculated in the same approximation. We thus write the ordinary rate Λ (again neglecting velocity terms) as

$$\Lambda = \frac{m_\mu^2}{2\pi}\,|\phi_\mu|^2 \sum_{ab} (G_F^2 + 3G_{GT}^2)_{ab} M_{ab}^2(\nu_{ab}) \quad (12)$$

with

$$M_{ab}^2 = (\nu_{ab}^2/m_\mu^2)\int (d\Omega_\nu/4\pi)\,|\langle b|\sum_j \tau^-(j) e^{-i\vec{\nu}_{ab}\cdot\vec{r}_j}|a\rangle|^2. \quad (13)$$

Proceeding as for radiative capture we find for the corrected rate

$$\tilde{\Lambda}(\nu) = \frac{m_\mu^2}{2\pi}\,|\phi_\mu|^2 (G_F^2 + 3G_{GT}^2)\tilde{M}^2(\nu), \quad (14)$$

where

$$\tilde{M}^2(\nu) = \left[1 + (E+E_c-\nu)\frac{d}{d\nu}\right]M^2(\nu) - 2(1+\alpha)Z\frac{m_\mu}{m}\left(\frac{\nu}{m_\mu}\right)^3, \quad (15)$$

with $M^2(\nu)$ the usual closure result. This is essentially the result of Bernabeu,[4] with Coulomb and exchange corrections added as above.

So far in this and previous work the velocity terms have been neglected. This has been necessary for the sum rule piece because an operator $\sim p$ does not satisfy the assumptions necessary to obtain the simple sum rule of Eq. (9). Observe, however, that both the velocity terms and the sum rule corrections to the main terms are $O(1/m)$. Thus, presumedly a sum rule correction to the velocity terms, which would involve commutators with the kinetic energy $p^2/2m$, would be of $O(1/m^2)$, i.e., of the same order as other terms neglected. Corrections of $O(1/m)$ are obtained, however, by including the velocity terms in the usual closure results $\sum_n C_n \dot{I}_n(k_m, k)$ and $(G_F^2 + 3G_{GT}^2)M^2(\nu)$ appearing in the first two terms of Eqs. (11) and (15). So henceforth we shall incorporate the velocity terms in $\sum_n C_n I_n(k_m, k)$ and $(G_F^2 + 3G_{GT}^2)M^2(\nu)$, obtaining a result which we expect will be accurate to $O(1/m)$.

III. SUM RULE RELATIONS FOR AVERAGE EXCITATION ENERGIES

Before actually evaluating the expressions derived above in the context of a specific model, we want to discuss the meaning of the parameters k_m and ν. In the process we derive, by using the same sort of sum rule techniques applied above, a consistency relation which allows an explicit estimate of the "physical" values of k_m and ν.

So far we have somewhat loosely referred to k_m and ν as "average" maximum photon energy and "average" neutrino energy corresponding to an "average" nuclear excitation of the residual nucleus. Strictly speaking, however, they are not averages in the physical sense but simply parameters defined formally by $N(k) = N(k_m, k)$ and $\Lambda = \Lambda(\nu)$, where $N(k_m, k)$ and $\Lambda(\nu)$ are the closure approximations to the actual rates $N(k)$ and Λ. Thus k_m and ν are the values which, when used to evaluate the rates in closure approximation, give the correct answer.

In the previous section we obtained improved approximations to the correct rates, i.e., $\tilde{N}(k_m, k)$ and $\tilde{\Lambda}(\nu)$, and we shall show in the ensuing discussion that at least in a simple model these are relatively independent of k_m and ν. Thus the appropriate values of k_m and ν to use in a closure calculation can be estimated by solving the equations $\tilde{N}(k_m, k) = N(k_m, k)$ and $\tilde{\Lambda}(\nu) = \Lambda(\nu)$, that is, by determining the intersection point of the closure and corrected calculations. This is just the point where the two correction terms in Eq. (11) or those in Eq. (15) exactly cancel. The resulting parameters are then those which must be used in a closure approximation calculation for consistent results. They are of course somewhat dependent on the model used for the nuclear matrix elements and thus if drastically different values are required to fit the data, one should view the model with suspicion.

The physical values of average maximum photon energy and average neutrino energy (which we shall write as $\bar{k}_m$ and $\bar{\nu}$) are in principle different from the parameters k_m and ν. Their pth moments can be defined formally as

$$\bar{k}_m^p = \frac{\sum_{\bar{a}b} (k_m^{ab})^p N_{ab}(k)}{\sum_{\bar{a}b} N_{ab}(k)}$$

and

$$\bar{\nu}^p = \frac{\sum_{\bar{a}b} \nu_{ab}^p \Lambda_{ab}}{\sum_{\bar{a}b} \Lambda_{ab}}, \qquad (16)$$

where the denominators are just the rates $N(k)$ and Λ, and the numerators are the appropriate quantities for the transition $a \to b$ weighted by the probability of that transition and summed over all states b.

In closure approximation, $\bar{k}_m = k_m$ and $\bar{\nu} = \nu$, which is what is normally assumed. We can however calculate corrections to these relations in exactly the same way that we have calculated corrections to the closure approximation for the rates, so that when such corrections are included, these equalities will no longer hold. To do this we expand both numerator and denominator of Eq. (16) about $k_m^{ab} = k_m$ and $\nu_{ab} = \nu$ and carry through the sum rule evaluation as done in Eqs. (6)–(11). The results, keeping only the first order correction, are

$$\bar{k}_m^p = k_m^p + p k_m^{p-1}\left[(E + E_c - k_m) - \frac{(1+\alpha)(Zk/mm_\mu^3)(k_m - k)^2}{\sum_n C_n I_n(k_m, k)} \times \sum_n C_n \times \begin{cases} [k_m^2 - 2k(k_m - k)]/(n+1) & n \text{ even} \\ 2k(k_m - k)/(n+2) & n \text{ odd} \end{cases}\right] \qquad (17)$$

and

$$\bar{\nu}^p = \nu^p + p\nu^{p-1}\left((E + E_c - \nu) - \frac{(1+\alpha)\nu^2 Z/2m}{M^2(\nu)}\right). \qquad (18)$$

Thus, for $p = 1$ these relations express the physical average $\bar{k}_m$ and $\bar{\nu}$ as the closure parameters k_m and ν plus a correction term. Note that the correction is essentially the same, only without the derivatives, as that appearing in the equations for $\tilde{N}(k_m, k)$ and $\tilde{\Lambda}(\nu)$.

A more useful result can be obtained by expanding the right sides of Eq. (16) about $\bar{k}_m$ and $\bar{\nu}$ instead of k_m and ν. The resulting equations (identical to those above with $k_m \to \bar{k}_m$ and $\nu \to \bar{\nu}$ everywhere) now provide consistency relations which can be solved for $\bar{k}_m$ and $\bar{\nu}$. Since the leading terms cancel, the solutions correspond to the zero of the term in brackets and provide an estimate of the physical averages corresponding to the physical average excitation energy of the residual nucleus. Note that the results will depend on the model used for the nuclear matrix elements in $I_n(k_m, k)$ and $M^2(\nu)$. We will evaluate these consistency relations for a simple model in the next section.

IV. APPLICATION TO THE CLOSED SHELL NUCLEUS ^{40}Ca

For the purpose of illustrating the calculations outlined in the previous section, we make an application of our method to the nucleus ^{40}Ca, evaluating the nuclear matrix elements using harmonic oscillator shell model wavefunctions. Such a model is perhaps too simple, but has been used for most comparisons with data and in any case will illustrate the results. In this model the closure result for I_n is

$$I_n(k_m, k) = \frac{k(k_m - k)^2}{m_\mu^{\ 3}}\, \theta(k_m - k) \int_{-1}^{1} y^n dy\, \mathfrak{M}^2 , \qquad (19)$$

where for ^{40}Ca with $\eta^2 = (sb)^2$ we have

$$\mathfrak{M}^2 = 20\left[1 - \left(1 + \frac{\eta^4}{8} - \frac{\eta^6}{80} + \frac{\eta^8}{640}\right) e^{-\eta^2/2}\right].$$

The oscillator parameter is taken to be $b = 2.03$ fm. We evaluate expressions (5) and (10) for the differential photon spectrum using the C_n from the appendix in Ref. 8 and the set of weak couplings $g_V = 1.0$, $g_M = 3.7$, $g_A = -1.25$, $g_P = 7g_A$, $g_T = 0$, $g_S = 0$. From Ref. 14 we have $E_c = 7.13$ MeV. and from Ref. 15, $E_{BE} = 1.066$ MeV, which give the value $E + E_c = 110.4$ MeV. As noted above, the velocity terms have been included as has the phenomenological correction for exchange effects $(1 + \alpha)$.

It has been customary to present results for radiative muon capture as a ratio of the differential photon spectrum to the ordinary rate, as presumedly some of the model dependence of the nuclear matrix elements will then cancel, though factors which independently affect the overall scale of the amplitudes for radiative and for ordinary capture will of course affect also the ratio. In Figs. 1 and 2 we show such plots which illustrate the main features of our results. We see that the ratio of corrected quantities $\tilde{N}(k_m, k)/\tilde{\Lambda}(\nu)$, shown as solid and short-dashed curves for two different values of α, is generally much less dependent on k_m than the usual closure result $N(k_m, k)/\Lambda(\nu)$, shown as a long-dashed curve. Thus the important qualitative result is that for radiative capture, just as for ordinary capture,[4] the sum rule technique allows one to obtain a result more or less independent of the closure parameter k_m over a reasonable range of k_m. Note that since $\tilde{\Lambda}(\nu)$ is itself nearly independent of ν in this approach,[4] these qualitative features hold for the absolute rate $\tilde{N}(k_m, k)$ just as for the ratio $\tilde{N}(k_m, k)/\tilde{\Lambda}(\nu)$.

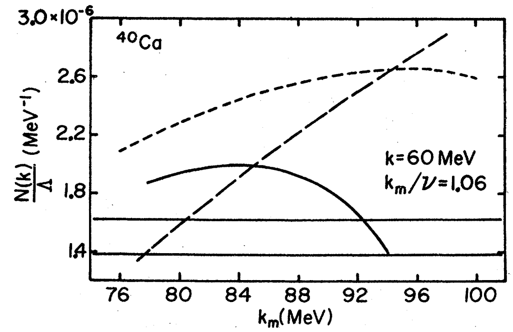

FIG. 1. The relative capture rate $N(k)/\Lambda$ for ^{40}Ca for $k = 60$ MeV and $k_m/\nu = 1.06$. The usual closure result (long-dashed curve) is compared to the corrected result for $\alpha = 0$ (short-dashed curve) and for $\alpha = 1.15$ (solid curve). The horizontal lines are experimental bounds on $N(k)/\Lambda$ from Ref. 16.

We also observe that the curves for the corrected ratio $\tilde{N}(k_m, k)/\tilde{\Lambda}(\nu)$ and the closure ratio $N(k_m, k)/\Lambda(\nu)$ intersect in a region where $\tilde{N}(k_m, k)/\tilde{\Lambda}(\nu)$, and due to the stability of $\tilde{\Lambda}(\nu)$, $\tilde{N}(k_m, k)$ itself is stable. Thus we can make the intersection of $\tilde{N}(k_m, k)$ and $N(k_m, k)$ a criterion in the selection of a value of k_m to be used in an ordinary closure calculation of $N(k)$. Similarly, the appropriate value of ν can be determined from the intersection of $\tilde{\Lambda}(\nu)$ and $\Lambda(\nu)$. As emphasized in Sec. III, k_m and ν determined this way are basically just parameters which force the closure approximation to give results for $N(k_m, k)$ and $\Lambda(\nu)$ equal to the sum rule corrected values, which in turn approximate the correct results.

In Figs. 1–3 we have shown results for two different values of α, $\alpha = 0$ and $\alpha = 1.15$. The value $\alpha = 0$ corresponds to no exchange contribution correction to the sum rule piece, whereas $\alpha = 1.15$ is the lower limit on α given by Ahrens *et al.*,[13] as

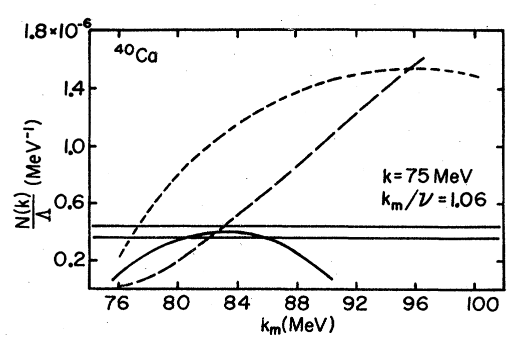

FIG. 2. The relative capture rate $N(k)/\Lambda$ for ^{40}Ca for $k = 75$ MeV and $k_m/\nu = 1.06$. The curves have the same meaning as in Fig. 1.

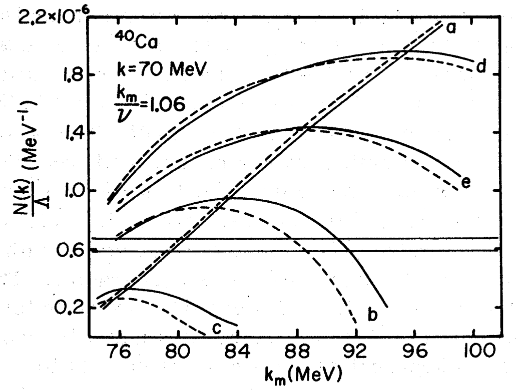

FIG. 3. Influence of the Coulomb energy shift and velocity terms on the relative capture rate for ^{40}Ca for $k=70$ MeV and $k_m/\nu=1.06$. Results including velocity terms (solid curves) and not including them (dashed curves) are shown for two choices of the phenomenological factor α and with and without the Coulomb energy shift E_c: a—usual closure result; b—$\alpha=1.15$, $E_c=7.1$ MeV; c—$\alpha=1.15$, $E_c=0$; d—$\alpha=0$, $E_c=7.1$ MeV; e—$\alpha=0$, $E_c=0$. The horizontal lines are experimental bounds on $N(k)/\Lambda$ from Ref. 16.

determined from the ratio of the experimental photoabsorption cross section to the classical dipole sum. We observe that the introduction of the correction α is numerically important, particularly for large k. The correction is also important in principle since a consistent treatment of both photoabsorption and radiative muon capture using sum rules requires that the information available from photoabsorption be used to constrain the parameters for radiative muon capture. It seems reasonable to take the values for α, $1.15 \leq \alpha \leq 1.4$, directly from Ahrens *et al.*,[13] since the operators in the nuclear matrix elements for photoabsorption are essentially the same as for muon capture,[5,10] and since for ^{40}Ca the dipole transition dominates both ordinary and radiative capture. For the minimum correction $\alpha=1.15$ the absolute rates are in much better agreement with the data[16] than for no correction (cf. Table I). As α is increased from 1.15 to 1.4, the relative rate decreases, in most cases improving the agreement with the data. The current experimental uncertainty in α corresponds to an uncertainty in k_m,

TABLE I. Ordinary and radiative muon capture rates Λ and N, and corresponding average parameters ν and k_m, obtained from the intersection of the closure and corrected rate expressions for ^{40}Ca in a simple harmonic oscillator model. Also presented are values of the physical averages $\bar{\nu}$ and $\bar{k}_m$ obtained by solving the consistency relations [Eqs. (17) and (18)] for these quantities. The cases are I: $\alpha=1.15$, $E_c=7.1$ MeV, velocity terms included; II: $\alpha=1.15, E_c=7.1$ MeV, no velocity terms; III: $\alpha=1.15, E_c=0.0$, velocity terms included; IV: $\alpha=0.0, E_c=7.1$ MeV, velocity terms included; V: $\alpha=0.0, E_c=7.1$ MeV, no velocity terms.

	k (MeV)	ν (MeV)	$\bar{\nu}$ (MeV)	k_m (MeV)	$\bar{k}_m$ (MeV)	k_m/ν	$\bar{k}_m/\bar{\nu}$	$10^{-7}\Lambda$ (sec^{-1})	N (sec^{-1} MeV^{-1})	$10^6 N/\Lambda$ (MeV^{-1})
I	60	78.72	82.76	84.08	84.93	1.068	1.026	0.333	6.55	1.97
	65			83.92	84.42	1.066	1.020		4.96	1.49
	70			83.53	83.73	1.061	1.012		3.15	0.95
	75			82.75	82.81	1.051	1.000		1.32	0.40
II	60	77.12	80.78	82.22	83.30	1.066	1.031	0.283	5.24	1.85
	65			82.31	83.02	1.067	1.028		3.96	1.40
	70			82.17	82.57	1.065	1.022		2.48	0.88
	75			81.74	81.89	1.060	1.014		0.98	0.35
III	60	73.41	76.94	77.86	78.22	1.061	1.017	0.259	3.65	1.40
	65			77.44	77.54	1.055	1.008		2.24	0.86
	70			76.67	76.64	1.044	0.996		0.83	0.32
	75			75.80	75.51	1.033	0.981		0.01	0.00
IV	60	93.12	96.05	96.99	97.94	1.042	1.020	0.599	15.98	2.67
	65			97.04	97.81	1.042	1.018		13.96	2.33
	70			97.04	97.62	1.042	1.016		11.73	1.96
	75			96.99	97.38	1.042	1.014		9.23	1.54
V	60	92.31	95.12	95.91	97.00	1.039	1.020	0.540	13.87	2.57
	65			96.01	96.96	1.040	1.019		12.20	2.26
	70			96.14	96.87	1.041	1.018		10.33	1.91
	75			96.18	96.71	1.042	1.017		8.17	1.51

determined from the intersection criterion, of about 1.5 MeV.

Figure 3 shows in addition the sensitivity of the results to the two other corrections we have included, Coulomb energy shifts and velocity terms. The Coulomb correction is very important. Its inclusion tends to increase $\tilde{N}(k_m,k)/\tilde{\Lambda}(\nu)$ and the radiative rate $\tilde{N}(k_m,k)$ by amounts which can be as much as a factor of two or three for some k in extreme cases. It also increases $\tilde{\Lambda}(\nu)$, though by a smaller amount, and increases k_m and ν obtained from the intersection criterion by roughly the Coulomb energy shift. Thus in this approach it is clearly important to include this effect.

The velocity terms are somewhat less important, but still should be included. They increase the ordinary rate by 15–20%, the radiative rate by 30–35% and the ratio by 5–15%. This is a somewhat larger effect than found in the usual closure approximation [where the velocity terms make very little difference in the ratio $N(k_m,k)/\Lambda(\nu)$], both because the corrected rates $\tilde{N}(k_m,k)$ and $\tilde{\Lambda}(\nu)$ are more sensitive to the velocity terms than their closure counterparts and because these changes in the rates change the values of k_m and ν obtained from the intersection criterion by about 1 MeV (cf. Table I) which in turn produces a change in the contribution of the leading terms.

Finally, we should discuss the results obtained for k_m and ν from the intersection criterion and those for the physical averages $\bar{k}_m$ and $\bar{\nu}$ obtained by solving the consistency relations of Eqs. (17) and (18). These results are given in Table I where we show ν, k_m, k_m/ν, the ordinary and radiative rates and their ratio evaluated at k_m and ν, and the values $\bar{\nu}$, $\bar{k}_m$, and $\bar{k}_m/\bar{\nu}$, for several combinations of the corrections discussed above and for several values of k.

The most striking result is that for fixed α, the ratio k_m/ν is essentially constant even though k_m and ν vary individually by sizeable amounts and for some k the rates can vary by as much as factors of two. Thus it seems sensible to fit data by fixing k_m/ν and treating, say, k_m as a parameter, as was done in Ref. 16. For the standard Rood and Tolhoek[2] model, using simple harmonic oscillator wave functions and $\alpha = 1.15$, we see from the table that $k_m/\nu = 1.06 \pm 0.01$ is an appropriate value. For the less desirable choice $\alpha = 0$, k_m/ν is still fairly constant, but a bit lower, ~1.04.

These values of k_m/ν are somewhat larger than those used before. We can understand this by looking at the ratio of physical averages $\bar{k}_m/\bar{\nu}$ obtained by solving the consistency relations starting from the definitions of Eq. (16). The values are not quite as constant as was the case for k_m/ν, but still vary by only ±0.02. They are generally lower than k_m/ν. Values of k_m/ν used in previous fits to data have usually been obtained from calculations such as that of Rood and Tolhoek,[2] who got 1.02 for ^{16}O by comparing the closure result with that obtained by summing over partial transitions. Such a calculation starts with the definitions of physical averages [Eq. (16)] however, and so is really a calculation of $\bar{k}_m/\bar{\nu}$. Hence the agreement with our consistency result for $\bar{k}_m/\bar{\nu}$ is encouraging. The effect of the additional sum rule correction terms we have included is to increase slightly the value of k_m/ν appropriate for a closure calculation.

We should emphasize that the specific numbers, e.g., for k_m/ν, may depend on the model used, in this case the standard Rood and Tolhoek[2] harmonic oscillator model, though the qualitative features we presume are general. In a more complete calculation a number of additional corrections should be included, in particular the propagator corrections of Rood, Yano, and Yano,[17] the higher order terms suggested in Ref. 18, perhaps better wave functions, and perhaps the requirement of consistency with electromagnetic matrix elements as done in the GDR model.[8,10]

It is interesting to note however that, with $\alpha = 1.15$, the ordinary rate $3.33 \times 10^6\ s^{-1}$ is in fair agreement with the experimental values[16,19] $(2.29 \pm 0.06) \times 10^6\ s^{-1}$ and $(2.53 \pm 0.02) \times 10^6\ s^{-1}$. The radiative rate is also somewhat high, but presumedly will be reduced on the order of 20% by the RYY[17] correction, so that it may be possible to achieve qualitative agreement here as well. Alternatively, an increase in α ($\alpha = 1.15$ is a lower limit) reduces ordinary, radiative, and relative rates.

Thus to summarize, we have shown that the differential photon spectrum for radiative muon capture can be calculated in a way which is nearly independent of the value of the maximum photon energy in a fashion analogous to that for ordinary capture.[4] The procedure consists of expanding the expression for the photon spectrum to first order about k_m, then using closure to perform the sum over final states. Evaluation of one of the resulting correction terms is done using a modified TRK sum rule, leading to an expression for the photon spectrum which exhibits only slight dependence on k_m. The intersection of this result with the usual closure result determines values of k_m and ν for use in closure approximation calculations. From a practical point of view the most useful result may be that in the standard RT model[2] the sum rule correction gives values for k_m/ν which are essentially constant, and slightly larger than those used previously.

*Work supported in part by the National Research Council of Canada.
[1]H. Primakoff, Rev. Mod. Phys. 31, 802 (1959).
[2]H. P. C. Rood and H. A. Tolhoek, Nucl. Phys. 70, 658 (1965).
[3]G. Do Dang, Phys. Lett. 38B, 397 (1972).
[4]J. Bernabeu, Nucl. Phys. A201, 41 (1973); J. Bernabeu and F. Cannata, *ibid.* A215, 411 (1973).
[5]B. Goulard and H. Primakoff, Phys. Rev. C 10, 2034 (1974); 11, 1894 (1975).
[6]P. Christillin, A. Dellafiore, and M. Rosa-Clot, Phys. Rev. Lett. 31, 1012 (1973); Nuovo Cimento, 34A, 272 (1976); Phys. Rev. C 12, 691 (1975); F. Cannata and N. C. Mukhopadhyay, *ibid.* 10, 379 (1974); 12, 1096(E) (1975); R. Rosenfelder, Nucl. Phys. A290, 315 (1977); A. Mekjian, Phys. Rev. Lett. 36, 1242 (1976).
[7]T. E. O. Ericson and M. P. Locher, Nucl. Phys. A148, 1 (1970).
[8]H. W. Fearing, Phys. Rev. 146, 723 (1966).
[9]For a discussion of the validity of the generally assumed relation $M_V^2=M_A^2=M_P^2$, see Ref. 10 and also the comments in Ref. 5.
[10]L. L. Foldy and J. D. Walecka, Nuovo Cimento 34, 1026 (1964).
[11]J. S. Levinger and H. A. Bethe, Phys. Rev. 78, 115 (1950).
[12]J. S. Bell, Nucl. Phys. 4, 295 (1957).
[13]J. Ahrens, H. Borchert, K. H. Czock, H. B. Eppler, H. Gimm, H. Gundrum, M. Kröning, P. Riehn, G. Sita Ram, A. Zeiger, and B. Ziegler, Nucl. Phys. A251, 479 (1975).
[14]J. A. Nolen, Jr., and J. P. Schiffer, Annu. Rev. Nucl. Sci. 19, 471 (1969).
[15]R. Engfer, H. Schneuwly, J. L. Vuilleumier, H. K. Walter, and A. Zehnder, At. Data Nucl. Data Tables 14, 509 (1974).
[16]R. D. Hart, C. R. Cox, G. W. Dodson, M. Eckhause, J. R. Kane, M. S. Pandey, A. M. Rushton, R. T. Siegel, and R. E. Welsh, Phys. Rev. Lett. 39, 399 (1977).
[17]H. P. C. Rood, A. F. Yano, and F. B. Yano, Nucl. Phys. A228, 333 (1974).
[18]H. W. Fearing, Phys. Rev. Lett. 35, 79 (1975).
[19]L. DiLella *et al.*, Phys. Rev. Lett. 27, 830 (1971); W. A. Cramer *et al.* Nuovo Cimento 24, 546 (1962); J. C. Sens, Phys. Rev. 113, 679 (1959).