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Minimal four-body equations

Sadhan K. Adhikari
Departmento de Física,[†] *Universidade Federal de Pernambuco, 50.000 Recife, Pe, Brazil*
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The constraints of unitarity and analyticity on four-body final state amplitudes are studied in the quasi-two-body scheme. The implementation of unitarity with total energy analyticity yields the minimal set of scattering equations for the problem consistent with constraints of quantum mechanics. The minimal set we obtain gives a dynamical scheme which is distinct from the full four-body scattering scheme. Nevertheless, with the assumption of separable approximation for two- and three-body interactions we get simple Lippmann-Schwinger type equations for four identical bosons for the following two-to-two processes: $nt \rightarrow nt$, $nt \rightarrow dd$, $dd \rightarrow dd$, and $dd \rightarrow nd$. Here n, d, and t refer to nucleon, deuteron, and triton type states. The amplitudes for the breakup processes can also be related to these amplitudes.

[NUCLEAR REACTIONS Four-body final state interaction theory. Useful scattering equations derived from constraints of quantum mechanics.]

I. INTRODUCTION

Recently it has been shown[1-5] that the elementary constraints of quantum mechanics, unitarity and analyticity, when applied to three- and four-body final states in the quasiparticle or isobar picture force singularity structure and interdependence of amplitudes usually taken as independent and constant in phenomenological analysis. Implementation of these unitarity constraints by a dispersion relation, along with some simple ideas about total energy analyticity, leads to a set of integral equations for few-body amplitudes. In the nonrelativistic three-body problem[2] the Faddeev equation can be recovered as a result of implementation. Similarly the relativistic three-body problem[3] yields a set of equations, which is similar to the usual Blankenbecler-Sugar equation.[6] In the four-body problem—both in relativistic[5] and nonrelativistic[4] cases—one can similarly recover the full dynamical scheme as a result of implementation.

Recently[4,5] we formulated the problem of four-body final states in terms of quasi-three-body states. This is because in the quasiparticle picture the four-body problem can be formulated in terms of these quasi-three-body states. The equations we obtained are in two vector variables and would be very difficult to solve even after partial wave decomposition. Here we formulate the problem in terms of quasi-two-body states, apply unitarity on these amplitudes, and derive important unitarity constraints on these amplitudes. The dynamical equations we get after implementation of unitarity constraints are in one vector variable and will be easy to solve after partial wave decomposition.

Here we apply unitarity to these quasi-two-body states focusing particularly on a special type of two-body unitarity, "independent pair two-body unitarity," which depends on the interaction between independent pairs in four-body final states and on three-body subenergy unitarity. There are two-types of two-body fragmentation channels in the four-body problem. In the first type, the quasiparticles are each two-body correlated states and in the second type they constitute a free particle and a three-body correlated state. Unitarity constraint on the first type of amplitude is derived by a consideration of independent pair two-body unitarity and that on the second type of amplitude is derived by considering three-body subenergy unitarity. As in other similar few-body problems[1-5] we find that unitarity forces the amplitudes to vary over their phase spaces, be singular on their edges, and be coherent and interrelated. The unitarity relation itself can be used to determine the numerical importance of these effects in any problem. If they are important, they must be implemented by considering analyticity as well. Since we are considering a part of the full four-body unitarity, the implementation of unitarity is ambiguous.[2] Out of these various ambiguous ways, we choose the one that preserves total energy analyticity as well, in view of various problems one faces upon neglecting it.[1] In this way, we get a set of equations for four-body amplitudes.

Very little is known or understood about the structure of four-body final states—especially how the two- and the three-body information is distributed over these states. The present and a previous analysis[4] will throw some light on the structure of four-body final states—especially the dynamical aspects of the problem. In the three-body problem, unitarity corrections are crucial in cases[7] where there are strong overlapping resonances in a certain region of phase space or where there are threshold enhancements. This is true in the case of three nucleons. In the case of three α particles interacting in the final state, unitarity corrections are negligible.[8] There are also cases where the unitarity corrections are reasonable.[9] No doubt a similar spectrum of four-body examples exists and the present plus previous work give the techniques to analyze them. Systematic examination and study of four-body final states by the present method should lead to useful approximation techniques in the future.

Though formally correct four-body equations exist in the literature,[10] their complexity is so great that it is difficult to solve the equations even with separable two- and three-body interactions. Here we have a simplified model for the four-body problem which we derive from the general constraints of quantum mechanics. This simple model will give us an idea about the important dynamical aspects of the problem, whereas the full dynamical scheme is very complicated and hence is difficult to visualize.

To illustrate the usefulness of the present equations, we apply these equations to the problem of four nucleons. To simplify the algebra, we work with four identical bosons. If needed, it is possible to introduce spin and other internal quantum numbers in our formalism.[1] In addition to the nucleon n, two composite particles meant to approximate the deuteron and triton are introduced with the couplings $d \rightarrow n+n$ and $t \rightarrow d+n$. With the separable approximation for two- and three-body interactions, we obtain four-body integral equations for the processes $nt \rightarrow nt$, $nt \rightarrow dd$, $dd \rightarrow dd$, and $dd \rightarrow nt$ with a certain assumption about the Born or the driving term. This is because implementation of unitarity constraints gives only the dynamics and cannot give the Born term. The Born term does not have the unitarity cut and hence is not defined by the unitarity relation. Different approximation schemes that emerge from implementation of unitarity constraints are discussed. After partial wave decomposition, these equations reduce to one-variable partial-wave Lippmann-Schwinger type equations which can be easily solved numerically.

In the case of four identical bosons, the equations we get are very similar to the dynamical equations obtained by Fonseca and Shanley,[11] who considered a simplified version of the generalization of the Lee model[12] to the four-body problem. We compare our equations to those obtained by Fonseca and Shanley.

In Sec. II we define two- and three-body separable t matrices that we shall be using in subsequent sections. We also develop the unitarity relations for independent-pair two-body unitarity and three-body subenergy unitarity in four-body final states. In Sec. III we derive the unitarity constraints for the four-body amplitudes. In Sec. IV we discuss their implementation and their relation to dynamics, stressing the importance of the "arbitrary" choices that are made there. In Sec. V we discuss the problem of four identical spinless bosons, derive the useful approximate equations for the different amplitudes, and compare them with the four-body soluble model of Fonseca and Shanley. In Sec. VI we summarize our results and discuss possible applications.

II. UNITARITY

A. Two-body unitarity

Before applying unitarity to three- and four-body systems, we review some of our conventions and definitions. (For a more complete review, see Ref. 1.). We define the S and the T matrix by

$$S = 1 - 2\pi i \delta(E) T \tag{1}$$

so that

$$\mathrm{Im}\langle\alpha\,|\,T\,|\,\beta\rangle = -\pi \sum_{\gamma} \langle\alpha\,|\,T\,|\,\gamma\rangle\langle\gamma\,|\,T\,|\,\beta\rangle^{*}\delta(E_\alpha - E_\gamma) \tag{2}$$

as long as $E_\alpha = E_\beta$. For the two-body t matrix, we write (neglecting spin, isospin, etc.)

$$\langle\vec{p}_1,\vec{p}_2\,|\,T\,|\,\vec{p}_1{}',\vec{p}_2{}'\rangle = (2\pi)^3\delta(\vec{p}_1+\vec{p}_2-\vec{p}_1{}'-\vec{p}_2{}') \times \langle\vec{q}_{12}\,|\,t\,|\,\vec{q}_{12}{}'\rangle, \tag{3}$$

where $(m_1+m_2)\vec{q}_{12} = m_2\vec{p}_1 - \vec{m}_1\vec{p}_2$. In partial wave form, we have

$$\langle\vec{q}\,|\,t\,|\,\vec{q}\,'\rangle = \sum_{l,m} Y_{lm}(\hat{q})t_l(q^2/2\mu)Y^{*}_{lm}(\hat{q}\,'), \tag{4}$$

where $\mu(m_1+m_2) = m_1 m_2$. Unitarity (2) gives

$$\mathrm{Im}t_l(E) = -\frac{\mu}{8\pi^2}k\,|\,t_l(E)\,|^2\,, \tag{5}$$

where $k^2 = 2\mu E$.

The two-body separable t matrix is written in the form

$$\langle p|\,t_l(E)\,|q\rangle = v_l(p)\,\frac{1}{D_l(E)}\,v_l(q) \tag{6}$$

and consequently the unitarity relation takes the form

$$\mathrm{Im}\,\frac{1}{D_l(E)} = -\frac{\mu}{8\pi^2}\,k\,\frac{v_l{}^2(k)}{|D_l(E)|^2}\,. \tag{7}$$

In the four-body amplitude there is a term which involves no interaction between two independent pairs and hence is disconnected. The unitarity relation for this disconnected piece of the amplitude is just a manipulation of two-body unitarity and can be written for four particles of equal mass as

$$\operatorname{Im}\left[\frac{1}{D_{l_1}(E_1)D_{l_2}(E-E_1-p^2/4\mu)}\right] = -\frac{1}{\pi}\left(\frac{\mu}{8\pi^2}\right)^2 k_1k_2\,\frac{v_{l_1}{}^2(k_1)v_{l_2}{}^2(k_2)}{|D_{l_1}(E_1)D_{l_2}(E-E_1-p^2/4\mu)|^2} \tag{8}$$

where $k_1{}^2=2\mu E_1$, $k_2{}^2=2\mu(E-E_1-p^2/4\mu)$, E is the total four-body energy, and $\vec{p}$ is the relative momentum between the two pairs. Here l_1 and l_2 denote the partial waves for each pair, respectively. This unitarity relation is diagrammatically represented in Fig. 1. With this definition and convention for the two-body unitarity we turn to the problem of three-body unitarity.

To make the algebra simple, from now on we shall be considering the case of four distinguishable particles of equal mass m ($2m=1$). This restriction keeps the algebra simple but otherwise has no effect on the result. If needed, this restriction can be removed very easily.

B. Three-body unitarity

To proceed to the four-body problem with no further approximation would lead to the numerical difficulties inherent in multivariable integral equations, so that to keep the algebra managable, we introduce the three-body separable interaction at this point. We assume that the three-body t matrix is dominated by the presence of an S-wave pole. We introduce the correlated state of three particles with vertex form factor ω and propagator D' in S wave as shown in Fig. 2. The separable t matrix T_{33} for the $3\to3$ process can be written as

$$\langle\vec{p}_1,\vec{q}_1|T_{3,3}(E)|\vec{p}_2,\vec{q}_2\rangle = \frac{1}{2}\sum_{a,b,c=1}^{3}\frac{1}{2}\sum_{d,e,f=1}^{3}\frac{v_{ab}(p_1)}{D_{ab}(2p_1{}^2)}\,\omega_{ab,c}(q_1) \times\frac{1}{D'(E)}\,\omega_{de,f}(q_2)\,\frac{v_{de}(p_2)}{D_{de}(2p_2{}^2)}\,. \tag{9}$$

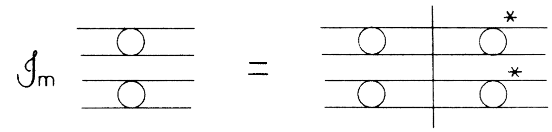

FIG. 1. Schematic representation for independent pair two-body unitarity. This is a disconnected part of the four-body unitarity, where each of the disconnected pieces involve a two-body t matrix. The vertical line represents an energy conserving δ function.

Here we assume that the two- and the three-body correlated states occur only in S wave. Generalization to other partial waves and to cases of particles with spin and isospin can be done in a similar way as in Ref. 1. In this paper, we shall consider only spinless bosons, with no isospin, interacting in relative S-waves only. In this model, as shown in Fig. 2, in the initial state two particles a and b in a relative momentum state $\vec{p}_1$ form a quasi-two-body state ab, which then interacts with particle c in relative momentum state $\vec{q}_1$ with respect to ab and forms a three-body correlated state. The state then propagates and decays first to a free particle f and a correlated state de, which subsequently decays to particles d and e. Here $\omega_{ab,c}$ is the vertex function for interaction between particle c and correlated state ab in S wave and $D'(E)$ is the three-body denominator function at energy E. v and D are the corresponding quantities for the two-body system and have been already defined. An expression for $D'(E)$ can be easily found by summing the series of self-energy bubbles. It has been discussed in detail elsewhere[11] and hence we shall not consider it here. The factors of $\frac{1}{2}$ in front of summation in Eq. (9) are there to take care of symmetries like $v_{ab}=v_{ba}$ or $\omega_{ab,c}=\omega_{ba,c}$ for distinguishable particles.

The connected three-body unitarity relation for the amplitude (9) has the form

$$\begin{aligned}
\frac{1}{2}\sum_{a,b,c=1}^{3}\frac{1}{2}\sum_{d,e,f=1}^{3}\Bigg[\operatorname{Disc}\left(\frac{v_{ab}(p_1)}{D_{ab}(2p_1{}^2)}\,\omega_{ab,c}(q_1)\,\frac{1}{D'_{jkl}(E_{jkl})}\,\omega_{de,f}(q_2)\,\frac{v_{de}(p_2)}{D_{de}(2p_2{}^2)}\right)
&= -\frac{\pi}{2}\sum_{s,t,u=1}^{3}\int\frac{v_{ab}(p_1)}{D_{ab}(2p_1{}^2)}\,\omega_{ab,c}(q_1)\,\frac{1}{D'_{jkl}(E_{jkl})}\,\omega_{st,u}(q'_{st,u})\,\frac{v_{st}(q'_{st})}{D_{st}(2q'^2_{st})}\,\delta(\vec{p}_i-\vec{p}'_i)\\
&\quad\times\delta(\vec{p}_j+\vec{p}_k+\vec{p}_l-\vec{p}'_j-\vec{p}'_k-\vec{p}'_l)\,\delta(p_i{}^2+p_j{}^2+p_k{}^2+p_l{}^2-p'^2_i-p'^2_j-p'^2_k-p'^2_l)\\
&\quad\times\frac{1}{(2\pi)^6}\,d^3p'_i\,d^3p'_j\,d^3p'_k\,d^3p'_l\\
&\quad\times\omega_{jk,l}(q'_{jk,l})\,\frac{v_{jk}(q'_{jk})}{D_{jk}(2q'^2_{jk})}\,\frac{1}{D'^{*}_{jkl}(E_{jkl})}\,\omega_{de,f}(q_2)\,\frac{v_{de}(p_2)}{D_{de}(2p_2{}^2)}\Bigg]\,.
\end{aligned} \tag{10}$$

This unitarity relation is shown diagrammatically[13] in Fig. 3. A similar unitarity relation has been considered elsewhere.[14] Equation (10) represents the discontinuity across the three-body cut. For the two-body D function D, we have used the identity $D=D^*$, because it does not have the three-body cut. Here q'_{ab} is the relative momentum of particles a and b, $q'_{ab,c}$ is the relative momentum of particle c and the correlated state ab, and E_{jkl} is the energy of the correlated state jkl. Canceling common factors from both sides of Eq. (10), we get

$$\operatorname{Disc} \frac{1}{D'_{jkl}(E_{jkl})} = -\frac{\pi}{2} \sum_{s,t,u=1}^{3} \frac{1}{(2\pi)^6} \int \frac{1}{D'_{jkl}(E_{jkl})} \omega_{st,u}(q'_{st,u}) \frac{v_{st}(q'_{st})}{D_{st}(2q'^2_{st})} \delta(\vec{p}_i - \vec{p}'_i)$$
$$\times \delta(\vec{p}_j + \vec{p}_k + \vec{p}_l - \vec{p}'_j - \vec{p}'_k - \vec{p}'_l) \delta(E_{jkl} - p'^2_j - p'^2_k - p'^2_l)$$
$$\times d^3p'_i d^3p'_j d^3p'_k d^3p'_l \omega_{jk,l}(q'_{jk,l}) \frac{v_{jk}(q'_{jk})}{D_{jk}(2q'^2_{jk})} \frac{1}{D'^*_{jkl}(E_{jkl})} \tag{11}$$

With these conventions and definitions for two- and three-body unitarity we turn to the problem of four-body unitarity in the next section.

III. FOUR-BODY UNITARITY

Let us consider the amplitude $T_{2,4}$ for a reaction going from a two-body state to a four-body state. Here we postulate a simple form for $T_{2,4}$. This form is suggested by the sequential decay or quasiparticle model of nuclear physics and isobar model of particle physics. We assume that the four-body final state is dominated by correlated states of two or three particles in S wave. The four-body state is then a sum of two types of terms. In one type, two correlated states of two particles each are formed and they subsequently propagate and decay. In the second type, a free particle and a three-body correlated state is formed. The three-body correlated state propagates and decays to a free particle and a two-body correlated state, which propagates and decays to two free particles. We then have for a reaction of two particles of relative momentum $\vec{q}$ going (in the center of mass frame) to four particles of momentum $\vec{p}_i$ and center of mass energy E,

$$\langle \vec{q} | T_{2,4}(E) | \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4 \rangle = (2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)$$
$$\times \sum_{i,j,k,l=1}^{4} \left(\tfrac{1}{8} \langle \vec{q} | F_{ij,kl}(E) | \vec{p}_i + \vec{p}_j \rangle \frac{v_{ij}(q_{ij})}{D_{ij}(2q_{ij}^2)} \frac{v_{kl}(q_{kl})}{D_{kl}(2q_{kl}^2)} \right.$$
$$\left. + \tfrac{1}{2} \langle \vec{q} | G_{i,jkl}(E) | \vec{p}_i \rangle \frac{\omega_{jk,l}(q_{jk,l})}{D'_{jkl}(\epsilon_{jkl})} \frac{v_{jk}(q_{jk})}{D_{jk}(2q_{jk}^2)} \right). \tag{12}$$

Here q_{ij} and $q_{ij,k}$ refer to relative momenta as in Eq. (10). This relation is diagrammatically represented in Fig. 4. The factors of $\frac{1}{8}$ and $\frac{1}{2}$ in front of the two terms account for the fact that $F_{ij,kl} = F_{kl,ij} = F_{ji,kl} = F_{ij,lk}$, etc., and $v_{ij} = v_{ji}$, $D_{ij} = D_{ji}$, $\omega_{ij,k} = \omega_{ji,k}$, etc. Consequently there are 3 terms of the first type and 12 of the second type in the summation in Eq. (12). Here $F_{ij,kl}$ is the quasi-two-body amplitude for going to a state of correlated pair ij and kl. The propagation and subsequent decays of these correlated pair for four particles i, j, k, and l are represented by the two factors of v/D multiplying F in Eq. (12). Similarly $G_{i,jkl}$ re-

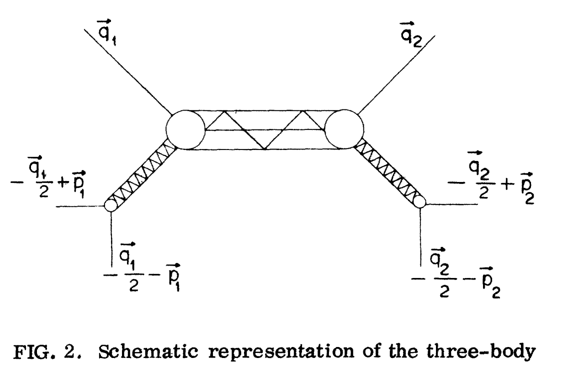

FIG. 2. Schematic representation of the three-body t matrix. The crossed line means that the corresponding propagator is fully dressed.

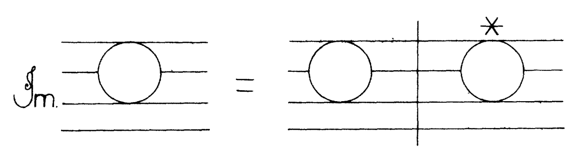

FIG. 3. Schematic representation of connected three-body unitarity in four-body space. The vertical line represents an energy conserving δ function.

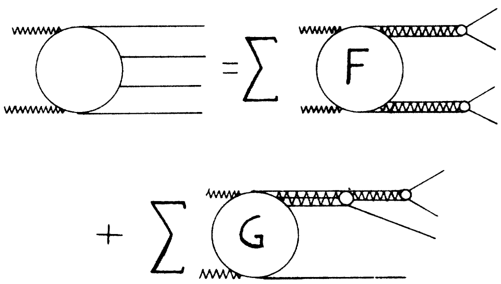

FIG. 4. Schematic representation of Eq. (12).

presents the quasi-two-body amplitude for going to a state, where the particle i is free and the three others form a correlated state jkl. The factors of ω, v, D, and D' multiplying G represent propagation and subsequent decay of the three-body correlated state to a free particle and a two-body correlated state, which then propagates and decays to two free particles as shown in Fig. 4. Other forms for the quasiparticle amplitude involving the full two-body t matrix, rather than v/D type functions, have been used in the three-body case.[1] The various problems they cause, primarily with total energy analyticity when implementing unitarity, led us to consider only the form presented in Eq. (12).

The unitarity relation for $T_{2,4}$, which has been discussed in detail elsewhere,[4,5] contains many terms. Assuming that only two-, three-, and four body intermediate states are energetically allowed, unitarity for $T_{2,4}$ can be written as[13]

$$\begin{aligned}\mathrm{Im}\,T_{2,4} &= -\pi \sum_{2'} T_{2,2'}\delta(E-E_{2'})T^*_{2',4}\\ &\quad -\pi \sum_{3'} T_{2,3'}\delta(E-E_{3'})T^*_{3',4}\\ &\quad -\pi \sum_{4'} T_{2,4'}\delta(E-E_{4'})T^*_{4',4}\,. \qquad (13)\end{aligned}$$

Contribution to the special types of four-body discontinuity we are interested in will come from the disconnected parts of unitarity. Hence we decompose the amplitudes in Eq. (13) into disconnected and totally connected parts. Eq. (13) and this decomposition is shown in Fig. 5. Each term in unitarity represents a singularity at the threshold of that term. Strictly speaking, each term in unitarity contributes to the discontinuity across the singularity beginning at that particular threshold. We are interested in finding the constraints of unitarity on

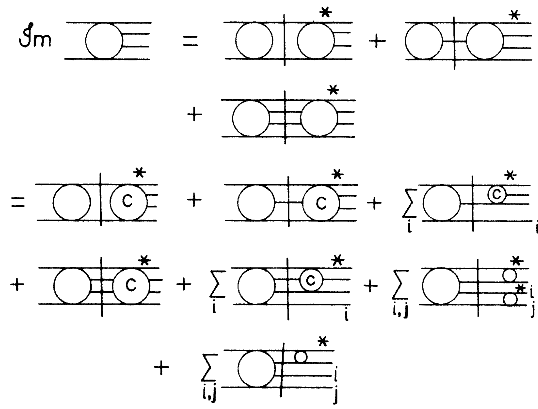

FIG. 5. Schematic representation of the unitarity relation in Eq. (13). The second line shows the amplitudes decomposed into fully connected (represented by a C) and disconnected parts.

the amplitudes F and G defined by Eq. (12). Because the two terms in Eq. (12) have distinct singularity structure, Eq. (13) will be satisfied separately by the two terms in Eq. (12).

A. Unitarity constraints on F

First let us consider constraints of unitarity on F. This is a quasi-two-body amplitude, where two independent pairs interact in the final state. The information about independent pair interaction in the final state will be contained in the last term in the last but one line in the unitarity relation in Fig. 5. In this term, a given pair threshold depends on the energy of the other pair. As this involves only a two particle interaction, the last term in Fig. 5 also appears to contribute to this singularity. We can neglect this term from our consideration, because this term has a different threshold—the pair subenergy threshold which was considered in detail in previous work,[4-5] where we concentrated on this term to analyze the problem of four-body final states. Hence, to consider unitarity constraints on F, we shall be limited to the consideration of the peculiar term—where two independent pairs interact in the final state—of four-body unitarity.

If we keep only this term in four-body unitarity (13) we have for the discontinuity across this cut schematically

$$\mathrm{Disc}\,T_{2,4} = -\pi \sum T_{2,4'}\delta(E-E')T^*_{2',2}T^*_{2',2}\,, \qquad (14)$$

where $T_{2',2}$ is the two-body amplitude. We now substitute the first term of Eq. (12) into Eq. (14) to obtain

$$\frac{(2\pi)^3\delta(\sum\vec{p}_i)}{8}\sum_{i,j,k,l=1}^{4}\left[\mathrm{Disc}\left(\langle\vec{q}\,|F_{ij,kl}(E)\,|\vec{p}_i+\vec{p}_j\rangle\,\frac{v_{ij}(q_{ij})}{D_{ij}(2q_{ij}{}^2)}\,\frac{v_{kl}(q_{kl})}{D_{kl}(2q_{kl}{}^2)}\right)\right.$$

$$=-\frac{\pi}{8}\int\sum_{a,b,c,d=1}^{4}\langle\vec{q}\,|F_{ab,cd}(E)\,|\vec{p}_a'+\vec{p}_b'\rangle\,\frac{v_{ab}(q_{ab}')}{D_{ab}(2q_{ab}'^2)}\,\frac{v_{cd}(q_{cd}')}{D_{cd}(2q_{cd}'^2)}\delta(\vec{p}_i+\vec{p}_j-\vec{p}_i'-\vec{p}_j')\delta(\vec{p}_k+\vec{p}_l-\vec{p}_k'-\vec{p}_l')$$

$$\left.\times\,\delta\left(E-\sum_{i=1}^{4}p_i'^2\right)f\,\frac{1}{(2\pi)^6}\left(\prod_{i=1}^{4}d^3p_i'\right)\,\frac{v_{kl}(q_{kl}')v_{kl}(q_{kl})}{D_{kl}^*(2q_{kl}{}^2)}\,\frac{v_{ij}(q_{ij}')v_{ij}(q_{ij})}{D_{ij}^*(2q_{ij}{}^2)}\right], \tag{15}$$

where

$$f=1+[\delta(2q_{ij}{}^2-2q_{ij}'^2)-1](\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi}+\delta_{ak}\delta_{bl}+\delta_{al}\delta_{bk}), \tag{16}$$

and takes care of the fact that $q_{ij}=q_{ij}'$ when $ab=ij$ or $ab=kl$. In other words, it accounts for the fact that in the unitary relation in Fig. 1 relative energies of the two independent pairs are separately conserved. The extra factor of $\frac{1}{8}$ on the right comes from the symmetry under interchange of the pairs ij and kl as well as ab and cd. Using

$$\mathrm{Disc}\left(\frac{F}{D_{ij}D_{kl}}\right)=F\,\mathrm{Disc}\,\frac{1}{D_{ij}D_{kl}}+\frac{1}{D_{ij}^*D_{kl}^*}\mathrm{Disc}F \tag{17}$$

as in the three-body case and noting that $\mathrm{Disc}[1/(D_{ij}D_{kl})]=\mathrm{Im}[1/(D_{ij}D_{kl})]$ and using Eq. (8) for $\mathrm{Im}[1/(D_{ij}D_{kl})]$, we find that $\mathrm{Im}[1/(D_{ij}D_{kl})]$ terms on the left of Eq. (15) just cancel the terms on the right with $ab=ij$, $cd=kl$, $ab=kl$, $cd=ij$, and permutations. Equating appropriate coefficients, taking account of symmetries, and canceling common factors, we derive

$$\mathrm{Disc}\langle\vec{q}\,|F_{ij,kl}(E)\,|\vec{p}_i+\vec{p}_j\rangle=-\frac{\pi}{8}\,\frac{1}{(2\pi)^6}\int\left(\prod_{i=1}^{4}d^3p_i'\right)\delta(\vec{p}_i+\vec{p}_j-\vec{p}_i'-\vec{p}_j')\delta(\vec{p}_k+\vec{p}_l-\vec{p}_k'-\vec{p}_l')\delta\left(E-\sum_{i=1}^{4}p_i'^2\right)v_{kl}(q_{kl}')v_{ij}(q_{ij}')$$

$$\times\sum_{\substack{a,b,c,d=1\\ ab,cd\neq ij,kl}}^{4}\langle\vec{q}\,|F_{ab,cd}(E)\,|\vec{p}_a'+\vec{p}_b'\rangle\,\frac{v_{ab}(q_{ab}')}{D_{ab}(2q_{ab}'^2)}\,\frac{v_{cd}(q_{cd}')}{D_{cd}(2q_{cd}'^2)}\,. \tag{18}$$

For the special case of identical particles, we make a transformation of integration variables to $\vec{p}_i'=\frac{1}{2}(\vec{p}+\vec{k}+\vec{s})$, $\vec{p}_j'=\frac{1}{2}(\vec{p}-\vec{k}-\vec{s})$, $\vec{p}_k'=\frac{1}{2}(-\vec{p}+\vec{k}-\vec{s})$, and $\vec{p}_l'=\frac{1}{2}(-\vec{p}-\vec{k}+\vec{s})$. Here, we take $\vec{p}_i+\vec{p}_j=-(\vec{p}_k+\vec{p}_l)=\vec{p}$ to evaluate two integrals with the two momentum δ functions and obtain

$$\mathrm{Disc}\langle\vec{q}\,|F(E)\,|\vec{p}\rangle=-\frac{\pi}{(2\pi)^6}\int d^3k\,d^3s\,\delta(E-p^2-k^2-s^2)v(|\tfrac{1}{2}(\vec{k}+\vec{s})|)v(|\tfrac{1}{2}(\vec{k}-\vec{s})|)$$

$$\times\left(\langle\vec{q}\,|F(E)\,|\vec{k}\rangle\,\frac{v(|\frac{1}{2}(\vec{p}+\vec{s})|)}{D(\frac{1}{2}(\vec{p}+\vec{s})^2)}\,\frac{v(|\frac{1}{2}(\vec{p}-\vec{s})|)}{D(\frac{1}{2}(\vec{p}-\vec{s})^2)}\right. \tag{19}$$

$$\left.+\langle\vec{q}\,|F(E)\,|\vec{s}\rangle\,\frac{v(|\frac{1}{2}(\vec{p}+\vec{k}|)}{D(\frac{1}{2}(\vec{p}+\vec{k})^2)}\,\frac{v(|\frac{1}{2}(\vec{p}-\vec{k})|)}{D(\frac{1}{2}(\vec{p}-\vec{k})^2)}\right).$$

It is easy to see that the two terms in Eq. (19) are equal to one another and in the special case of identical particles Eq. (19) becomes

$$\mathrm{Disc}\langle\vec{q}\,|F(E)\,|\vec{p}\rangle=-\frac{2\pi}{(2\pi)^6}\int d^3k\langle\vec{q}\,|F(E)\,|\vec{k}\rangle$$

$$\times\int d^3s\,\delta(E-p^2-k^2-s^2)v(|\tfrac{1}{2}(\vec{k}+\vec{s})|)v(|\tfrac{1}{2}(\vec{k}-\vec{s})|)\,\frac{v(|\frac{1}{2}(\vec{p}+\vec{s})|)}{D(\frac{1}{2}(\vec{p}+\vec{s})^2)}\,\frac{v(|\frac{1}{2}(\vec{p}-\vec{s})|)}{D(\frac{1}{2}(\vec{p}-\vec{s})^2)}\,. \tag{20}$$

Before going into detailed discussion about implementation of Eq. (20), we turn to the consideration of unitarity constraints on G.

B. Unitarity constraints on G

Next we would like to find the constraints of unitarity on G. This forms a quasi-two-body state, where one of the four particles is free and the three others appear in a correlated quasiparticle state. The information about the interaction of these three particles in the final state will be contained in the three-body subenergy term of the four-body unitarity relation. This is the next to last term in the last but one line in the unitarity relation in Fig. 5. This has the three-body subenergy unitarity cut, which will give the correct unitarity constraint on G.

If we keep only this term in four-body unitarity, we have for the discontinuity across this cut schematically

$$\mathrm{Disc}T_{2,4}=-\pi\sum T_{2,4'}\delta(E-E')T^*_{3',3}\delta\,, \tag{21}$$

where $T_{3'3}$ is the connected three-body amplitude and δ refers to the "fly-by" particle. We now substitute the second term of Eq. (12) into Eq. (21) to obtain

$$\frac{(2\pi)^3\delta(\sum\vec{p}_l)}{2}\sum_{i,j,k,l=1}^{4}\left[\mathrm{Disc}\left(\langle\vec{q}\,|G_{i,jkl}(E)|\vec{p}_i\rangle\frac{\omega_{jk,l}(q_{jk,l})}{D'_{jkl}(\epsilon_{jkl})}\frac{v_{jk}(q_{jk})}{D_{jk}(2q_{jk}^{\,2})}\right)\right.$$
$$=-\frac{\pi}{2}\int\sum_{a,b,c,d=1}^{4}\langle\vec{q}\,|G_{a,bcd}(E)|\vec{p}_a'\rangle\frac{\omega_{bc,d}(q'_{bc,d})}{D'_{bcd}(\epsilon_{bcd})}\frac{v_{bc}(q'_{bc})}{D_{bc}(2q'^2_{bc})}\,\delta(\vec{p}_i-\vec{p}_i')\delta(\vec{p}_j+\vec{p}_k+\vec{p}_l-\vec{p}_j'-\vec{p}_k'-\vec{p}_l')$$
$$\times\,\delta\left(E-\sum_{i=1}^{4}p_i'^2\right)\frac{1}{(2\pi)^6}\left(\prod_{i=1}^{4}d^3p_i'\right)\omega_{jk,l}(q'_{jk,l})\frac{v_{jk}(q'_{jk})}{D_{jk}(2q'^2_{jk})}\frac{1}{D'^*_{jkl}(\epsilon_{jkl})}$$
$$\left.\times\,\omega_{jk,l}(q_{jk,l})\frac{v_{jk}(q_{jk})}{D_{jk}(2q_{jk}^{\,2})}\right]. \tag{22}$$

Here as in Eq. (10) we have used $D=D^*$ across this cut. Now using $\mathrm{Disc}(G/D')=G\,\mathrm{Disc}(1/D')+(1/D'^*)\mathrm{Disc}G$ as before and using Eq. (11) we find that $\mathrm{Disc}(1/D')$ term on the left just cancels the term on the right with $a=i$. Equating appropriate coefficients and canceling common factors, then gives

$$\mathrm{Disc}\langle\vec{q}\,|G_{i,jkl}(E)|\vec{p}_i\rangle=-\frac{\pi}{2}\frac{1}{(2\pi)^6}\int\left(\prod_{i=1}^{4}d^3p_i'\right)\delta(\vec{p}_i-\vec{p}_i')\delta(\vec{p}_j+\vec{p}_k+\vec{p}_l-\vec{p}_j'-\vec{p}_k'-\vec{p}_l')\delta\left(E-\sum_{i=1}^{4}p_i'^2\right)$$
$$\times\,\frac{v_{jk}(q'_{jk})}{D_{jk}(2q'^2_{jk})}\omega_{jk,l}(q'_{jk,l})\sum_{\substack{a,b,c,d=1\\ a\neq i}}^{4}\langle\vec{q}\,|G_{a,bcd}(E)|\vec{p}_a'\rangle\frac{\omega_{bc,d}(q'_{bc,d})}{D'_{bcd}(\epsilon_{bcd})}\frac{v_{bc}(q'_{bc})}{D_{bc}(2q'^2_{bc})}\,. \tag{23}$$

In the special case of identical particles the nine terms in the summation in Eq. (23) will reduce to five terms, because four of them are repeated twice. Then we have for identical particles (see Appendix for detail)

$$\mathrm{Disc}\langle\vec{q}\,|G(E)|\vec{p}\rangle$$
$$=-\frac{2\pi}{(2\pi)^6}\int d^3k\,\frac{\langle\vec{q}\,|G(E)|\vec{k}\rangle}{D'(E-\frac{4}{3}k^2)}\int d^3s\,\delta(E-p^2-k^2-s^2-(\vec{p}+\vec{k}+\vec{s}\,)^2)$$
$$\times\left[\left(\frac{\omega(|\frac{1}{3}(\vec{k}+3\vec{p})|)\,v(|\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s}\,)|)}{D(\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s}\,)^2)}+\frac{\omega(|\frac{1}{3}(3\vec{p}+3\vec{s}+2\vec{k}\,)|)\,v(|\frac{1}{2}(\vec{p}-\vec{s}\,)|)}{D(\frac{1}{2}(\vec{p}-\vec{s})^2)}\right.\right.$$
$$\left.+\,\frac{\omega(|\frac{1}{3}(3\vec{s}+\vec{k})|)\,v(|\frac{1}{2}(2\vec{p}+\vec{k}+\vec{s}\,)|)}{D(\frac{1}{2}(2\vec{p}+\vec{k}+\vec{s})^2)}\right)$$
$$\times\,\frac{v(|\frac{1}{2}(\vec{k}-\vec{s})|)\,\omega(|\frac{1}{3}(2\vec{p}+3\vec{k}+3\vec{s}\,)|)}{D(\frac{1}{2}(\vec{k}-\vec{s})^2)}+\frac{v(|\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s}\,)|)\,\omega(|\frac{1}{3}(\vec{p}+3\vec{k})|)}{D(\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s})^2)}$$
$$\left.\times\left(\frac{v(|\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s}\,)|)\,\omega(|\frac{1}{3}(3\vec{p}+\vec{k})|)}{2D(\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s})^2)}+\frac{v(|\frac{1}{2}(2\vec{p}+\vec{k}+\vec{s}\,)|)\,\omega(|\frac{1}{3}(\vec{k}+3\vec{s}\,)|)}{D(\frac{1}{2}(2\vec{p}+\vec{k}+\vec{s})^2)}\right)\right]. \tag{24}$$

Before we go to actual implementation of these constraints of unitarity (20) and (24), we give a brief discussion about their physical significance.

As in other similar problems unitarity relations, Eqs. (20) and (24) show that constraints of quantum mechanics introduce coherence and variation on amplitudes usually taken as independent and constant in phenomenological analysis. In particular, the amplitudes F and G have complicated branch points at the edge of the phase space. As in Eq. (8), F has two square-root branch points, where the argument of one square root depends on that of the other square root. The term G has the well-known $\epsilon^2 \log(-\epsilon)$ singularity where ϵ is the three-body subenergy. This comes from the phase space consideration of the unitarity relation in Fig. 3 and has been discussed elsewhere[15] for the three-body problem. In the case of the four-body problem, the same algebra goes through with three-body energy replaced by three-body subenergy. A test of the importance of the unitarity constraint in a particular problem is easily made with Eqs. (20) and (24). The F's and G's are assumed constant and the right-hand sides of Eqs. (20) and (24) are calculated. If they generate small discontinuities for F and G (measured against the assumed scale of F or G, respectively) the assumption of constant F or G is a good approximation. If Disc F or Disc G are large by the scale of F or G, the constraints of unitarity are important and must be implemented.

IV. IMPLEMENTATION

In the last section we saw unitarity forces F and G to be singular with the discontinuity across the singularity given by Eqs. (20) and (24), respectively. In order to be able to implement the conditions of unitarity (to make the algebra simple) we shall be limited to the consideration of four identical bosons each of mass $m(=\frac{1}{2})$. The best way to implement unitarity (20) and (24), so that it will give us information about the discontinuity is to disperse in E as in Refs. 2–5. Because the essential feature of the discontinuities of Eqs. (20) and (24) is a simple δ function in E, it is then trivial to disperse in E and maintain total energy analyticity. This method of implementation will not interfere with other singularities of the functions F and G.

We write the dispersion relation for $F(E)$ or $G(E)$ in schematic partial-wave form. Let us call the function $A(E)$ and assume that $A(E)$ goes to zero sufficiently rapidly as $E\to\infty$. Then we can write the dispersion relation for partial wave $A(E)$ as

$$A(E)=R(E)+\frac{1}{\pi}\int\frac{dE'}{E'-E}\,\mathrm{Disc}A(E')\,, \tag{25}$$

where $R(E)$ is a term, that does not have the discontinuity. Schematically the discontinuity is represented by

$$\mathrm{Disc}A(E')=-\pi\mathcal{A}(E')\delta(E'-E_0). \tag{26}$$

Substituting (26) into (25) we get

$$A(E)=R(E)+\frac{\mathcal{A}(E_0)+a(E,E_0)}{E-E_0}\,, \tag{27}$$

where $a(E_0,E_0)=0$ and hence does not contribute to the discontinuity of $A(E)$. The function a is arbitrary except for this condition and could be included in the definition of R, but it is more convenient to keep it explicitly. For example, if we take $a(E,E_0) = \mathcal{A}(E)-\mathcal{A}(E_0)$, Eq. (27) becomes

$$A(E)=R(E)+\frac{\mathcal{A}(E)}{E-E_0}\,. \tag{28}$$

In general, if in Eq. (26)

$$\mathcal{A}(E')=\mathcal{A}_1(E')\mathcal{A}_2(E')\mathcal{A}_3(E')\cdots\mathcal{A}_N(E'), \tag{29}$$

then after implementation with proper choices of $a(E,E_0)$, we can have E as an argument in some of the factors and E_0 as an argument in the rest of the factors of $\mathcal{A}$ in Eq. (28). The way to achieve this has been illustrated in Refs. 2, 4. The dispersion integral essentially puts the argument of the δ function in the denominator and in the multiplying factors we may or may not implement the constraint of the δ function (i.e., write in terms of E_0 or E).

From this it is clear that if the discontinuities of Eqs. (20) and (24) are dispersed in E, we get

$$\langle\vec{q}\,|F(E)|\,\vec{p}\rangle=\langle\vec{q}\,|R_1(E)|\,\vec{p}\rangle+\frac{2}{(2\pi)^6}\int d^3k\langle\vec{q}\,|F(E)|\,\vec{k}\rangle$$
$$\times\int\frac{d^3s}{E-p^2-k^2-s^2}\,\frac{v(|\frac{1}{2}(\vec{k}+\vec{s})|)\,v(|\frac{1}{2}(\vec{k}-\vec{s})|)\,v(|\frac{1}{2}(\vec{p}+\vec{s})|)\,v(|\frac{1}{2}(\vec{p}-\vec{s})|)}{D(E-k^2-\frac{1}{2}(\vec{p}+\vec{s})^2)D(E-k^2-\frac{1}{2}(\vec{p}-\vec{s})^2)}\,, \tag{30}$$

and

$$
\begin{aligned}
\langle\vec{q}|G(E)|\vec{p}\rangle = {} & \langle\vec{q}|R_2(E)|\vec{p}\rangle + \frac{2}{(2\pi)^6}\int d^3k \, \frac{\langle\vec{q}|G(E)|\vec{k}\rangle}{D'(E-\frac{4}{3}k^2)} \int \frac{d^3s}{E-k^2-s^2-p^2-(\vec{k}+\vec{s}+\vec{p})^2} \\
& \times\Bigg[\Bigg(\frac{\omega(|\frac{1}{3}(\vec{k}+3\vec{p})|)\,v(|\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s})|)}{D(E-\frac{3}{2}p^2-\frac{3}{2}k^2-\vec{p}\cdot\vec{k})} + \frac{\omega(|\frac{1}{3}(3\vec{p}+3\vec{s}+2\vec{k})|)\,v(|\frac{1}{2}(\vec{p}-\vec{s})|)}{D(E-\frac{3}{2}k^2-\frac{3}{2}(\vec{k}+\vec{s}+\vec{p})^2+\vec{k}\cdot(\vec{k}+\vec{s}+\vec{p}))} \\
& \quad + \frac{\omega(|\frac{1}{3}(3\vec{s}+\vec{k})|)\,v(|\frac{1}{2}(2\vec{p}+\vec{k}+\vec{s})|)}{D(E-\frac{3}{2}k^2-\frac{3}{2}s^2-\vec{k}\cdot\vec{s})}\Bigg) \frac{v(|\frac{1}{2}(\vec{k}-\vec{s})|)\,\omega(|\frac{1}{3}(2\vec{p}+3\vec{k}+3\vec{s})|)}{D(E-\frac{3}{2}p^2-\frac{3}{2}(\vec{k}+\vec{s}+\vec{p})^2+\vec{p}\cdot(\vec{k}+\vec{s}+\vec{p}))} \\
& \quad + \frac{v(|\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s})|)\,\omega(|\frac{1}{3}(\vec{p}+3\vec{k})|)}{D(E-\frac{3}{2}p^2-\frac{3}{2}k^2-\vec{p}\cdot\vec{k})} \\
& \quad \times\Bigg(\frac{1}{2}\,\frac{v(|\frac{1}{2}(\vec{p}+\vec{k}+2\vec{s})|)\,\omega(|\frac{1}{3}(3\vec{p}+\vec{k})|)}{D(E-\frac{3}{2}p^2-\frac{3}{2}k^2-\vec{p}\cdot\vec{k})} + \frac{v(|\frac{1}{2}(2\vec{p}+\vec{k}+\vec{s})|)\,\omega(|\frac{1}{3}(\vec{k}+3\vec{s})|)}{D(E-\frac{3}{2}k^2-\frac{3}{2}s^2-\vec{k}\cdot\vec{s})}\Bigg)\Bigg]\,. \qquad (31)
\end{aligned}
$$

The choice of implementation in Eqs. (30) and (31) is motivated by the fact that in the final integral equations we wish to get D, D', F, and G as functions of E, while v's and ω's should not depend on E. In other words in the language of Eq. (29) we have taken v's and ω's as functions of E_0 and D, D', F, and G as functions of E. We write v's and ω's as functions of momentum and that is what is physically expected of them. This has the added advantage that the left-hand cuts of v's and ω's do not get involved in the k and s integration of Eqs. (30) and (31) at a fixed value of E. These choices are, of course, all equivalent at the zero of the δ function so that the discontinuities (20) and (24) are independent of these choices. Any way of implementing unitarity by the dispersion relation satisfies that particular "subenergy analyticity," but our particular choice also satisfies total energy analyticity. Another motivation for this choice is that we get a set of dynamical equations for F and G and not just an integral representation for them.

Equations (30) and (31) are the "minimal" implementation of unitarity and analyticity, provide useful phenomenology for F and G, and do not contain any spurious singularity in E. However, this simple minimal implementation of unitarity and analyticity gives the minimal four-body dynamical scheme with the specific assumptions we made about the interaction. The only assumption made about the interaction is that contained in Eq. (12) through the introduction of v's and ω's. In the definition of F, we introduced only two-body separable interactions and in that of G have both two- and three-body separable interactions. Here, unlike in other similar implementations, the limited consideration of unitarity and analyticity does not give the full dynamical scheme.[1-5] It is clear from other similar implementations that if we formulate the N-body problem in terms of quasi-$(N-1)$-body amplitudes, simple implementation of unitarity and analyticity will give the full dynamical scheme, but may not necessarily give simple equations to solve. But the present analysis shows that other formulations of the problem are possible which do not give the full dynamics, but give useful equations for making approximations. It is interesting to recall that for the three-body problem there is[1,2] only one way of implementing unitarity and analyticity which gives the full dynamical scheme, the Faddeev equation with separable interactions, or the Amado model.[16] In order to fully understand the content of Eqs. (30) and (31), we need to make some assumption about the Born terms $R_1(E)$ and $R_2(E)$. We do this in the next section and try to develop certain useful approximation schemes for the problem.

V. USEFUL MODELS

If we look at Eqs. (30) and (31) we see that they are really integral equations in one vector variable and after partial-wave decomposition are simple equations to solve. Equations (30) and (31) are diagrammatically represented in Fig. 6. Here the incident state is either an nt or a dd type state. Diagrammatically the structure of the equations is very similar to the Amado model[16] for the three-body problem. Hence the complexity involved to solve these equations is similar to that involved to solve the Amado model. To build up any calculational scheme, we have to make assumptions about the incident state and also the Born term. Let us suppose we take the incident state to be an nt type state. Then we can easily rewrite the equations in Fig. 6 with certain assumptions about the Born term. For $R_1(E)$ we take a simple one particle exchange term, which is the so called driving term for the process. For $R_2(E)$, we do not have one such simple term. But the important driving terms for the process $nt\to nt$, if unitarity effects are important, are given by Eq. (31) itself or by Fig. 6(b) and are exhibited explicitly in Fig. 6(c). So we make the assumption that $R_2(E)$ is given by Fig. 6(c). Then the equations for the processes $nt\to nt$ and $nt\to dd$ are given by Fig. 7(a). In the first term on the right in Fig. 6(c), there is a bubble in the already dressed two-body propagator. This means a repetition of the two-body t matrices. This is redundant and in Fig. 7(a) we replace this term in Fig. 6(c) by a simple d particle exchange without any bubble. This term is shown in Fig. 7(b). In Fig. 7(a) the circle is the amplitude for the process $nt \to dd$ and the square represents the amplitude for the process $nt \to nt$.

At this point we have in principle completed the construction of $2 \to 2$ reactions initiated by nt states. We can similarily construct the reaction amplitudes initiated by a dd state. Now $R_1(E)$ is dictated by the dynamical equation in Fig. 7(a) and the simplest form of $R_2(E)$ is the one particle exchange term. This is pictorially represented in Fig. 8. Here the rectangle represents a dd type final state and the rotated square represents an nt type final state. Figures 7(a) and 8 give all the equations for the $2 \to 2$ process.

The equations in Figs. 7 and 8 are derived from limited consideration of unitarity and analyticity. But they suffer from an obvious defect. In one type, only the dd intermediate state is allowed and in the other only the nt intermediate state is allowed. It is easy to develop a calculational model for these amplitudes in such a way that for each of them, both dd and nt type intermediate states are allowed. This can be achieved by introducing some *ad hoc* coupling between the two equations in Fig. 7(a) and in Fig. 8. This is represented in Fig. 9, which gives the equations for all possible $2 \to 2$ pro-

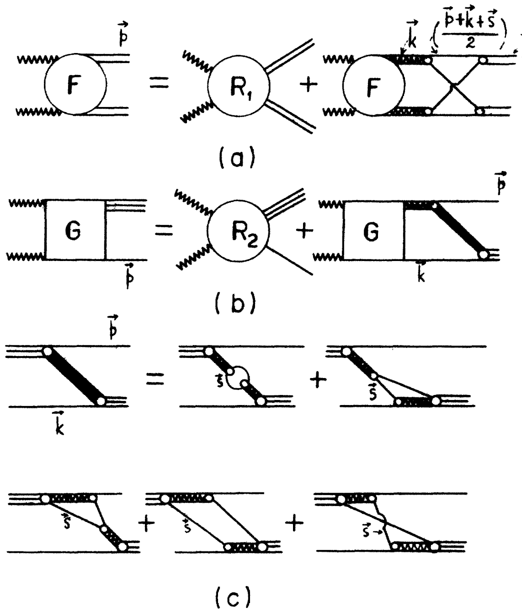

FIG. 6. Schematic representation of (a) Eq. (30) and (b) Eq. (31) with (c) as the driving term of Eq. (31). The curly lines represent that they could be either an dd or nt type states.

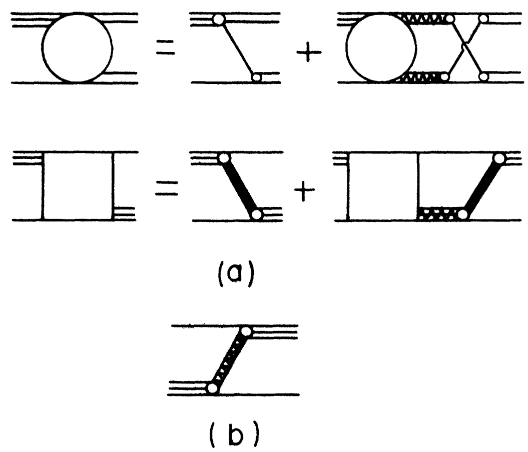

FIG. 7. (a) Schematic representation of $2 \to 2$ reactions induced by a nt type initial state. (b) The main d exchange driving term in $nt \to nt$ reaction (see text for detail). Here the circle refer to a $nt \to dd$ amplitude and the square to an $nt \to nt$ amplitude.

cesses. These equations can be easily solved after partial-wave projections. Fonseca and Shanley[11] solved very similar equations for four identical particles. They found their model from a consideration of a simplified version of the generalization of the Lee model to the four-body problem.[12] They completely neglected the interaction mechansim shown in Fig. 6(a). Their equations are a special case of the equations shown in Fig. 9 and are exhibited explicitly in Fig. 10. Apart from the obvious difference in the nature of the equations in Figs. 9 and 10, the driving term in $nt \to nt$ process to be used in Fig. 10 is quite distinct from the one we obtained from consideration of unitarity and shown in Fig. 6(c). The first terms on the right in Figs. 6(c) and 10(c) are the same; of course, we have to consider the modification suggested in Fig. 7(b). The next two terms in Fig. 6(c) are the first-order vertex correction to the two vertices of the

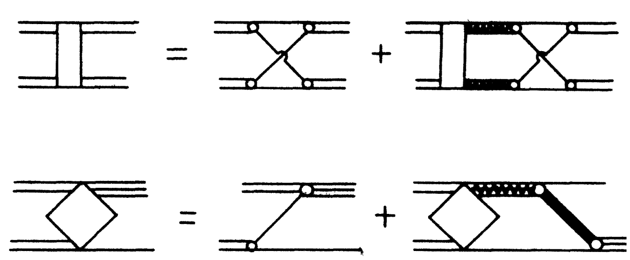

FIG. 8. Schematic representation of $2 \to 2$ reactions initiated by a dd type initial state. Here the rectangle refers to a $dd \to dd$ amplitude and the rotated square to a $dd \to nt$ amplitude.

first term on the right. This first-order correction is the first of an infinite series of correction to an ω vertex. This has been discussed by Fonseca and Shanley and is shown in Fig. 11. The first-order correction to this ω vertex is to be included if unitarity effects are important and will be included in our dynamical scheme. The second and third terms of Fig. 10(c) are not included in Fig. 6(c) but it is easy to see that they can be easily generated by iteration of the equations in Fig. 9. The fourth terms of Figs. 6(c) and 10(c) are the same and the last term of Fig. 6(c) is the exchange diagram corresponding to this term. Hence Eq. (9) is superior to Eq. (10) in driving terms as well as richer in dynamics. Of course, there are more complicated terms in the dynamics, which are not included in Fig. 9. Numerically the dimension of equations in Fig. 9 is twice that of equations of Fonseca and Shanley. Otherwise the two sets of equations are similar. Fonseca and Shanley solved their model for all possible processes including breakups. The present sets of equations can be solved exactly in a similar way.

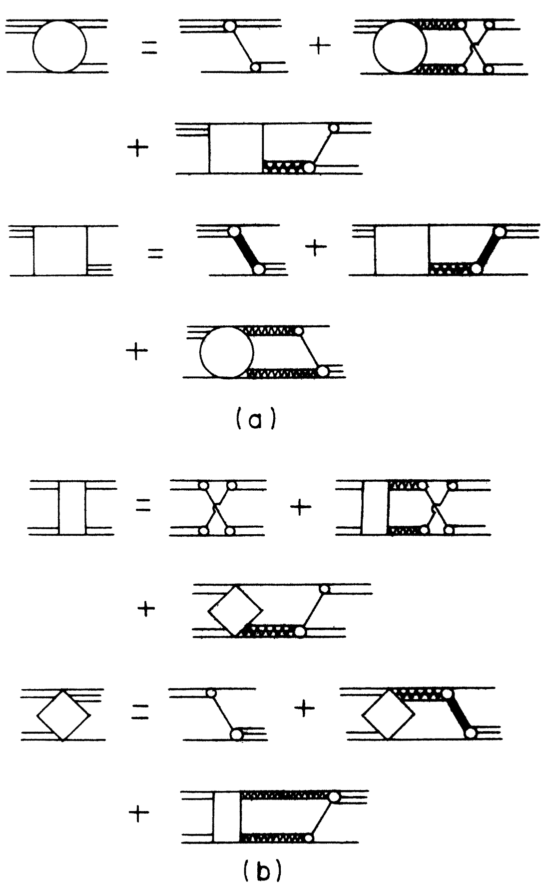

FIG. 9. Schematic representation of the following: (a) $nt \to dd$, $nt \to nt$; (b) $dd \to dd$, $dd \to nt$ reactions. The circle, square, etc. have the same meaning as in Figs. 7 and 8.

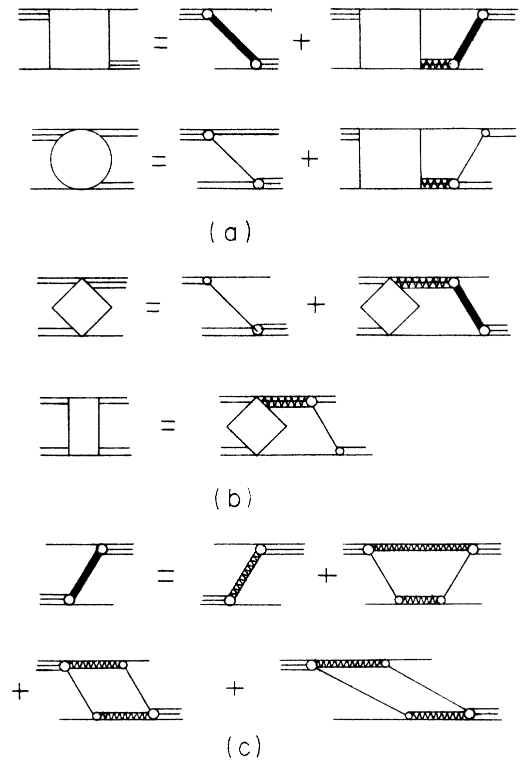

FIG. 10. Schematic representation of the following: (a) $nt \to nt$, $nt \to dd$; (b) $dd \to nt$, $dd \to dd$ reactions; and (c) a driving term to be used in them according to a unitary model of Fonseca and Shanley. The circle, square, etc. have the same meaning as in Figs. 7 and 8.

VI. SUMMARY AND DISCUSSION

We have considered the effects of unitarity, independent pair two-body unitarity and three-body subenergy unitarity, and analyticity on four-body final states. In the framework of the quasiparticle or isobar model, the four-body states are thought of as quasi-two-body states, where the quasi-particle states are either 2 two-body resonances or 1 three-body resonance and a free particle. Though the amplitudes for forming these quasi-two-body states—F and G—are usually taken to be constants

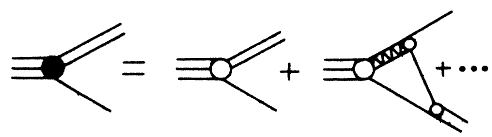

FIG. 11. First few terms contributing to the nd vertex correction. The two terms shown in the figure are retained in our formulation.

(up to kinematic factors) and independent in phenomenological analysis, we have seen that unitarity constraints force them to have branch points and be interrelated and coherent. The different F amplitudes taken together contain any information about independent-pair interactions in final states. Similarly the different G's unite coherently to contribute to three-body interactions in final states. When the resonances in four-body final states are narrow and nonoverlapping, constraints of unitarity are not important. Otherwise the constraints of unitarity should be implemented by writing a dispersion relation for the amplitudes in terms of their discontinuities. This automatically satisfies "subenergy" unitarity and analyticity. We also show how to maintain total energy analyticity. This yields the minimal four-body equations consistent with unitarity and analyticity. The equations are in one vector variable and are easy to solve after partial-wave analysis. Similar equations have been constructed and solved by Fonseca and Shanley. We apply our equations to the case of four identical bosons and get very useful equations for the problem. We calculate the amplitudes for all possible $2 \to 2$ processes and the breakup amplitudes can be related to them.[11]

At the expense of computer time and algebraic complexity the present model can be generalized to include spin, isospin, etc. We did not include that in our present formulation. We developed our model stressing unitarity and analyticity and this must work well when unitarity effects are important. We expect these effects to be important for two classes of strongly overlapping final state interactions, threshold enhancements as one encounters in the nucleon-nucleon system, particularly for s wave pairs, and resonance interactions. Thus these effects could be important in reactions leading to four nucleons in the final state and other similar problems.

The equations shown in Figs. 7(a) and 8 arose from the minimal implementation of unitarity and analyticity. The suggested equation in Fig. 9 contains more information than required by unitarity and analyticity. But all of them suffer from the defect that the three-body scattering amplitude is approximated by a single separable amplitude. To do a realistic calculation the effects of higher partial waves and more terms in each partial wave have to be included at least in a perturbative way. Some finite rank method for solving the equations is desirable because that minimizes the computer time substantially. One such method developed by the present author and Sloan[17] for the three-body problem appears to be very attractive for this problem.

The author would like to acknowledge gratefully a critical communication from Professor R. D. Amado and some informal discussions about few-body problems with Professor W. Glöckle, Professor H. T. Coelho, and Professor F. A. B. Coutinho.

APPENDIX

Equation (23) can be written as

$$
\begin{aligned}
\mathrm{Disc}\langle \vec{q}\,|G_{i,jkl}(E)|\,\vec{p}_i\rangle = &-\frac{\pi}{(2\pi)^6}\int d^3p_j\, d^3p_k\, \delta(E-p_i^2-p_j^2-p_k^2-(\vec{p}_i+\vec{p}_j+\vec{p}_k)^2) \\
&\frac{v_{jk}(|\tfrac{1}{2}(\vec{p}_j-\vec{p}_k)|)\,\omega_{jk,l}(|\tfrac{1}{3}(3\vec{p}_j+3\vec{p}_k+2\vec{p}_l)|)}{D_{jk}(\tfrac{1}{2}(\vec{p}_j-\vec{p}_k)^2)} \\
&\times\Bigg\{\Bigg[\frac{\langle \vec{q}\,|G_{j,kli}(E)|\,\vec{p}_j\rangle}{D'_{kli}(E-\tfrac{4}{3}p_j^2)}\Bigg(\frac{\omega_{kl,i}(|\tfrac{1}{3}(3\vec{p}_i+\vec{p}_j)|)\,v_{kl}(|\tfrac{1}{2}(2\vec{p}_k+\vec{p}_j+\vec{p}_i)|)}{D_{kl}(E-\tfrac{3}{2}p_i^2-\tfrac{3}{2}p_j^2-\vec{p}_i\cdot\vec{p}_j)} \\
&\qquad+\frac{\omega_{il,k}(|\tfrac{1}{3}(3\vec{p}_k+\vec{p}_j)|)\,v_{il}(|\tfrac{1}{2}(2\vec{p}_i+\vec{p}_j+\vec{p}_k)|)}{D_{il}(E-\tfrac{3}{2}p_k^2-\tfrac{3}{2}p_j^2-\vec{p}_k\cdot\vec{p}_j)} \\
&\qquad+\frac{\omega_{ik,l}(|\tfrac{1}{3}(3\vec{p}_i+3\vec{p}_k+2\vec{p}_j)|)\,v_{ik}(|\tfrac{1}{2}(\vec{p}_i-\vec{p}_k)|)}{D_{ik}(E-\tfrac{3}{2}p_j^2-\tfrac{3}{2}(\vec{p}_i+\vec{p}_j+\vec{p}_k)^2+\vec{p}_j\cdot(\vec{p}_i+\vec{p}_j+\vec{p}_k))}\Bigg)\Bigg] \\
&+\{k\leftrightarrow j\}+\frac{\langle \vec{q}\,|G_{l,ijk}(E)|-\vec{p}_i-\vec{p}_j-\vec{p}_k\rangle}{D'_{ijk}(E-\tfrac{4}{3}(\vec{p}_i+\vec{p}_j+\vec{p}_k)^2)} \\
&\times\Bigg(\frac{\omega_{jk,i}(|\tfrac{1}{3}(\vec{p}_j+\vec{p}_k-2\vec{p}_i)|)\,v_{jk}(|\tfrac{1}{2}(\vec{p}_j-\vec{p}_k)|)}{D_{jk}(\tfrac{1}{2}(\vec{p}_j-\vec{p}_k)^2)} \\
&\qquad+\frac{\omega_{ij,k}(|\tfrac{1}{3}(\vec{p}_i+\vec{p}_j-2\vec{p}_k)|)\,v_{ij}(|\tfrac{1}{2}(\vec{p}_j-\vec{p}_i)|)}{D_{ij}(\tfrac{1}{2}(\vec{p}_j-\vec{p}_i)^2)} \\
&\qquad+\frac{\omega_{ik,j}(|\tfrac{1}{3}(\vec{p}_i+\vec{p}_k-2\vec{p}_j)|)\,v_{ik}(|\tfrac{1}{2}(\vec{p}_k-\vec{p}_i)|)}{D_{ik}(\tfrac{1}{2}(\vec{p}_k-\vec{p}_i)^2)}\Bigg)\Bigg\}\,,
\end{aligned}
\tag{A1}
$$

where $\{k \leftrightarrow j\}$ denotes the first three terms with k and j interchanged. So the two sets of terms, whose coefficients are $G_{j,kli}$ and $G_{k,lij}$, when taken together are symmetric with respect to $k \leftrightarrow j$ interchange. Similarly the last two terms, whose coefficient is $G_{l,ijk}$ are symmetric with respect to $k \leftrightarrow j$ interchange. But $\vec{p}_j$ and $\vec{p}_k$, are integration variables. For identical particles only the distinct terms in Eq. (A1) are supposed to be taken. That is what has been considered in Eq. (24), after a change of integration variable in some of the terms.

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