# Scattering integral equations for distorted transition operators

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(Received 19 December 1977)

Methods for embedding phenomenological distorted-wave techniques for rearrangement and inelastic scattering within well-defined theories of multiparticle scattering are developed. The essential point of contact between the two approaches is in the definition and choice of distorting potential. It is shown that the concept of a channel coupling scheme allows a comparative freedom of choice for these potentials; if they are connected operators, such as optical potentials, then it is possible to obtain connected-kernel equations for the distorted transition operators. The latter are introduced in the course of exploiting the two-potential formula for the full transition operator and have the property that their matrix elements with respect to distorted waves are the physical scattering amplitudes. It is found that the distorted counterparts of the Kouri, Levin, and Tobocman and the Bencze-Redish integral equations maintain their connected-kernel and minimally coupled properties. These equations can be used to derive other integral equations with the same properties for the distorted-wave operators which consist of the product of the distorted transition operators and the wave operators corresponding to distorted waves. These simplifications are not realized for arbitrary channel coupling schemes. In order to deal with the general situation an alternative approach employing a subtraction technique which involves projections on the bound two-cluster channel states is introduced. When the distorting potentials are essentially the optical potentials in the entrance and exit channels a set of multichannel two-particle Lippmann-Schwinger integral equations for the two-cluster distorted-wave transition operators are obtained. The input into these two-particle integral equations involves the solution of a modified $N$-particle equation. Approximations to the latter are discussed in the particular cases of the Kouri, Levin, and Tobocman and Bencze-Redish channel coupling schemes. The inhomogeneous terms of all of the integral equations investigated for the distorted transition operator are of particular interest in connection with the distorted-wave Born approximation and modifications to it.

[NUCLEAR REACTIONS Distorted-wave techniques, $N$-particle scattering, integral equations, channel coupling, distorted-wave Born-approximation, scattering theory.]

## I. INTRODUCTION

The distorted-wave Born approximation (DWBA)[1] and its many variants and extensions[2] are highly useful phenomenological tools for the description of nuclear rearrangement and inelastic collisions. Attempts at the justification of this class of approximations are legion.[1-3] However, only recently have we begun to learn just what is involved in doing this within a completely well-defined formulation of multiparticle scattering.[4-7] It is only in the latter context that the magnitude of neglected terms can be reliably estimated—at least in principle. In practice it is unlikely that this opportunity will be exploited very frequently and thereby provide compelling reasons for the use of such formalisms.

However, successful approximation methods in collision theory ultimately depend upon the recognition of the dominant structural aspects of the underlying scattering integral equations for the class of processes of interest. A well-defined set[8-11] of such equations would seem to be indispensable for the formulation of optimal versions of techniques such as the DWBA as well as improvements upon them.

In view of the practical success of distorted-wave methods, multiparticle scattering theories which retain some contact with the phenomenological spirit of such approaches are of evident interest. We have in mind particularly the flexibility present in early formulations with regard to the introduction of distorting potentials.[1-3] This is important for two related reasons. First, the distorting potentials are customarily regarded as subsidiary to the calculation of interest, i.e., they are either determined phenomenologically or obtained elsewhere. Second, in any meaningful comparison between a typical heuristically formulated distorted-wave prescription and the type of complete scattering formalism we have in mind, the meaning and use of the distorting potential must be identical in the two instances.

The introduction of arbitrary distorted potentials into the Kouri, Levin, and Tobocman (KLT)[12,13] scattering integral equations was suggested by Tobocman[13] and elaborated upon in more detail by Kouri and Levin.[6] The starting point is es-

 

sentially the same as in earlier treatments.[1-3] Namely, a generalization of the two-potential formula[3,14-16] is first used to derive an expression for the rearrangement amplitude which consists of a distorted transition operator sandwiched between incoming and outgoing distorted waves. The distorted transition operator is essentially the usual transition operator in which the interactions between the fragments (usually two) in the entrance and exit channels have been modified by distorting potentials. The next step is to obtain a set of well-defined integral equations for the distorted transition operators. This was done by KLT,[6,13] but the attributes of the distorting potentials needed to preserve the connected kernel properties of these equations were not explored extensively.

A very interesting distorted-wave theory of rearrangement collisions has been formulated by Redish[5] using the connected-kernel scattering integral equations of Bencze and Redish (BR).[17,18] Only a restricted class of distorting potentials enter into Redish's formalism. One of the original motivations for the present work was to investigate how this class could be enlarged to include, for example, the type of optical potential used in many practical calculations of rearrangement collisions.

Following an exposition of our notation in Sec. II, we demonstrate in Sec. III that the concept of a *channel-coupling scheme*[10] allows the combination of a comparative freedom of choice of distorting potential with the possibility of well-defined scattering integral equations for the distorted transition operators. The BR and KLT scattering integral equations correspond to special choices of such schemes.

The original (undistorted) KLT and BR scattering integral equations have the very desirable property of being minimally coupled.[19] We mean by this the situation where the integral equations for the two-cluster transition operators are closed with respect to other types of channels, and the transition operators representing entrance or exit channels with more than two clusters can be obtained without the solution of any additional integral equations. Evidently this is a desirable property to retain for the integral equations satisfied by the distorted transition operators, but it certainly will not be realized without some restrictions on the distorting potentials. In view of the dominant interest in reactions with two fragments in the initial and final channels, it is reasonable to limit the distorting potentials to those which vanish in all channels corresponding to more than two clusters. In Sec. IV it is shown that with these constraints the distorted counterparts of the KLT and BR integral equations remain minimally coupled. The proof for the KLT-type equations is straightforward and is implicit in the work of Refs. 6 and 13. The BR case is more complex and requires the properties of the BR kernel established in Ref. 19.

Moreover, in both the KLT and BR cases it is demonstrated in Sec. IV that if the array for the distorting potential is diagonal, then minimal, closed equations obtain for the so-called distorted-wave transition operator. The latter is the product operator consisting of the distorted transition operator lying between the wave operators which generate the distortions in the entrance and exit channels. This is the object of direct relevance in practical distorted-wave calculations since its matrix elements with respect to channel states, rather than distorted channel states, yield the scattering amplitudes. The kernels in the integral equations for the distorted transition operator are shown to be the same as in the case without distortion.

The simplifications of Sec. IV are not realized for an arbitrary channel coupling scheme. However, when the distorting interactions are essentially the optical potentials within the relevant channels it is possible to derive closed, coupled-channel equations for the distorted-wave transition operator. This is done in Sec. V. These integral equations are of the multichannel Lippmann-Schwinger variety with only two-particle channels. The input into these equations is expressed in terms of an auxiliary quantity which is itself the solution of an $N$-particle integral equation. The structure of this subsidiary $N$-particle equation and some classes of approximations to it are considered in Sec. VI.

The primary purpose of this paper is to deal with the technical problems associated with the introduction of distorting potentials into well-defined theories of multiparticle scattering. The important subsequent questions concerning the justification and improvement of the various types of distorted-wave approximations are considered here only briefly. However, the present work provides a well-defined basis for these investigations.

## II. PARTITIONS, INTERACTIONS, AND SCATTERING OPERATORS

A nuclear rearrangement collision even of the simplest sort is a multiparticle scattering problem. This fact is the starting point of any complete description of such a process. One's practical objective, of course, is to learn how to suppress all the relatively unimportant aspects of the full complexity of the $N$-particle problem.

A related goal is to construct an approach which is capable of utilizing information about the system and its subsystems obtained in other contexts and perhaps by phenomenological means. We have in mind, in particular, various bound-state wave functions and optical potentials.

The discussion of $N$-particle scattering is made much more convenient by the notion of a partition, $a_m$, of the $N$ particles into $m$ clusters, where $m=1,\ldots,N$.[20-21] This is merely a subdivision of the $N$ particles into groupings called clusters within which the arrangement of the particles is irrelevant. As an illustration as well as to facilitate the later use of this case as an example we list and identify the distinct partitions for $N=4$: $a_1=(1234)$, $a_2=(1)(234)$, $b_2=(2)(134)$, $c_2=(3)(124)$, $d_2=(4)(123)$, $e_2=(12)(34)$, $f_2=(13)(24)$, $g_2=(14)(23)$, $a_3=(1)(2)(34)$, $b_3=(1)(3)(24)$, $c_3=(1)(4)(23)$, $d_3=(12)(3)(4)$, $e_3=(13)(2)(4)$, $f_3=(14)(2)(3)$, and $a_4=(1)(2)(3)(4)$.

We associate with each partition $a_m$ a channel[22] Hamiltonian

$$H_{a_m}=H_0+V_{a_m}\,, \tag{2.1}$$

which consists of $H_0$, the kinetic energy operator of the $N$ particles plus $V_{a_m}$, the sum of the interactions of the particles within each of the $m$ clusters among themselves. No interactions between particles in different clusters are included in $V_{a_m}$. For example $V_{e_2}$ consists only of the pair interactions between particles 1 and 2 and between particles 3 and 4. Thus, each partition $a_m$ represents a physical system consisting of $m$ disjoint groups of interacting particles.

The particles within a given cluster may or may not be capable of forming a bound state. However, all of the asymptotic states of an $N$-particle system correspond to partitions in which every cluster containing more than one particle is in a bound state of the particles within that cluster. The partition index itself, however, does not distinguish among possible different bound states of a cluster.

The single partitions $a_1$ and $a_N$ correspond to the extreme situations of the fully interacting $N$-particle system and a system of $N$ noninteracting particles, respectively. Clearly

$$H_{a_1}=H\,,$$

where $H$ is the full Hamiltonian, while

$$H_{a_N}=H_0\,.$$

Evidently $a_1$ cannot correspond to an asymptotic state of the system while $a_N$ can.

In general arbitrary partitions are denoted by lower case Latin letters $a,b,c,\ldots$. When it is necessary, the number of clusters is identified by a subscript as in $a_m$. Two-cluster and $(N-1)$-cluster partitions play special roles in our work; they are denoted, alternatively, by lower case Greek letters $\alpha,\beta,\ldots$, and primed Latin letters $i',j',\ldots$, respectively.[19]

The total $N$-particle interaction $V$ can be decomposed into the partition degenerate form[19]

$$V=V^b+V_b\,, \tag{2.2}$$

where we regard (2.2) as the definition of the interaction $V^b$ among the clusters appropriate to the partition $b$. Then the transition operators for the scattering from channel $a$ to channel $b$ can be taken as any one of the on-shell equivalent operators

$$T^{(+)}_{b,a}=V^b+V^bGV^a\,, \tag{2.3a}$$

$$T^{(-)}_{b,a}=V^a+V^bGV^a\,, \tag{2.3b}$$

or

$$\begin{aligned}U_{ba}&=\bar{\delta}_{b,a}G_a^{-1}+T^{(+)}_{b,a}\\&=G_b^{-1}\bar{\delta}_{b,a}+T^{(-)}_{b,a}\,,\end{aligned} \tag{2.3c}$$

where $\bar{\delta}_{b,a}=1-\delta_{b,a}$ and

$$G=(z-H)^{-1}$$

and

$$G_a=(z-H_a)^{-1}$$

refer to the total and channel Green's functions, respectively. We consistently suppress the dependence upon the complex energy parameter $z$.

The distorted-wave viewpoint begins upon rewriting any one of Eqs. (2.3) in terms of a distorting potential $V_1^b$ which is subtracted off from $V^b$. Then, for example, as shown in Ref. 15, $U_{b,a}$ admits of the fully symmetric two-potential form

$$U_{b,a}=V_1^b\Omega_b(1)\,\delta_{b,a}+\Omega_b^{tr}(1)\,U^{(2)}_{b,a}\Omega_a(1)\,, \tag{2.4}$$

where

$$\Omega_a(1)\equiv G_a(1)\,G_a^{-1} \tag{2.5a}$$

and

$$\Omega_b^{tr}(1)\equiv G_b^{-1}G_b(1) \tag{2.5b}$$

are wave operators corresponding to the distorting potentials $V_1^a$ and $V_1^b$, respectively with, e.g.,

$$G_a(1)^{-1}=G_a^{-1}-V_1^a\,.$$

$U^{(2)}_{b,a}$ is the symmetric version of what we refer to as a *distorted transition operator*. It is defined by

$$U^{(2)}_{b,a}=\bar{\delta}_{b,a}G_a(1)^{-1}+T^{(+)}_{b,a}(2)\,, \tag{2.6a}$$

$$U^{(2)}_{b,a}=G_b(1)^{-1}\bar{\delta}_{b,a}+T^{(-)}_{b,a}(2)\,, \tag{2.6b}$$

where

$$T^{(+)}_{b,a}(2) = V^b_2 + V^b_2 G V^a_2, \tag{2.7a}$$

$$T^{(-)}_{b,a}(2) = V^a_2 + V^b_2 G V^a_2, \tag{2.7b}$$

and the potential $V^b_2$ is defined by

$$V^b_2 \equiv V^b - V^b_1. \tag{2.8}$$

Let $|\phi_{a_m}(\bar{\eta})\rangle$ denote an eigenstate of $H_{a_m}$ corresponding to the situation in which the groups of particles in each of the $m$ clusters form bound states. The index $\bar{\eta}$ refers to any other variables which are necessary to specify this state completely including $m$ indices which label the different possible bound states of each of the clusters. We call $|\phi_{a_m}(\bar{\eta})\rangle$ an *m-cluster bound state* of the $N$-particle system. A partition $a_m$ for which such a bound state exists is referred to as a *stable* partition.[23] Then the rearrangement amplitude $R_{b,a}$ corresponding to two unequal stable partitions is given by the equivalent on-shell distorted-wave matrix elements

$$R_{b,a} = \langle \chi^{(-)}_b(\bar{\eta}) | U^{(2)}_{b,a} | \chi^{(+)}_a(\bar{\eta}) \rangle$$

$$= \langle \chi^{(-)}_b(\bar{\eta}) | T^{(\pm)}_{b,a}(2) | \chi^{(+)}_a(\bar{\eta}) \rangle, \tag{2.9}$$

where the distorted-wave vectors are defined by

$$|\chi^{(+)}_a(\bar{\eta})\rangle \equiv \Omega_a(1) |\phi_a(\bar{\eta})\rangle, \tag{2.10a}$$

$$\langle \chi^{(-)}_b(\bar{\eta})| \equiv \langle \phi_b(\bar{\eta}) | \Omega^{tr}_b(1). \tag{2.10b}$$

Equations (2.9) express what is usually referred to as the on-shell equivalence of the distorted-wave matrix elements of $T^{(\pm)}_{b,a}(2)$ and $U^{(2)}_{b,a}$. This attribute is in general not preserved in the course of making approximations to each of the operators. This has given rise to a great deal of discussion in the past and it is certainly a point of some importance in specific applications, although it is not of direct concern to us in this paper. We do demonstrate that the integral equations satisfied by all three sets of operators have the same kernels so that the off-shell inequivalence is manifested solely in the Born terms of these equations.

## III. CONNECTED-KERNEL EQUATIONS FOR THE DISTORTED TRANSITION OPERATOR

The discussion of general classes of connected-kernel[24] $N$-particle scattering integral equations is facilitated by the introduction of the concept of the channel-coupling scheme.[9, 10, 19] We now adapt this idea to the distorted transition operators defined by Eqs. (2.6) and Eqs. (2.7).

We suppose that we have a channel-coupling scheme appropriate to the full transition operators defined by Eqs. (2.3). By this we mean that there is an array of partition-labeled operators $\mathcal{U}^{a,c}$ such that

$$V^a = \sum_c \mathcal{U}^{a,c}, \tag{3.1}$$

where the sum is over all partitions of the $N$-particle system. We assume that this array has the property that $(\mathcal{U}\hat{G})^p$ is a connected operator for some positive integer $p \geq 1$, where we have introduced a matrix notation with respect to the partition indices. Namely, $\mathcal{U}$ denotes the matrix whose elements are the operators $\mathcal{U}^{a,c}$ and $\hat{G}$ is the diagonal matrix with elements $G_b \delta_{b,a}$. Then, if we let $\overline{V}$ be the diagonal matrix $(V^b \delta_{b,a})$, Eq. (3.1) can be expressed in matrix form as[19]

$$\overline{V}(1 + \bar{\delta}) = \mathcal{U}(1 + \bar{\delta}), \tag{3.1'}$$

where it follows from the definition of $\bar{\delta} = (\bar{\delta}_{b,a})$ that $(1 + \bar{\delta})$ is a matrix in the partition indices all of whose elements are equal to unity.

Using Eqs. (2.3) and Eqs. (3.1), one can easily derive sets of coupled integral equations for each of the sets of operators $(T^{(+)}_{b,a})$, $(T^{(-)}_{b,a})$, and $(U_{b,a})$.[10, 19] In matrix form each of these sets of equations possesses the same kernel $\mathcal{U}\hat{G}$ so that, by hypothesis, these are connected-kernel equations.[24] Corresponding sets of equations but of the "other way around" variety with kernels $\hat{G}\mathcal{U}^t$ also obtain. Here $\mathcal{U}^t$ denotes the transpose of $\mathcal{U}$ with respect to the partition indices.

The similarity of the forms (2.7) for $T^{(\pm)}_{b,a}(2)$ with Eqs. (2.3) for $T^{(\pm)}_{b,a}$ as well as the similarity of Eqs. (2.3c) and Eqs. (2.6) suggests that we introduce a channel-coupling scheme $\mathcal{U}_2$ appropriate to $\overline{V}_2 = (V^b_2 \delta_{b,a})$ so that instead of (3.1') we have

$$\overline{V}_2(1 + \bar{\delta}) = \mathcal{U}_2(1 + \bar{\delta}). \tag{3.2}$$

In view of definition (2.8) we see that the introduction of $\mathcal{U}_2$ entails the consideration of an array $\mathcal{U}_1$ corresponding to the distorting potential $\overline{V}_1 = (V^b_1 \delta_{b,a})$ so that

$$\overline{V}_1(1 + \bar{\delta}) = \mathcal{U}_1(1 + \bar{\delta}) \tag{3.3}$$

and consequently

$$\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2. \tag{3.4}$$

The choices of the arrays $\mathcal{U}$ and $\mathcal{U}_{1,2}$ are nonunique but are constrained by Eqs. (3.1)–(3.4) as well as connected-kernel requirements.

One finds from Eq. (2.7a) using Eq. (3.2) that $T^{(+)}_2 = [T^{(+)}_{b,a}(2)]$ satisfies a set of coupled integral equations which can be written concisely in matrix form as

$$T^{(+)}_2 = \mathcal{U}_2 G_1 (1 + \bar{\delta}) G_1^{-1} + \mathcal{U}_2 G_1 T^{(+)}_2, \tag{3.5}$$

where $G_1 = [G_b(1)\delta_{b,a}]$. The operators $T^{(-)}_2$ and $U_2$ satisfy equations differing from (3.5) only in the inhomogeneous terms.

In general we have no assurance that $\mathcal{U}_2 G_1$ will be a connected kernel.[24] However, if the distorting potential $\overline{V}_1$ is a connected operator we can confine ourselves only to those arrays for which $\mathcal{U}_1$ is a connected operator. Then given that $\mathcal{U}\hat{G}$ is a connected kernel it follows that $\mathcal{U}_2 G_1$ possesses this attribute as well. It is instructive to demonstrate this. Let us write $G_1$ in the form

$$G_1 = \hat{G} + \hat{G}\, T_1 \hat{G}\,, \tag{3.6a}$$

where

$$T_1 = \overline{V}_1 + \overline{V}_1 G_1 \overline{V}_1\,. \tag{3.6b}$$

Then the kernel $\mathcal{U}_2 G_1$ decomposes into

$$\mathcal{U}_2 G_1 = \mathcal{U}\hat{G} + K_{\rm con}\,,$$

where

$$K_{\rm con} = \mathcal{U}\hat{G}\, T_1 \hat{G} - \overline{\mathcal{U}}_1 G_1$$

is a fully connected operator. Thus, if $(\mathcal{U}\hat{G})^p$ is connected for some $p \geq 1$ then it follows that $(\mathcal{U}_2 G_1)^p$ is connected as well since the cross terms involving $K_{\rm con}$ all have this property.

We now have sufficient conditions for $T_2^{(+)}$ to satisfy connected-kernel equations of the form (3.5). Moreover, these conditions can be satisfied in practical applications. First, arrays of the form $\mathcal{U}$ exist with the stipulated properties.[10,19] Second, because most applications of distorted-wave approximations are confined to two-cluster entrance and exit channels corresponding to stable partitions we can take

$$V_1^{b_k} = 0$$

For $k > 2$ or for any two-cluster partitions $b_2$ which are not stable. Typical choices for $V_1^\alpha$ are connected operators. We can then confine ourselves to arrays $\mathcal{U}_1$ such that

$$\mathcal{U}_1^{b_k, a_m} = 0 \tag{3.7}$$

if $k > 2$ or $m > 2$.

One choice for $V_1^\alpha$ deserves special mention. This is the case in which we identify $V_1^\alpha$ for a stable two-cluster partition $\alpha$ with the full optical potential, $V_{\rm opt}^\alpha$, for elastic scattering in channel $\alpha$. $V_{\rm opt}^\alpha$ is an operator with the structure

$$V_{\rm opt}^\alpha = P_\alpha \mathcal{U}_\alpha P_\alpha\,, \tag{3.8}$$

where $P_\alpha$ is the projection operator onto the space spanned by the two-cluster bound states $|\phi_\alpha(\overline{\eta})\rangle$.[25,26] We recall that the latter state includes the relative momentum of the two bound clusters so that $P_\alpha$ involves an integration over this variable. The operator $\mathcal{U}_\alpha$ which appears in (3.8) consists only of various interaction linkages between particles lying in different clusters and is such that the operator $\tau_\alpha$ defined by

$$\tau_\alpha = V_{\rm opt}^\alpha + V_{\rm opt}^\alpha g_\alpha \tau_\alpha \tag{3.9a}$$

$$= V_{\rm opt}^\alpha + \tau_\alpha g_\alpha V_{\rm opt}^\alpha\,, \tag{3.9b}$$

where

$$g_\alpha \equiv P_\alpha G_\alpha = G_\alpha P_\alpha\,,$$

has on-shell matrix elements equal to the elastic scattering amplitude in channel $\alpha$:

$$\langle \phi_\alpha(\overline{\eta}') | T_{\alpha,\alpha}^{(\pm)} | \phi_\alpha(\overline{\eta}) \rangle = \langle \phi_\alpha(\overline{\eta}') | \tau_\alpha | \phi_\alpha(\overline{\eta}) \rangle\,,$$

where $\overline{\eta}'$ refers to the same bound states of each of the two clusters as does $\overline{\eta}$. A particular realization of $\mathcal{U}_\alpha$ is[26,27]

$$\mathcal{U}_\alpha = V^\alpha + V^\alpha Q_\alpha \left( \frac{1}{G_\alpha^{-1} - Q_\alpha V^\alpha Q_\alpha} \right) Q_\alpha V^\alpha\,, \tag{3.10}$$

where $Q_\alpha = 1 - P_\alpha$. The connected structure of the operator $V_{\rm opt}^\alpha$ is illustrated diagrammatically in Fig. 1.

We let $V_{\rm opt}$ refer to a diagonal matrix whose only nonzero elements $V_{\rm opt}^\alpha$ correspond to stable partitions $\alpha$. In Sec. V specific choices for the arrays $\mathcal{U}_1$ corresponding to $V_{\rm opt}$ are proposed.

When $V_{\rm opt}$ is identified with the distorting potential $\overline{V}_1$ in practical distorted-wave calculations of rearrangement scattering its elements $V_{\rm opt}^\alpha$ are usually either determined phenomenologically or in any case are regarded as given. In accord with this we do not take up the question of calculation of this quantity in the present work.

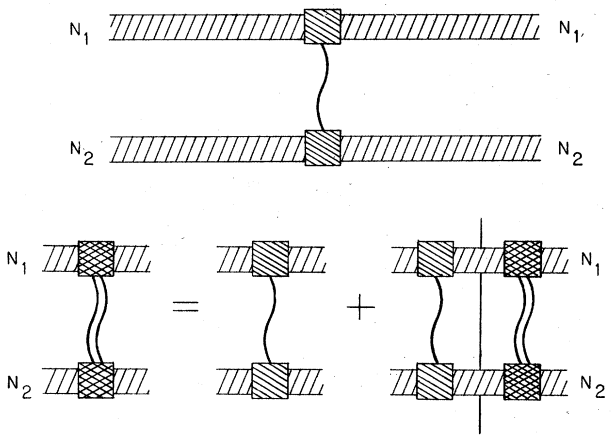


FIG. 1. (a) Connectedness of the optical potential operator $V_{\rm opt}^\alpha$. (b) The relation of $V_{\rm opt}^\alpha$ to the elastic scattering transition operator $\tau_\alpha$. The cross hatched lines represent bound clusters of $N_1$ and $N_2$ particles with $N_1 + N_2 = N$. Cross-hatched squares connected by a single wavy line correspond to the action of the operator $\mathcal{U}_\alpha$, while cross-hatched squares connected by a double wavy line represent the operator $\tau_\alpha$. The vertical line depicts the Green's operator $g_\alpha$.

## IV. MINIMALLY COUPLED DISTORTED TRANSITION OPERATOR EQUATIONS

We suppose, henceforth, that $\mathcal{U}$ and $\mathcal{U}_1$ have been chosen so that Eq. (3.5) represents a set of connected-kernel integral equations for the operators $T^{(+)}_{b,a}(2)$. However, the kernel term of (3.5) contains a sum over all partitions so that these equations couple together transition operators to all possible channels. On the other hand, for the KLT or BR coupling schemes the full transition operators $T^{(+)}_{b,a}$ defined by Eq. (2.3a) satisfy integral equations similar to (3.5) with this sum limited to two-cluster channels. As a consequence the integral equations for the two-cluster transition operators are closed among themselves and operators such as $T^{(+)}_{b_k,\alpha}$, $k>2$, can be calculated without solving any additional integral equations.[10,12,13,17-19] Such sets of integral equations are called *minimally coupled*[19,28] and it is evident that this property is of major practical importance.

There are implicit criteria of simplicity attached to this nomenclature as well as the requirement that the connected-kernel property be preserved. It appears difficult to characterize the generation of minimally coupled equations in a general fashion and still satisfy these criteria. For example, if we define a projector onto the space of two-cluster partitions,

$$(\mathfrak{z})_{b_k,a_m} \equiv \delta_{b_k,a_m}\delta_{k,2},$$

we can rewrite (3.5) as

$$T^{(+)}_2 = \Lambda G_1(1+\bar{\delta})G_1^{-1} + \Lambda G_1 \mathfrak{z}\, T^{(+)}_2, \tag{4.1a}$$

where $\Lambda$ is defined as the solution of

$$\Lambda = \mathcal{U}_2 + \mathcal{U}_2\hat{G}(1-\mathfrak{z})\Lambda. \tag{4.1b}$$

Here we have presumed the validity of Eq. (3.7), so that $[G_1,\mathfrak{z}]=0$ and $G_1(1-\mathfrak{z})=\hat{G}(1-\mathfrak{z})$. We observe that Eq. (4.1a) is certainly minimally coupled from the point of view of closure. However, we have picked up another (nonminimal) integral equation in Eq. (4.1b) and we have no assurance that the kernels of Eqs. (4.1) will be of the connected variety. Also, one can reduce Eqs. (3.5) to equations which are minimally coupled, by the substitution of the (Lippmann-Schwinger) identities implied by Eqs. (3.5) with diagonal $\mathcal{U}_2$. The resultant equations, however, have more complicated kernels which do not necessarily retain the connected-kernel property and may admit spurious solutions.[10,28]

As a consequence of the preceding remarks we confine ourselves in this section to the special cases of the KLT and BR coupling schemes which are represented by the arrays $\mathcal{U}_{\mathrm{KLT}}$ and $\mathcal{U}_{\mathrm{BR}}$, respectively. In Sec. V we develop a method similar to that represented by Eqs. (4.1) for dealing with the problem of an arbitrary channel-coupling scheme as well as these particular arrays.

### A. KLT channel-coupling scheme

In their original work KLT[12,13] consider only two-cluster channel coupling schemes. This is easily extended, however, and we take

$$\mathcal{U}^{b_k,a_m}_{\mathrm{KLT}} = V^{b_k} W_{b_k,a_m}, \tag{4.2}$$

where $W_{b_k,a_m}$ is a channel-coupling array which by definition[29] satisfies

$$\sum_a W_{b_k,a} = 1$$

and the sum is over all partitions. In order to ensure the connected-kernel property of $\mathcal{U}_{\mathrm{KLT}}\hat{G}$ as well as the minimal coupling of the equations for $T^{(+)}$ we take the two-cluster subarray $W_{\beta,\alpha}$ to form a channel permuting array[6,12,13,30] and for $k>2$ we require that

$$W_{b_k,a_m} = 0$$

unless $m=2$. We stipulate that for each $b_k$ with $k>2$, there is one and only one two-cluster partition, $\gamma(b_k)$, such that[31]

$$W_{b_k,\gamma(b_k)} = 1.$$

If we choose $\mathcal{U}_1$ so that (3.7) is satisfied, then Eq. (4.2) along with our choice for $W_{b,a}$ implies that

$$\mathcal{U}^{b_k,a_m}_{2(\mathrm{KLT})} = 0, \quad \text{if } m>2 \tag{4.3a}$$

which can be expressed as

$$\mathcal{U}_{2(\mathrm{KLT})} = \mathcal{U}_{2(\mathrm{KLT})}\mathfrak{z}. \tag{4.3b}$$

In this instance Eq. (3.5) becomes

$$T^{(+)}_2 = \bar{\mathcal{U}}_{2(\mathrm{KLT})} + \mathcal{U}_{2(\mathrm{KLT})} G_1 \mathfrak{z}\, T^{(+)}_2, \tag{4.4}$$

where

$$\bar{\mathcal{U}}_{2(\mathrm{KLT})} = \mathcal{U}_{2(\mathrm{KLT})}\mathfrak{z}G_1(1+\bar{\delta})G_1^{-1}. \tag{4.5}$$

From (4.4) we easily see that we obtain a closed system of equations for the two-cluster operators, namely,

$$\mathfrak{z}\, T^{(+)}_2\, \mathfrak{z} = \mathfrak{z}\bar{\mathcal{U}}_{2(\mathrm{KLT})}\mathfrak{z} + \mathfrak{z}\mathcal{U}_{2(\mathrm{KLT})} G_1 \mathfrak{z}\, T^{(+)}_2\, \mathfrak{z}, \tag{4.6a}$$

or, when written out explicitly in terms of the partition indices,

$$T^{(+)}_{\beta,\alpha}(2) = \sum_\gamma \mathcal{U}^{\beta,\gamma}_{2(\mathrm{KLT})} G_\gamma(1) G_\alpha(1)^{-1} + \sum_\gamma \mathcal{U}^{\beta,\gamma}_{2(\mathrm{KLT})} G_\gamma(1)\, T^{(+)}_{\gamma,\alpha}(2). \tag{4.6b}$$

Equations (4.6) can be simplified if we keep in mind that our ultimate goal is the calculation of the on-shell matrix elements $R_{\beta,\alpha}$, which are defined by (2.9). First, as a consequence of the half-on-shell Lippmann identity,[32]

$$G_b G_\alpha^{-1} |\phi_\alpha(\overline{\eta})\rangle = \delta_{b,\alpha} |\phi_\alpha(\overline{\eta})\rangle , \tag{4.7a}$$

and its analog for the distorted-wave vectors,

$$G_b(1) G_\alpha(1)^{-1} |\chi_\alpha^{(+)}(\overline{\eta})\rangle = \delta_{b,\alpha} |\chi_\alpha^{(+)}(\overline{\eta})\rangle , \tag{4.7b}$$

the Born terms in Eqs. (4.6) simplify significantly as they do in the undistorted case.[6,30] Indeed, one finds that

$$(\overline{v}_{2(\mathrm{KLT})})_{b,\alpha} |\chi_\alpha^{(+)}(\overline{\eta})\rangle = (v_{\mathrm{KLT}} - v_1)_{b,\alpha} |\chi_\alpha^{(+)}(\overline{\eta})\rangle . \tag{4.8}$$

As a result of (4.8) we can consider instead of $T_2^{(+)}$ the half-on-shell equivalent operators, which satisfy Eqs. (4.6) with $\overline{v}_{2(\mathrm{KLT})}$ replaced by $(v_{\mathrm{KLT}} - v_1)$.

A more important simplification derives from the observation that Eqs. (4.6) do not directly involve what we call the *distorted-wave transition operators*,

$$T_{\mathrm{DW}} \equiv \omega^{(-)\dagger} T_2^{(+)} \omega^{(+)} , \tag{4.9}$$

whose matrix elements with respect to the channel states are the amplitudes of ultimate interest. Here

$$\omega^{(+)} \equiv G_1 \hat{G}^{-1} \mathfrak{I} = \mathfrak{I}(1 + G_1 \overline{V}_1) \tag{4.10a}$$

and

$$\omega^{(-)\dagger} \equiv \mathfrak{I} \hat{G}^{-1} G_1 = (1 + \overline{V}_1 G_1) \mathfrak{I} \tag{4.10b}$$

are wave operators corresponding to the distorted-wave vectors $|\chi^{(+)}\rangle$ and $\langle\chi^{(-)}|$, respectively. We note that on-shell

$$R_{\beta,\alpha} = \langle\phi_\beta(\overline{\eta}')| T_{\mathrm{DW}} |\phi_\alpha(\overline{\eta})\rangle .$$

It is easily demonstrated starting from (4.6a) that $T_{\mathrm{DW}}$ satisfies the minimal, connected-kernel integral equation

$$T_{\mathrm{DW}} = \omega^{(-)\dagger} \overline{v}_{2(\mathrm{KLT})} \omega^{(+)} + \omega^{(-)\dagger} v_{2(\mathrm{KLT})} (\hat{G} \mathfrak{I}) T_{\mathrm{DW}} . \tag{4.11}$$

We note, relevant to the connected-kernel aspect of (4.11), that $\omega^{(-)\dagger}$ differs from $\mathfrak{I}$ by a connected operator so that the arguments of Sec. III may be applied to show that (4.11) possesses a connected kernel.

If $v_1$ is diagonal, viz.,

$$v_1 = \overline{V}_1 , \tag{4.12}$$

which is by far the most common implicit choice for $v_1$,[1-3] one can obtain an integral equation for $T_{\mathrm{DW}}$ alternative to (4.11). We can best appreciate the implications of (4.12) by starting from Eq. (4.4) which can be rewritten as

$$[(1 - \mathfrak{I}) + \omega^{(-)\dagger}] T_2^{(+)} = \overline{v}_{2(\mathrm{KLT})} + v_{\mathrm{KLT}} (\hat{G} \mathfrak{I}) \omega^{(-)\dagger} T_2^{(+)} \tag{4.13}$$

through the use of Eqs. (4.10b) and (4.12). Multiplication of (4.13) on the right by $\omega^{(+)}$ and on the left by $\mathfrak{I}$ yields

$$T_{\mathrm{DW}} = \mathfrak{I} \overline{v}_{2(\mathrm{KLT})} \omega^{(+)} + \mathfrak{I} v_{\mathrm{KLT}} (\hat{G} \mathfrak{I}) T_{\mathrm{DW}} . \tag{4.14}$$

The minimal, connected-kernel integral equations, (4.11) and (4.14), for $T_{\mathrm{DW}}$ have several interesting properties. We note that the distorting potential appears only in the inhomogeneous term of (4.14) in constrast to Eqs. (4.6) and (4.11). As a consequence the kernel of (4.14) is precisely the same as for the undistorted transition operator. Equations (4.8) and (4.12) imply that the inhomogeneous term in (4.14) can be taken to be $(v_{\mathrm{KLT}} - \overline{V}_1)$ so that the on-shell "Born approximation," as dictated by Eq. (4.14), is given by the asymmetrical expression

$$R_{\beta,\alpha}^{(\mathrm{Born})} = \langle\phi_\beta(\overline{\eta}')| v_{\mathrm{KLT}}^{\beta,\alpha} - V_1^\beta \delta_{\beta,\alpha} |\chi_\alpha^{(+)}(\overline{\eta})\rangle . \tag{4.15a}$$

However the "Born approximation" to $R_{\beta,\alpha}$ as defined by Eq. (4.6) or Eq. (4.11) is, by way of comparison in the case that (4.12) is valid, given by

$$R_{\beta,\alpha}^{(\mathrm{Born})} = \langle\chi_\beta^{(-)}(\overline{\eta}')| v_{\mathrm{KLT}}^{\beta,\alpha} - V_1^\beta \delta_{\beta,\alpha} |\chi_\alpha^{(+)}(\overline{\eta})\rangle . \tag{4.15b}$$

Equation (4.15b) is more in accord with the usual DWBA prescriptions than is (4.15a). From the point of view of the integral equation (4.14), the approximation (4.15b) involves the inclusion of higher-order corrections, although it is not possible to state which of the two expressions (4.15) constitutes the better approximation to $R_{\beta,\alpha}$ without further information.

It is worth pointing out that the only property of the KLT coupling scheme, besides the connected-kernel attribute, which is essential to the preceding development is (4.3b). Therefore, Eqs. (4.6), (4.11) and (4.14) realize for *any* scheme for which (4.3b) holds. From the standpoint of Eqs. (4.1) we note that (4.3b) corresponds to the case $\Lambda = v_2$.

The KLT scheme itself is highly asymmetric. In fact, for a given $\alpha$ and choice of channel permuting array there will be only one $\beta \neq \alpha$, say $\overline{\beta}$, for which $v_{\mathrm{KLT}}^{\overline{\beta},\alpha} = V^{\overline{\beta}} \neq 0$. Thus the Born terms (4.15) with $\beta \neq \overline{\beta}$, $\beta \neq \alpha$ all vanish identically. The normal Born expressions for the $\beta \neq \overline{\beta}$ channels are contained implicitly in the kernel terms of Eqs. (4.6), (4.11) and (4.14) in a manner discussed in great detail by Kouri and Levin.[6] Obviously one can always regard the channel permuting

array to be such that $\bar{\beta}$ is the channel of computational interest. So for all practical purposes we can regard $\mathcal{U}^{\beta,\alpha}_{\mathrm{KLT}} - V^{\beta}_{1}\delta_{\beta,\alpha}$ as replaced by $V^{\beta} - V^{\beta}_{1}\delta_{\beta,\alpha}$ in Eqs. (4.15).

Our discussion of the conditions sufficient to preserve the connected-kernel property and the derivation of Eqs. (4.11) and (4.14) for the distorted-wave transition operators constitute an extension of the work contained in Ref. 6 concerning $T^{(+)}_{2}$ in the KLT case.

### B. BR channel-coupling scheme

In at least two ways the BR channel coupling scheme provides a more interesting example than the KLT case. First, it is a more symmetrical scheme than the latter and, second, nothing as simple as Eq. (4.3b) obtains in this instance. The BR scheme is defined by the identification[10, 19, 33, 34]

$$\mathcal{U}^{b_k,a_m}_{\mathrm{BR}} = V^{b_k}_{a_m}(-1)^m(m-1)!\,, \tag{4.16}$$

where $V^{b_k}_{a_m}$ is defined next.

We confine ourselves, for the sake of simplicity, to the situation where there are only pair interactions among the $N$ particles. In accordance with our notation introduced in Sec. II, $i'$ is a generic pair index and $V_{i'}$ represents the interaction between that pair of particles. Then $V^{b_k}_{a_m}$ is the sum of pair potentials such that both particles are entirely within one of the clusters of the partition $a_m$ while at most one particle is in any cluster of $b_k$. It is useful to express this as[19]

$$V^{b_k}_{a_m} = \sum_{i'} \bar{\delta}(b_k|i')\,V_{i'}\,\delta(a_m|i')\,,$$

where the sum is over all pairs. Here

$$\delta(a_m|b_k) = 1 \text{ if } a_m \supset b_k$$
$$= 0 \text{ otherwise}.$$

The notation $a_m \supset b_k$ means that the clusters which define $b_k$ are contained entirely within those of $a_m$, which can occur if $m \leq k$. Finally,

$$\bar{\delta}(a_m|b_k) \equiv 1 - \delta(a_m|b_k)\,.$$

It can be shown that the array $\mathcal{U}_{\mathrm{BR}}$ defined by Eq. (4.16) satisfies the constraints (3.1). The crucial property that $(\mathcal{U}_{\mathrm{BR}}\hat{G})^2$ is a connected operator is proved in Ref. 19. In fact, if we call $K_{\mathrm{BR}} \equiv \mathcal{U}_{\mathrm{BR}}\hat{G}$, then[19]

$$(K_{\mathrm{BR}})^2 = M G_0 \,\mathring{\jmath}\, K_{\mathrm{BR}}\,. \tag{4.17}$$

Here $G_0$ is the free $N$-particle Green's function or, equivalently, the channel Green's function corresponding to the partition $a_N$. $M$ is a somewhat more complicated object.[17-19, 33, 34]

$M_{b_k,a_m}$ is the $a_m$-connected part of the operator $T^{(+)}_{b_k,a_N}$. Equivalently, $M_{b_k,a_m}$ is the sum of all diagrams of connectivity $a_m$ which end in pair interactions $i'$ such that $b_k \not\supset i'$. We consider a few explicit examples of $M$ operators later; we note that $M_{b_k,a_m} = 0$ if $b_k \supset a_m$.

Equation (4.17) implies that $K_{\mathrm{BR}}$ is a connected kernel. This follows from the fact that $K_{\mathrm{BR}}$ admits of the representation[19]

$$K_{\mathrm{BR}} = M G_0 D^t\,, \tag{4.18}$$

where

$$(D^t)_{b_k,a_m} \equiv (-1)^m(m-1)!\,\delta(a_m|b_k)\,.$$

Then the right-hand side of (4.17) consists of a sum of products of operators of the form $M_{b,\alpha}G_0M_{\alpha,d}G_0$. But $M_{\alpha,d}$ is of connectivity $d$ and $\alpha \not\supset d$, so as a consequence no pair $i'$ of particles lying entirely within any of the (connected) clusters of $d$ is such that $\alpha \supset i'$. Hence, each of these pairs of particles travels to a distinct cluster of $\alpha$. Therefore, every cluster of $d$ containing more than a single particle is connected to both clusters of $\alpha$ and every single-particle cluster is connected to one of the clusters of $\alpha$. Thus, each of these products of operators is connected. This is illustrated in Fig. 2.

The minimal character of the original (undistorted) BR equations[17, 18] is also implied by Eq. (4.17).[19]

We next derive the counterpart of (4.17) for the distorted kernel

$$K_{2(\mathrm{BR})} \equiv \mathcal{U}_{2(\mathrm{BR})}G_1\,, \tag{4.19a}$$

where

$$\mathcal{U}_{2(\mathrm{BR})} \equiv \mathcal{U}_{\mathrm{BR}} - \mathcal{U}_1\,. \tag{4.19b}$$

If we use (4.17), the square of $K_{2(\mathrm{BR})}$ can be put

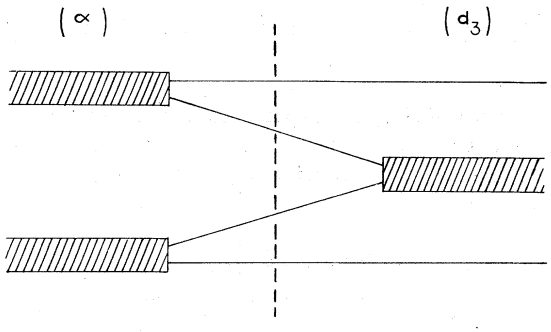


FIG. 2. Connectedness of $M_{b,\alpha}G_0M_{\alpha,d_3}$ for $\alpha \not\supset d_3$. The three clusters of $d_3$ consist of one $N-2$ particle cluster (cross hatched lines) and two single-particle clusters (horizontal solid lines). The two cross hatched lines on the left represent the two clusters of $\alpha$ and the dotted vertical line refers to $G_0$. The slanting lines emanating from the $N-2$ cluster of $d_3$ refer to a typical pair within that cluster and to their destination within the two clusters of $\alpha$.

into the form

$$(K_{2(\mathrm{BR})})^2 = A\,\mathfrak{z}\,K_{2(\mathrm{BR})} + B\,\mathfrak{z}\,, \tag{4.20}$$

where

$$A = MG_0 - \mathcal{V}_1 G_1 + K_{\mathrm{BR}} T_1 \hat{G}\,, \tag{4.21a}$$

$$B = (MG_0 - K_{\mathrm{BR}})\,\mathcal{V}_1 G_1\,, \tag{4.21b}$$

and we note that $T_1 = \mathfrak{z} T_1 \mathfrak{z}$ [cf. Eqs. (3.6)]. While it is obvious from (4.20) and (4.21) that $(K_{2(\mathrm{BR})})^2$ is a connected operator, it is somewhat less apparent that Eq. (4.20) also implies that Eq. (3.5) in the BR case, namely,

$$T_2^{(+)} = \overline{\mathcal{V}}_{2(\mathrm{BR})} + \mathcal{V}_{2\,\mathrm{BR})} G_1 T_2^{(+)}\,, \tag{4.22a}$$

where

$$\overline{\mathcal{V}}_{2(\mathrm{BR})} \equiv \mathcal{V}_{2(\mathrm{BR})} G_1 (1 + \bar{\delta}) G_1^{-1}\,, \tag{4.22b}$$

is minimally coupled.

The proof of the last assertion is carried out in a manner similar to the $\overline{V}_1 \equiv 0$ case considered in Ref. 19 and is facilitated by the introduction of the resolvent kernel, $\mathcal{R}_{2(\mathrm{BR})}$, corresponding to $K_{2(\mathrm{BR})}$:

$$\mathcal{R}_{2(\mathrm{BR})} = K_{2(\mathrm{BR})} + K_{2(\mathrm{BR})} \mathcal{R}_{2(\mathrm{BR})}\,, \tag{4.23}$$

so that

$$T_2^{(+)} = (1 + \mathcal{R}_{2(\mathrm{BR})})\,\overline{\mathcal{V}}_{2(\mathrm{BR})}\,. \tag{4.24}$$

Let us multiply Eq. (4.23) on the left by $K_{2(\mathrm{BR})}$. Using (4.20) we obtain the expression

$$K_{2(\mathrm{BR})}\,\mathcal{R}_{2(\mathrm{BR})} = (A + B)\,\mathfrak{z}\,\mathcal{R}_{2(\mathrm{BR})} + B\,\mathfrak{z}\,,$$

which inserted back into (4.23) transforms the latter into the minimally coupled equation

$$\mathcal{R}_{2(\mathrm{BR})} = K_{2(\mathrm{BR})} + B\,\mathfrak{z} + (A + B)\,\mathfrak{z}\,\mathcal{R}_{2(\mathrm{BR})}\,.$$

Equation (4.24) then implies that

$$T_2^{(+)} = [1 + (K_{2(\mathrm{BR})} - A\,\mathfrak{z})]\,\overline{\mathcal{V}}_{2(\mathrm{BR})} + (A + B)\,\mathfrak{z}\,T_2^{(+)}\,. \tag{4.25}$$

The kernel of this minimally coupled integral equation simplifies only when $\mathcal{V}_1$ is diagonal and in this instance we obtain using Eqs. (4.12) and (4.18)

$$T_2^{(+)} = [1 + MG_0(D^t - \mathfrak{z})]\,\overline{\mathcal{V}}_{2(\mathrm{BR})} + [MG_0 + (MG_0 - 1)\,\overline{V}_1 G_1]\,\mathfrak{z}\,T_2^{(+)}\,. \tag{4.25'}$$

We note that the structural form of the inhomogeneous term of (4.25) reduces to that of the corresponding term of Eq. (4.25′) whether or not (4.12) is valid.

We remark that Eqs. (4.1) do not lend themselves to a facile description of the reduction of Eq. (4.22a) to Eqs. (4.25). This is because the projector ($\mathfrak{z}$) complement ($1 - \mathfrak{z}$) split of the kernel $K_{2(\mathrm{BR})}$ is not an appropriate way of characterizing the cancellations implicit in Eq. (4.20).

As in the KLT case the inhomogeneous term of (4.25) can be simplified using (4.7b). Corresponding to Eq. (4.8) we have, half on shell,

$$(\overline{\mathcal{V}}_{2(\mathrm{BR})})_{b,\alpha}\,|\chi_\alpha^{(+)}(\overline{\eta})\rangle = (\mathcal{V}_{\mathrm{BR}} - \mathcal{V}_1)_{b,\alpha}\,|\chi_\alpha^{(+)}(\overline{\eta})\rangle\,. \tag{4.26}$$

Further simplification requires the use of the sum rule[19]

$$D^t\,\mathcal{V}_{\mathrm{BR}} = \mathfrak{z}\,\mathcal{V}_{\mathrm{BR}}\,.$$

Then if we call

$$\mathfrak{G} = [1 + MG_0(D^t - \mathfrak{z})]\,\mathcal{V}_{2(\mathrm{BR})}\,,$$

we see that

$$\mathfrak{G} = [\mathcal{V}_{2(\mathrm{BR})} - MG_0(D^t - \mathfrak{z})\,\mathcal{V}_1]\,, \tag{4.27}$$

so that half on shell the inhomogeneous term of Eq. (4.25) is

$$\mathfrak{G}_{c,\alpha}\,|\chi_\alpha^{(+)}(\overline{\eta})\rangle = \Big[\mathcal{V}_{\mathrm{BR}}^{c,\alpha} - \mathcal{V}_1^{c,\alpha} - \sum_{k=3}^{N-1} \sum_{(b_k)} \sum_{\gamma} M_{c,b_k} G_0\,\delta(\gamma\,|\,b_k)\,\mathcal{V}_1^{\gamma,\alpha}\Big] \times |\chi_\alpha^{(+)}(\overline{\eta})\rangle\,. \tag{4.28}$$

It follows then that $T_2^{(+)}$ is half-on-shell equivalent to an operator which satisfies (4.25) with its inhomogeneous term replaced by $\mathfrak{G}$ as given by (4.27). We note that $\mathcal{V}_{\mathrm{BR}}^{\beta,\alpha} = V_\alpha^\beta$ and $\mathcal{V}_1^{c,\alpha} = 0$ if $c$ is not a two-cluster partition.

If $N \geq 4$ the "Born" approximation,

$$\langle\chi_\beta^{(-)}(\overline{\eta}')\,|\mathfrak{G}_{\beta,\alpha}\,|\chi_\alpha^{(+)}(\overline{\eta})\rangle\,,$$

defined by Eqs. (4.25) and (4.28) contains in the various $M_{\beta,b_k}$ terms contributions from transition operators corresponding to subsystems of the $N$ particles. These operators possess a degree of disconnectedness ranging from three to $N-1$ clusters. Each of these terms represents the scattering by the distorting potential in the entrance channel followed by a quasi-final-state interaction of the particles within the clusters appropriate to an allowed partition $b_k$. Any pair of particles which subsequently emanates from one of these clusters splits up so that each particle ends up in a distinct cluster of particles in the exit channel $\beta$. Finally, there is an elastic interaction in the exit channel represented by $\langle\chi_\beta^{(-)}|$. Diagrammatic illustrations of a few of these terms for $N = 4$ are given in Fig. 3.

The generalization of the DWBA afforded by Eq. (4.28) involves the knowledge of the connected transition operators corresponding to subsystems of from two to $N-2$ particles. For $N = 4$ only the

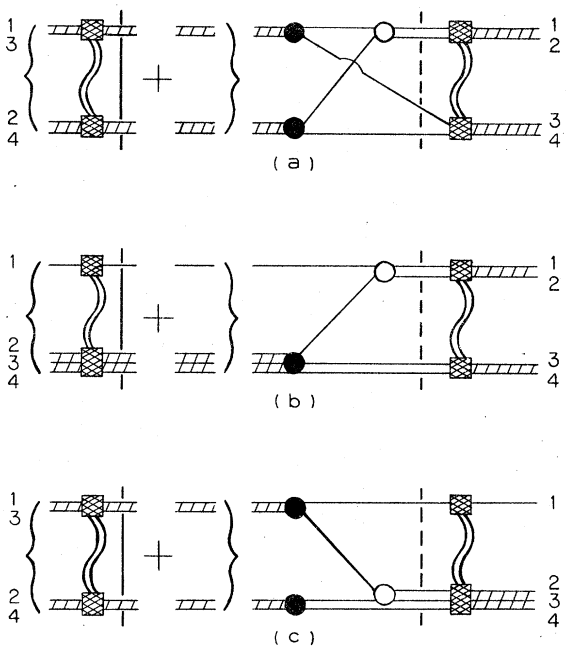


FIG. 3. Representation of $\langle\chi^{(-)}|MG_0\mathcal{U}_1|\chi^{(+)}\rangle$ terms for two-cluster to two-cluster transitions and $N=4$. The bound states, particle lines, Green's functions, and elastic scattering transition operators are denoted as in Figs. 1 and 2. The dark circles indicate bound-state vertices, while the open circles refer to the two-particle transition amplitudes which correspond to $M$ in this case. The distorted elastic interaction $\langle\chi^{(-)}|$ is pictured within the brackets. The action of $\mathcal{U}_1|\chi^{(+)}\rangle$ appears as an initial elastic scattering.

off-shell two-particle transition operators enter into (4.28), but for $N=5$ the completely off-shell connected three-particle amplitudes enter in as well. When $N$ is large the calculation of all of the terms in the sum over $k$ in (4.28) is nearly as difficult as the solution of the full $N$-particle scattering problem. Indeed, it is easily shown that[19]

$$\sum_{k=3}^{N-1}\sum_{(b_k)}\delta(\gamma|b_k)M_{\beta,b_k}=V_\gamma^\beta(1+G_\gamma V_\gamma)-M_{\beta,\gamma},$$

the evaluation of which is an $(N-2)$-body problem.

The exact calculation of the $k=N-1$ and $k=N-2$ terms is feasible. For $k<N-2$ the systematic approximation of the off-shell, multiparticle, subsystem amplitudes by their pole terms suggests itself as a practical possibility.[34,35]

Next let us develop integral equations for $T_{\mathrm{DW}}$. From Eq. (4.25) we find

$$T_2^{(+)}\omega^{(+)}=\overline{\mathfrak{G}}\,\omega^{(+)}+\overline{K}_{\mathrm{BR}}\,\mathfrak{z}\,T_2^{(+)}\omega^{(+)}, \tag{4.29}$$

where

$$\overline{\mathfrak{G}}\equiv\mathfrak{G}\,G_1(1+\overline{\delta})\,G_1^{-1} \tag{4.30a}$$

and

$$\overline{K}_{\mathrm{BR}}=[MG_0+(MG_0-1)\,\mathcal{U}_1G_1+K_{\mathrm{BR}}(T_1\hat{G}-\mathcal{U}_1G_1)]\,\mathfrak{z}\,. \tag{4.30b}$$

Because of the indirect relationship of $\mathcal{U}_1$ to $\omega^{(-)\dagger}$, in general, one does not obtain an equation closed with respect to $T_{\mathrm{DW}}$ from (4.29) analogous to Eq. (4.11). Therefore, let us confine ourselves for the remainder of this section to the case when $\mathcal{U}_1$ is diagonal as per Eq. (4.12).

If Eq. (4.12) is valid, $\overline{K}_{\mathrm{BR}}$ simplifies to [cf. Eq. (4.25′)]

$$\overline{K}_{\mathrm{BR}}=MG_0\omega^{(-)\dagger}-\omega^{(-)\dagger}+\mathfrak{z}\,,$$

which allows one to rewrite (4.29) in a form similar to Eq. (4.13):

$$[(1-\mathfrak{z})+\omega^{(-)\dagger}]\,T_2^{(+)}\omega^{(+)}=\overline{\mathfrak{G}}\,\omega^{(+)}+MG_0\,\mathfrak{z}\,T_{\mathrm{DW}}\,. \tag{4.31}$$

An equation closed with respect to $T_{\mathrm{DW}}$ follows from (4.31) upon multiplication by $\mathfrak{z}$ on the left, viz.,

$$T_{\mathrm{DW}}=\mathfrak{z}\,\overline{\mathfrak{G}}\,\omega^{(+)}+\mathfrak{z}\,MG_0\,\mathfrak{z}\,T_{\mathrm{DW}}\,. \tag{4.32}$$

The general remarks made in connection with Eq. (4.14) apply to the last equation as well. In particular, we note the absence of the distorting potential in the kernel of (4.32). Of course, the inhomogeneous terms of Eqs. (4.14) and (4.32) differ in detail. However, the asymmetry noted in Eqs. (4.15) appears here as well. We have in fact [cf. Eqs. (4.15)]

$$R_{\beta,\alpha}^{(\mathrm{Born})}=\left\langle\phi_\beta(\overline{\eta}')\left|\left[V_\alpha^\beta-V_1^\beta\delta_{\beta,\alpha}-\sum_{k=3}^{N-1}\sum_{(b_k)}M_{\beta,b_k}G_0\,\delta(\alpha|b_k)\,V_1^\alpha\right]\right|\chi_\alpha^{(+)}(\overline{\eta})\right\rangle, \tag{4.33a}$$

which should be contrasted to the "Born approximation" following from (4.28) under the same constraint (4.12) on the distorting potential:

$$R_{\beta,\alpha}^{(\mathrm{Born})}=\left\langle\chi_\beta^{(-)}(\overline{\eta}')\left|\left[V_\alpha^\beta-V_1^\beta\delta_{\beta,\alpha}-\sum_{k=3}^{N-1}\sum_{(b_k)}M_{\beta,b_k}G_0\,\delta(\alpha|b_k)\,V_1^\alpha\right]\right|\chi_\alpha^{(+)}(\overline{\eta})\right\rangle. \tag{4.33b}$$

Since Eqs. (4.14) and (4.29) possess the same kernels as the undistorted two-cluster integral equations some of the approximation methods proposed for the KLT[6,30] and BR[5,18,34] equations can be applied without essential modification.

## V. COUPLED-CHANNEL EQUATIONS FOR DISTORTED-WAVE OPERATORS

Perhaps the most interesting results of Sec. IV consist in Eqs. (4.14) and (4.29), which are closed equations for the distorted-wave operators $T_{\rm DW}$. These equations are derived for either the $\mathcal{U}_{\rm KLT}$ or $\mathcal{U}_{\rm BR}$ channel-coupling arrays, with $\mathcal{U}_1$ constrained to be diagonal. For a general $\mathcal{U}$, but with $\mathcal{U}_1$ still diagonal we have

$$\mathcal{U} G_1 = \mathcal{U} \hat{G}[(1-\mathcal{J}) + \omega^{(-)\dagger}]$$

and

$$\overline{V}_1 G_1 = \omega^{(-)\dagger} - \mathcal{J} ,$$

so that (3.5) becomes

$$[(1-\mathcal{J}) + \omega^{(-)\dagger}] T_2^{(+)} = \overline{\mathcal{U}}_2 + \mathcal{U} \hat{G}[(1-\mathcal{J}) + \omega^{(-)\dagger}] T_2^{(+)} .$$

This implies the relationship

$$T_{\rm DW} = \mathcal{J}\, \overline{\mathcal{U}}_2 \omega^{(+)} + \mathcal{J}\, \mathcal{U} \hat{G}(1-\mathcal{J}) T_2 \omega^{(+)} + \mathcal{J}\, \mathcal{U} \mathcal{J}\, \hat{G}\, T_{\rm DW} ,$$

which, in general, is not closed with respect to $T_{\rm DW}$ and so does not constitute an integral equation for that quantity. Evidently other methods must be sought to obtain suitable integral equations for $T_{\rm DW}$ for general $\mathcal{U}_1$. One such procedure is developed in this section.

Two circumstances suggest a means of constructing such equations. First, the most common choices for distorting potentials are related to, although not necessarily identical to, the optical potentials in the two-cluster entrance and exit channels. Second, a reasonable lowest-order approximation to the kernel term in Eq. (4.14) or Eq. (4.29) involves, in effect, the truncation of the intermediate propagator to include only the various two-cluster bound states with arbitrary relative momentum between the (bound) clusters.[5,6,18,30,34] As a consequence of such an approximation these many-particle equations can be reduced to a set of coupled two-particle integral equations for the matrix elements of $T_{\rm DW}$. The same projectors are involved in the definition of this truncation of Eqs. (4.14) and (4.29) as in the definition of the optical potential operators. This indicates that a unified projection operator treatment of Eq. (3.5) in the case when the distorting potentials are defined on the same subspaces as the corresponding optical potentials may provide a technique for developing integral equations for $T_{\rm DW}$ for arbitrary $\mathcal{U}_1$.

Let $P$ denote a projector matrix in the partition indices whose elements

$$(P)_{b_k, a_m} = P_{b_k} \delta_{b_k, a_m} \delta_{k,2} \tag{5.1a}$$

are proportional to the projection operators onto the two-cluster bound states, so that if $\alpha$ is a stable partition

$$P_\alpha = \sum_{s,\overline{\eta}} |\phi_\alpha^{(s)}(\overline{\eta})\rangle\langle\phi_\alpha^{(s)}(\overline{\eta})| . \tag{5.1b}$$

The index $s$ enumerates the possible pairs of bound states (ground plus excited) of the two clusters represented by $\alpha$; $s=0$ corresponds to the ground states of both clusters. The symbol $\overline{\eta}$ refers to any other pertinent indices and these include the momentum of the centers of mass of each of the clusters relative to one another. If $\alpha$ is not a stable cluster, $P_\alpha \equiv 0$.

The complement of $P$, namely $Q = I - P$, is the diagonal matrix

$$(Q)_{b_k, a_m} = Q_{b_k} \delta_{b_k, a_m} , \tag{5.2a}$$

$$Q_{b_k} = I - P_{b_2} \delta_{k,2} . \tag{5.2b}$$

The projector $Q_{b_k}$ differs from the identity operator only when $b_k$ is a stable two-cluster partition, but its detailed structure is taken to be that of a resolution of the identity with respect to some basis. We now illustrate this. Let $|\phi_{a_m}(\overline{\eta})\rangle$ denote an $m$-cluster bound state; i.e., $a_m$ is a stable partition and all of the particles in each of their clusters are bound. Then $|\phi_{a_m}(\overline{\eta})\rangle$ is an eigenstate of the channel Hamiltonian $H_{a_m}$ and so are the scattering states

$$|\psi_{d_j}^{(+)}(a_m, \overline{\eta})\rangle = \overline{\delta}_{m,j}\, \delta(a_m | d_j)(1 + G_{a_m} V_{a_m}^{d_j}) |\phi_{d_j}(\overline{\eta})\rangle .$$

For each partition $a_m$ the collection of $m$-cluster bound states $\{|\phi_{a_m}(\overline{\eta})\rangle\}$ along with the scattering states $\{|\psi_{d_j}^{(+)}(a_m, \overline{\eta})\rangle\}$ form a complete orthonormal set on the entire $N$-particle Hilbert space. Therefore,

$$Q_\alpha = \sum_{k=3}^{N} \sum_{(b_k)} \sum_{s,\overline{\eta}} |\psi_{b_k}^{(+)}(\alpha; s, \overline{\eta})\rangle \times \langle\psi_{b_k}^{(+)}(\alpha; s, \overline{\eta})| \delta(\alpha | b_k) , \tag{5.2c}$$

and for $m > 2$

$$Q_{a_m} = \sum_{\overline{\eta}} |\phi_{a_m}(\overline{\eta})\rangle\langle\phi_{a_m}(\overline{\eta})| + \sum_{k=m+1}^{N} \sum_{(b_k)} \sum_{\overline{\eta}} |\psi_{b_k}^{(+)}(a_m, \overline{\eta})\rangle \times \langle\psi_{b_k}^{(+)}(a_m, \overline{\eta})| \delta(a_m | b_k) , \tag{5.2d}$$

where the excitation index has been absorbed into $\overline{\eta}$.

In a circumstance where one is solely concerned with the inelastic scattering within a given two-cluster channel $\alpha_0$, e.g., a convenient choice for $P$ is

$$P^{(\alpha_0)} = (P_{b_k, a_m}\, \delta_{b_k, a_m}\, \delta_{a_m, \alpha_0}). \tag{5.3}$$

The complement of $P^{(\alpha_0)}$, namely $Q^{(\alpha_0)} = 1 - P^{(\alpha_0)}$, then contains the sum of the bound states of the other stable two-cluster partitions in contrast to the situation in Eqs. (5.2). An example of considerable interest is a proton induced reaction $A(p,p')A^*$, where $A^*$ is an excited state of the nucleus whose ground state is represented by $A$.

We now assume that the distorting potentials $\overline{V}_1$ correspond to a not necessarily diagonal array $\mathcal{U}_1$ which satisfies

$$\mathcal{U}_1 Q = Q\,\mathcal{U}_1 = 0. \tag{5.4}$$

This, along with the constraints of Sec. III, implies that

$$G_1 P = P G_1 = g\omega^{(-)\dagger} \tag{5.5a}$$

and

$$G_1 Q = Q G_1 = Q\hat{G}, \tag{5.5b}$$

where we recall that $g = P\hat{G}$.

The constraints (5.4) will certainly be satisfied for arrays which generate $\overline{V}_1 = V_{\rm opt}$, where the latter operator has the form (3.8). The general type of distorting potential which we have in mind has the characteristic connected structure portrayed in Fig. 1 along with a restriction to $P$ space. Specifically, such a $\overline{V}_1$ has the form of a sum of operators each of which consists of an operator representing an interaction between two clusters sandwiched between two appropriate $P_\alpha$ projectors. Each of the operators in the sum is connected. Another example is given by the decomposition

$$V_1^\alpha = \sum_s P_{\alpha(s)} \hat{\mathcal{U}}_\alpha P_{\alpha(s)}, \tag{5.6}$$

where $P_{\alpha(s)}$ is the projector on the bound-state configuration of the two clusters represented by $s$. We note that

$$P_\alpha = \sum_s P_{\alpha(s)}. \tag{5.7}$$

$\hat{\mathcal{U}}_\alpha$ corresponds to an interaction between the two clusters represented by $\alpha$ and it may or may not have the same content as the operator defined by Eq. (3.10). That is, $\hat{\mathcal{U}}_\alpha$ is not necessarily the full optical potential operator. Equation (5.6) is compatible with either a diagonal or a nondiagonal array $\mathcal{U}_1$. An example of both is given by

$$\mathcal{U}_1^{\alpha,\beta} = \sum_s P_{\alpha(s)} \hat{\mathcal{U}}_\alpha P_{\alpha(s)} W_{\alpha,\beta}, \tag{5.8}$$

where $W_{\alpha,\beta}$ is an element of a channel-coupling array. The choice $W_{\alpha,\beta} = \delta_{\alpha,\beta}$ yields a diagonal $\mathcal{U}_1$, while taking $W_{\alpha,\beta}$ to be a channel permuting array corresponds to the nondiagonal case. This last does not vanish in the Born approximation for the rearrangement amplitudes ($\beta \neq \alpha$).

The decomposition

$$G_1 = P\hat{G}\,\omega^{(-)\dagger} + Q\hat{G},$$

which follows from Eqs. (5.5), suggests a subtraction technique[36] in connection with Eq. (3.5) with the result

$$T_2^{(+)} = \Gamma G_1 (1+\bar{\delta}) G_1^{-1} + \Gamma G_1 P T_2^{(+)}, \tag{5.9a}$$

where $\Gamma$ is defined as the solution of the integral equation

$$\Gamma = \mathcal{U}_2 + \mathcal{U}\hat{G}Q\Gamma. \tag{5.9b}$$

Whether or not $\mathcal{U}\hat{G}Q$ is a connected kernel is a structural circumstance which should be independent of the strength of the interparticle interactions. We can imagine the latter being so weak that $Q = I$. However, by assumption $(\mathcal{U}\hat{G})^p$ is a connected operator for some $p \geq 1$. We conclude, therefore, that $(\mathcal{U}\hat{G}Q)^p$ is connected for arbitrary interaction strengths and thus $\mathcal{U}\hat{G}Q$ is a connected kernel. For $\Gamma$ as defined the kernel of (5.9a) will be a connected operator. Therefore, we can regard the subtraction technique yielding Eqs. (5.9) as being well defined.

Equation (5.5a) allows one to derive a closed integral equation for $T_{\rm DW}$ from Eq. (5.9a):

$$T_{\rm DW} = \omega^{(-)\dagger} \Gamma G_1 (1+\bar{\delta}) G_1^{-1} \omega^{(+)} + \omega^{(-)\dagger} \Gamma g T_{\rm DW}. \tag{5.10}$$

The Lippmann identity (4.7b) implies that $T_{\rm DW}$ is equivalent on-shell to the operator $F$ which satisfies

$$F = F^{(0)} + \kappa g F, \tag{5.11a}$$

where

$$F^{(0)} \equiv P\omega^{(-)\dagger} \Gamma \omega^{(+)} P \tag{5.11b}$$

and

$$\kappa \equiv P\omega^{(-)\dagger} \Gamma P. \tag{5.11c}$$

We note that $FQ = QF = 0$ while $T_{\rm DW}(1-Z) = (1-Z)T_{\rm DW} = 0$.

Equation (5.11a) represents a set of coupled-channel two-particle Lippmann-Schwinger integral equations for the off-shell two-cluster to two-cluster rearrangement amplitudes. It is very informative to explicate the structure of these equations. To this end we require a more detailed characterization of the information symbolized by the states $|\phi_\alpha^{(s)}(\overline{\eta})\rangle$.

For each stable $\alpha$ the internal states of each of the bound clusters, $i=1,2$, appropriate to this channel can be specified by a collection of discrete quantum numbers $\{c_i(\alpha)\}$ which include the internal energy, $\epsilon_i(\alpha)$, of cluster $i$. The internal states of both clusters are then completely defined by a pair, $\{c_1(\alpha), c_2(\alpha)\}$, of the collections $c_1(\alpha)$ and $c_2(\alpha)$. For the sake of notational simplicity we assign a distinct value of a single discrete index $s$ to each of these pairs of collections of quantum numbers. The only additional variable then needed to completely characterize the two-cluster bound state $|\phi_\alpha^{(s)}(\vec{\eta})\rangle$ is the relative momentum $\vec{p}_\alpha$ between the centers of mass of the two clusters. The total energy of such a state is

$$E_\alpha(s,\vec{p}_\alpha)=\epsilon_1(s,\alpha)+\epsilon_2(s,\alpha)+\frac{\vec{p}_\alpha^{\,2}}{2\mu_\alpha},$$

where $\mu_\alpha$ is the reduced mass of the two clusters.

With the preceding notational connections the matrix elements of $F$, for example, can be denoted by $F_{\beta,\alpha}(\vec{p}_\beta', s'|\vec{p}_\alpha, s)$. In addition to the momentum dependence, these amplitudes possess a discrete matrix structure with respect to the channel and internal state indices. If we consider scattering from the ground state, $s=0$, in the entrance channel $\alpha$ with a total energy $E$, Eq. (5.11a) then becomes in explicit form

$$F_{\beta,\alpha}(\vec{p}_\beta', s'|\vec{p}_\alpha, 0)=F^{(0)}_{\beta,\alpha}(\vec{p}_\beta', s'|\vec{p}_\alpha, 0)+\sum_\gamma\sum_{s''(\gamma)}\int d\vec{p}_\gamma''\,\frac{\kappa_{\beta,\gamma}(\vec{p}_\beta', s'|\vec{p}_\gamma'', s'')F_{\gamma,\alpha}(\vec{p}_\gamma'', s''|\vec{p}_\alpha, 0)}{E-E_\gamma(s'',\vec{p}_\gamma'')+i0}. \tag{5.12}$$

If we choose the projector (5.3) appropriate to inelastic scattering, we obtain instead of (5.12) the set of coupled equations

$$F_{\alpha,\alpha}(\vec{p}_\alpha', s'|\vec{p}_\alpha, 0)=F^{(0)}_{\alpha,\alpha}(\vec{p}_\alpha', s'|\vec{p}_\alpha, 0)+\sum_{s''}\int d\vec{p}_\alpha''\,\frac{\kappa_{\alpha,\alpha}(\vec{p}_\alpha', s'|\vec{p}_\alpha'', s'')F_{\alpha,\alpha}(\vec{p}_\alpha'', s''|\vec{p}_\alpha, 0)}{E-E_\alpha(s'',\vec{p}_\alpha'')+i0}. \tag{5.13}$$

While Eqs. (5.12) and (5.13) are manifestly a set of coupled two-particle integral equations, their solution is far from trivial except possibly when $\Gamma$ is approximated by $\mathcal{U}_2$ in $F^{(0)}$ and $\kappa$. This is because the input functions $F^{(0)}$ and $\kappa$ possess, in general, singularities which greatly complicate the integration procedure. However, by far the most difficult problem connected with the use of Eqs. (5.11)–(5.13) is the determination of $\Gamma$ and thereby $F^{(0)}$ and $\kappa$; we take up this question in the next section.

## VI. $\Gamma$ EQUATIONS: STRUCTURE AND APPROXIMATIONS

The input to be inserted into the coupled channel Eqs. (5.11)–(5.13) ultimately consists in a statement concerning the operator $\Gamma$ defined by Eq. (5.9b). This integral equation will inevitably be solved only after considerable approximation unless the number of particles is very small. In this section we explore some aspects of the structure of the $\Gamma$ integral equations attendant to the formulation of approximations. Some examples of the latter are also considered.

From Eqs. (5.11) it is clear that only $\mathcal{J}\Gamma\mathcal{J}$ is needed as input into the coupled-channel equations. It then suffices to consider

$$\Gamma\mathcal{J}=(\mathcal{U}-\mathcal{U}_1)\mathcal{J}+\mathcal{U}\hat{G}Q\Gamma\mathcal{J} \tag{6.1}$$

rather than (5.9b) as the basic $\Gamma$ equation. Evidently whether or not (6.1) is minimally coupled is also of some relevance.

The operator $\Gamma$ depends on the choices of the arrays $\mathcal{U}$ and $\mathcal{U}_1$. Even if Eqs. (5.9b) and (6.1) were solved exactly different arrays would yield different $\Gamma$ operators. The array $\mathcal{U}_1$ appears only in the inhomogeneous terms of Eqs. (5.9b) and (6.1) and so is essentially irrelevant to the approximation of the kernel terms in these equations. This is obviously not the case with $\mathcal{U}$ and the ease of formulating and applying approximation methods as well as the question of their accuracy can be expected to be dependent upon the choice of channel-coupling scheme.

Different exact $\Gamma$ operators resulting from distinct choices for $\mathcal{U}$ and/or $\mathcal{U}_1$ yield the same values for the on-shell $F$ amplitudes. However, this will not remain true once approximations are introduced into the $\Gamma$ integral equations. One consequence is that the same approximation prescription, e.g. the second Born approximation, will yield different scattering amplitudes for different choices of arrays. This is true even if only $\mathcal{U}$ is being varied.

For any array which satisfies

$$\mathcal{U}=\mathcal{U}\mathcal{J}, \tag{6.2}$$

Eq. (6.1) becomes minimally coupled and we find that

$$\mathcal{J}\Gamma\mathcal{J}=\mathcal{J}\mathcal{U}_2\mathcal{J}+\mathcal{J}\mathcal{U}\hat{G}Q\mathcal{J}\Gamma\mathcal{J}. \tag{6.3}$$

The sum over intermediate states in the kernel of Eq. (6.3) is given by the relatively simple two-

cluster $Q$ operator of Eq. (5.2c). Equation (6.3) yields a closed integral equation for $Q\mathfrak{d}\Gamma P$, namely,

$$Q\mathfrak{d}\Gamma P = Q\mathfrak{d}\mathcal{U}_2 P + Q\mathfrak{d}\mathcal{U}\hat{G}Q\mathfrak{d}\Gamma P . \tag{6.4}$$

The solution of (6.4) can be used with (6.3) to provide a determination of $P\Gamma P$ "by quadrature":

$$P\Gamma P = P\mathcal{U}_2 P + P\mathcal{U}\hat{G}Q\mathfrak{d}\Gamma P . \tag{6.5}$$

The array $\mathcal{U}_{\mathrm{KLT}}$ satisfies (6.2) and therefore Eqs. (6.3)–(6.5) apply to this case. The highly asymmetric character of the KLT coupling scheme must be taken into account in the course of determining approximation solutions of Eqs. (6.4) and (6.5). The simplest of these, namely

$$P\Gamma P \simeq P\mathcal{U}_{2(\mathrm{KLT})}P , \tag{6.6}$$

is similar to the bound-state approximation proposed in Ref. 6.

Our work in Sec. IV indicates that a more complex example will be provided by the BR channel-coupling scheme. We next analyze Eq. (6.1) when the kernel is $\hat{K}_{\mathrm{BR}} \equiv K_{\mathrm{BR}}Q$.

The square of this last quantity is

$$(\hat{K}_{\mathrm{BR}})^2 = (MG_0 - K_{\mathrm{BR}}P)\mathfrak{d}\hat{K}_{\mathrm{BR}} , \tag{6.7}$$

which, by the arguments of Sec. IV and the fact that $P_\alpha$ has connectivity $\alpha$, is a connected operator. Equation (6.7) implies that the resolvent kernel $\hat{\mathcal{R}}_{\mathrm{BR}}$ corresponding to $\hat{K}_{\mathrm{BR}}$ satisfies

$$\hat{\mathcal{R}}_{\mathrm{BR}} = \hat{K}_{\mathrm{BR}} + (MG_0 - \mathcal{U}_{\mathrm{BR}}g)\mathfrak{d}\hat{\mathcal{R}}_{\mathrm{BR}} . \tag{6.8}$$

Since $\Gamma\mathfrak{d} = (1+\hat{\mathcal{R}}_{\mathrm{BR}})\mathcal{U}_{2(\mathrm{BR})}\mathfrak{d}$, Eq. (6.8) allows us to reduce (6.1) to a minimally coupled integral equation for $\mathfrak{d}\Gamma P$, viz.,

$$\mathfrak{d}\Gamma P = \mathfrak{d}[\mathcal{U}_{2(\mathrm{BR})} - MG_0(D^t - \mathfrak{d})\mathcal{U}_1]P + \mathfrak{d}(MG_0 - \mathcal{U}_{\mathrm{BR}}g)(\mathfrak{d}\Gamma P) . \tag{6.9}$$

The inhomogeneous term of (6.9) is typical of the minimal BR equations derived in Sec. IV. The kernel, a component of which is

$$K_{\beta,\alpha} \equiv M_{\beta,\alpha}G_0 - V^\beta_\alpha g_\alpha , \tag{6.10}$$

is a bit different. The $V^\beta_\alpha g_\alpha$ term in (6.10) in effect subtracts off the $\alpha$-cluster bound-state cut from $M_{\beta,\alpha}G_0$. However, it is important to note that

$$K_{\beta,\alpha}P_\alpha \neq 0 .$$

Therefore, part of the price of reducing Eq. (6.1) to the minimal form (6.9) is to lose the closure with respect to $Q\Gamma P$ possessed by (6.1). The remainder of the price is the comparatively complicated inhomogeneous term in Eq. (6.9).

The form (6.10) of the kernel (6.9) allows us to identify $K_{\beta,\alpha}$ as being precisely the kernel of a subsidiary integral equation similar to (6.9) introduced by Redish.[5,34] In point of fact, if we let $\mathcal{U}_1 \equiv 0$, the present $F/\Gamma$ complex of equations reduces to the $U/U^{(1)}$ set of Ref. 5. In this limit the $\Gamma$ operators take over the role of the distorting potentials.

The preceding observation has the important implication that the program of systematic approximations introduced by Redish[34] can be applied to Eq. (6.9). For example, as the next step Eq. (6.9) can be reduced to an integral equation whose kernel has a three-body structure and which involves as input the solution of another integral equation whose kernel possesses no two- and three-cluster cuts. This is a continuation of the process initiated in Sec. V where Eq. (3.5) is reduced to an integral equation (5.10) whose kernel has a two-body structure and which involves as input the solution of an integral equation (6.9) whose kernel does not possess the two-cluster cuts.

In practice corresponding simplifications of the inhomogeneous term of (6.9) are evidently necessary. These were discussed in Sec. IV. In lowest order, of course,

$$\Gamma_{\beta,\alpha} \simeq \Gamma^{(0)}_{\beta,\alpha}$$

where

$$\Gamma^{(0)}_{\beta,\alpha} = V^\beta_\alpha - \mathcal{U}^{\beta,\alpha}_1 .$$

For instance, if channel $\alpha$ represents the two-cluster partition $(p,n)(A)$, where $p$, $n$, and $A$ refer to a proton, a neutron, and the remaining $N-2$ particles, respectively, and $\beta$ corresponds to the channel $(p)$ $(nA)$, then the on-shell matrix elements of $P_\beta \omega^{(-)\dagger}_\beta \Gamma^{(0)}_{\beta,\alpha} \omega^{(+)}_\alpha P_\alpha$ [cf. Eqs. (5.11)] can be related to the stripping reaction $A(d,p)A'$. If we choose $\mathcal{U}^{\gamma,\alpha}_1 = V^\gamma_{\mathrm{opt}}\delta_{\gamma,\alpha}$, we have, since $V^\beta_\alpha = V_{n\text{-}p}$,

$$\langle \phi_{p\text{-}A'} | F^{(0)}_{\beta,\alpha} | \phi_{d\text{-}A} \rangle \simeq \langle \chi^{(-)}_{p\text{-}A'} | V_{n\text{-}p} | \chi^{(+)}_{d\text{-}A} \rangle , \tag{6.11}$$

which is the DWBA amplitude for this process. Here $|\chi^{(+)}_{d\text{-}A}\rangle$ is the incoming distorted wave generated by the elastic scattering of the deuteron $(d)$ and the bound nucleus $A$ while $\langle \chi^{(-)}_{p\text{-}A'}|$ is the outgoing distorted wave corresponding to the elastic scattering of $p'$ and the nucleus $A'$. Of course, in order to obtain $\langle \phi_{p\text{-}A'} | F_{\beta,\alpha} | \phi_{d\text{-}A} \rangle$ one requires the counterparts of (6.11) for all $\beta$ as the inhomogeneous terms in the two-body integral equations (5.12).

A possible alternative to the use of (6.9) is the direct exploitation of Eq. (6.1) in the form

$$\Gamma P = (\mathcal{U}_{\mathrm{BR}} - \mathcal{U}_1)P + \mathcal{U}_{\mathrm{BR}}\hat{G}Q\Gamma P .$$

Thus

$$Q\Gamma P = Q(\mathcal{U}_{BR} - \mathcal{U}_1)P + (Q\,\mathcal{U}_{BR}\,Q\hat{G})(Q\Gamma P)$$

and so

$$P\Gamma P = P(\mathcal{U}_{BR} - \mathcal{U}_1)P + P\mathcal{U}_{BR}\hat{G}(Q\Gamma P)$$

very much like what is obtained in the KLT case. However, considerable care must be exercised in formulating approximations to the kernel terms of these equations since the connected-kernel aspect may thereby be lost.


## ACKNOWLEDGMENTS

This work was supported in part by the NSF under Grant No. PHY74-02239A03. We would like to thank R. D. Koshel for several conversations on some of the topics of this paper.



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