Theoretical estimate of the probability of pair production in α decay

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A theoretical estimate has been made of the probability of electron-positron pairs being produced in the α decay process. The results are in agreement with the experimental value observed in the decay of ^{241}Am, which is the only experimental result currently available. Estimates of the probabilities of pair production in other possible α decays are also given.

[RADIOACTIVITY Calculation of internal pair formation in α decay.]

I. INTRODUCTION

Internal pair production (IPP) in β decay is well known.[1-4] An experimental observation of the phenomenon was made by Greenberg and Deutsch[3] who investigated the pair production associated with the decays of ^{32}P and ^{90}Y. A calculation by Huang[4] for the allowed transition of ^{32}P was in good agreement with the experimental results, and the calculations of Richards and Rose[5] for first forbidden unique β transitions are in good agreement with an extrapolated value of Huang's calculation. These calculations neglected possible detour contributions via intermediate nuclear states and the validity of this assumption is demonstrated in the bremsstrahlung calculations of Hubbard and Rose.[6]

The only report of IPP in α decay is that of Ljubičić and Logan[7] who observed pairs being produced in the α decay of ^{241}Am. A rate of $(3.1\pm 0.6) \times 10^{-9}$ pairs per α decay was measured. A rough estimate of the theoretical probability was made by multiplying the estimated probability for α-particle bremsstrahlung by the internal pair formation coefficient. The bremsstrahlung probability values were based on calculated values for bremsstrahlung associated with meson production and the internal pair coefficients of Rose were used. It was estimated that 1.2×10^{-9} pairs per α disintegration should be produced. It was also estimated that the electron-positron pair production mechanism proceeding via real nuclear levels in ^{237}Np is negligible. No other theoretical estimates of IPP in α decay have been reported. This work reports a more detailed calculation of the probability of IPP in α decay.

Although IPP in β decay does not depend explicitly on the atomic number Z, it is evident that the Coulomb field will play an important role in α decay. It is not, for example, possible to use the Born approximation for α decay because $2(Z-2)\alpha$ is not small for any α emitter, and because the velocity v of α particles is relatively small, the normal approximation $[2(Z-2)\alpha]/v \ll 1$ is certainly not applicable. In our calculations the effects of the Coulomb field are taken into account approximately by assuming the wave function of the α particle to be that for a free outgoing wave with amplitudes of the Coulomb wave functions evaluated at the classical turning point of the full-energy α particle, which is well outside the region where nuclear potentials are effective. The wave functions of the electron and positron are assumed to be plane waves normalized on the Coulomb values at the origin ($r=0$).

It is expected that these approximations will be sufficiently accurate for the calculation of the total number of pairs per α disintegration but they would be less suitable for a calculation of the angular distribution of the charged particles. As in the calculations of IPP in β decay the sum of the possible intermediate states is approximated through only one state.

II. MATRIX ELEMENTS OF THE IPP PROCESS

We consider a nucleus of atomic number Z emitting s-wave α particles. The Hamiltonian for the interaction of the nucleus, the emitted α particle, and the electron-positron pair through the electromagnetic field is

$$H_{\text{int}} = H_{\alpha N} + ie\int d\vec{x}\,\bar{\psi}(\vec{x})\gamma_\mu\psi(\vec{x})A_\mu(\vec{x}) - \int d\vec{x}\,j_\mu^{(\alpha)}(\vec{x})A_\mu(\vec{x}) - \int d\vec{x}\,J_\mu^{(N)}(\vec{x})A_\mu(\vec{x}), \quad (1)$$

where $H_{\alpha N}$ is the coupling for the α decay, $ie\bar{\psi}\gamma_\mu\psi$ is the current of the electron-positron field, $J_\mu^{(N)}$ is the current of the nucleus, and $j_\mu^{(\alpha)}$ is the current of the α-particle scalar field

$$j_\mu^{(\alpha)}(x) = -2ei\left[\alpha^*(x)\frac{\partial\alpha(x)}{\partial x_\mu} - \alpha(x)\frac{\partial\alpha^*(x)}{\partial x_\mu}\right]. \quad (2)$$

We employ the units in which $m_e = c = \hbar = 1$ (m_e is the mass of the electron), $e^2/4\pi \equiv \alpha = 1/137$, $x_\mu \equiv (\vec{x}, it)$, and the Dirac matrices are $\vec{\gamma} = -i\beta\vec{\alpha}$, $\gamma_4 = \beta$. The nonvanishing matrix elements of $H_{\alpha N}$ are between the nuclear states of charge Z and $Z-2$:

$$N_\alpha(E) = g_\alpha \int d\vec{x}\, \phi^*_{Z-2}(\vec{x})\alpha^*_E(\vec{x})\phi_Z(\vec{x}), \quad (3)$$

where g_α is the coupling constant, α_E is the wave function of the α particle of total energy E, and ϕ_Z, ϕ_{Z-2} are the wave functions of the nuclei of charges Z and $Z-2$, respectively.

The matrix element M for the transition between the states $|Z\rangle$ and $|Z-2, \alpha, e^+e^-\rangle$ with the interaction (1) is

$$M = 2\pi\delta(U_{Z-2} + E + \epsilon_+ + \epsilon_- - U_Z)\mathfrak{M}, \quad (4)$$

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3,$$

where U_Z and U_{Z-2} are the total energies of the initial and final nuclear states and ϵ_+, ϵ_- are the total energies of the electron and positron. Neglecting the recoil of the nucleus, and taking into account only the Coulomb interaction between the charged particles, the amplitudes $\mathfrak{M}_i$ are

$$\mathfrak{M}_1 = 2M_Z Z e^2 g_\alpha \int d\vec{x}\, d\vec{y}\, d\vec{z}\, \phi^*_{Z-2}(\vec{y})\alpha^*_E(\vec{y})G(U_1;\vec{y},\vec{x}) \times D(\omega;\vec{x}-\vec{z})\phi_Z(\vec{x})\bar{\psi}_-(\vec{z})\gamma_4\psi_+(\vec{z}),$$

$$\mathfrak{M}_2 = 2M_{Z-2}(Z-2)e^2 g_\alpha \int d\vec{x}\, d\vec{y}\, d\vec{z}\, \phi^*_{Z-2}(\vec{y})G(U_2;\vec{y},\vec{x}) \times D(\omega;\vec{y}-\vec{z})\alpha^*_E(\vec{x})\phi_Z(\vec{x}) \times \bar{\psi}_-(\vec{z})\gamma_4\psi_+(\vec{z}), \quad (5)$$

$$\mathfrak{M}_3 = 2e^2 g_\alpha (E_0 + E) \int d\vec{x}\, d\vec{y}\, d\vec{z}\, \phi^*_{Z-2}(\vec{y})\alpha^*_E(\vec{x}) \times S(E_0;\vec{x},\vec{y})\phi_Z(\vec{y}) \times D(\omega;\vec{x}-\vec{z})\bar{\psi}_-(\vec{z})\gamma_4\psi_+(\vec{z}),$$

where M_Z, M_{Z-2} are the masses of the initial and final nuclei, ψ_+ and ψ_- are the wave functions of the positron and electron, and G, S, and D are the propagators of the nucleus, α particle, and photon, respectively. The energies of the propagators are given by:

$$\begin{aligned} E_0 &= U_Z - U_{Z-2}, \\ \omega &= \epsilon_+ + \epsilon_-, \\ U_1 &= U_Z - \epsilon_+ - \epsilon_-, \\ U_2 &= U_{Z-2} + \epsilon_+ + \epsilon_-. \end{aligned} \quad (6)$$

The corresponding Feynman diagrams are shown in Fig. 1.

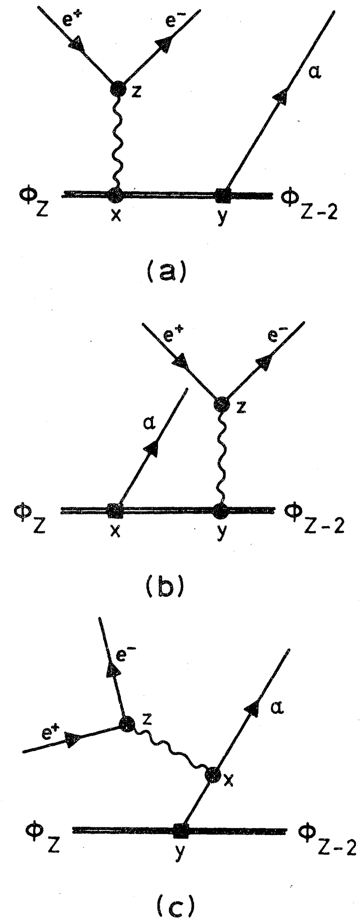

FIG. 1. Feynman diagrams for pair production in α decay.

It has been shown[5,6] that due to the resonance structure of nuclear propagators the contribution to the transition amplitudes $\mathfrak{M}_1$ or $\mathfrak{M}_2$ is dominated by only one intermediate nuclear state. Corresponding nuclear propagators could be approximated in that case by the expression:

$$\begin{aligned} G(U_1;\vec{y},\vec{x}) &\approx \frac{\phi_Z(\vec{y})\phi^*_Z(\vec{x})}{U_Z - U_1}, \\ G(U_2;\vec{y},\vec{x}) &\approx \frac{\phi_{Z-2}(\vec{y})\phi^*_{Z-2}(\vec{x})}{U_{Z-2} - U_2}. \end{aligned} \quad (7)$$

When Eqs. (5) and (7) are combined and only the monopole term in the interaction of the nucleus with the electromagnetic field is considered, it is seen that the sum of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is not explicitly Z dependent:

$$\mathfrak{M}_1 + \mathfrak{M}_2 = \frac{2e^2}{(\epsilon_+ + \epsilon_-)} N_\alpha(E) \frac{1}{(2\pi)^3} \times \int d\vec{z}\, d\vec{t}\, \frac{e^{i\vec{t}\vec{z}}}{\vec{t}^2 - \omega^2}\, \bar{\psi}_-(\vec{z})\gamma_4\psi_+(\vec{z}). \quad (8)$$

An analogous result has been obtained for IPP in β decay.

To calculate $\mathfrak{M}_3$ it is necessary to assume a value for the wave function of the emitted α particle. We have assumed that the α particles after emission [outside the nucleus at some point $\vec{x}$ in Fig. 1(c)] are free. However, the amplitudes of these free waves are influenced by the Coulomb field in such a way that they have the values of Coulomb wave functions for α particles of the full energy E_0 at the classical turning point R_0. Thus:

$$\alpha_E(\vec{x}) = C_{i,r}(E, E_0)\alpha_E^{\text{free}}(\vec{x}), \quad |\vec{x}| > R_N, \tag{9}$$

where R_N is the nuclear radius and α_E^{free} is the free wave function of the α particle. Typical values of R_N and R_0 for $Z=84$ and $W_0=6$ MeV are 7.5 and 45 fm, respectively. Nuclear forces are negligible in comparison to Coulomb forces at a distance of 45 fm from the nucleus. The factors $C_{i,r}$ are given in the Appendix.

In effect we assume that α particles of energy E are produced at the classical turning point for the energy E_0, with probabilities defined by the density of the Coulomb wave function at the turning point for the energy E_0. Using Eq. (9) the nuclear part of the $\mathfrak{M}_3$ amplitude can be written for s-wave emission as

$$g_\alpha \int d\vec{y}\, \phi^*_{Z-2}(\vec{y})\, S(E_0; \vec{y}, \vec{x})\phi_Z(\vec{y}) = \frac{C_i(E_0, E_0)}{4\pi} N_\alpha(E_0) \frac{e^{ig_0|\vec{x}|}}{|\vec{x}|}, \tag{10}$$

where $g_0 = (E_0{}^2 - M_\alpha{}^2)^{1/2}$ and C_i is given by (A4). Combining Eqs. (5), (9), and (10) we get for $\mathfrak{M}_3$

$$\mathfrak{M}_3 = \frac{2e^2}{4\pi} N_\alpha(E_0)\frac{E+E_0}{(2EV_0)^{1/2}} C_r(E, E_0) \times C_i(E_0, E_0)\frac{1}{(2\pi)^3}\int d\vec{x}\, d\vec{z}\, d\vec{t}\, \frac{e^{ig_0|\vec{x}|}}{|\vec{x}|} e^{-i\vec{g}\vec{x}} \frac{e^{-i\vec{t}(\vec{x}-\vec{z})}}{\vec{t}^2-\omega^2}\bar{\psi}_-(\vec{z})\gamma_4\psi_+(\vec{z}), \tag{11}$$

where $g = (E^2 - M_\alpha{}^2)^{1/2}$ and C_r is given by (A4). V_0 is the normalization volume.

Normalizing the free wave functions of the electron and positron of the Coulomb values at the origin we have

$$\psi_\pm(\vec{z}) = [F(Z, \epsilon_\pm)]^{1/2}\frac{u_\pm(\vec{p}_\pm)}{V_0^{1/2}} e^{-i\vec{p}_\pm\vec{z}}, \tag{12}$$

where $\vec{p}_\pm$ are the momenta of the electron and positron, respectively, and $u_\pm$ the corresponding spinors. The factors $F(Z, \epsilon_\pm)$ are given by (A5).

Combining Eqs. (8), (11), and (12) and integrating by neglecting the momenta of the pair in comparison with the momentum of the α particle (because of the mass difference and because the contribution of low energy α particles is small) we have for $\mathfrak{M}$:

$$\mathfrak{M} = \frac{2e^2}{(2E)^{1/2}}\frac{[F(Z,\epsilon_+)F(Z,\epsilon_-)]^{1/2}}{V_0^{3/2}} \times [N_\alpha(E) + C_i(E_0, E_0)C_r(E, E_0)N_\alpha(E_0)] \times \frac{\bar{u}_-(\vec{p}_-)\gamma_4 u_+(\vec{p}_+)}{(\epsilon_+ + \epsilon_-)(\vec{t}^2-\omega^2)}, \tag{13}$$

where $\vec{t}^2 - \omega^2 = 2(\vec{p}_+\vec{p}_- - \epsilon_+\epsilon_- - 1)$.

However, as the energy difference $E_0 - E$ must be at least 1.022 MeV in the IPP process, the first term in the bracket of Eq. (13) is negligible for large Z. This is because the ratio of the nuclear matrix elements $|N_\alpha(E)|^2/|N_\alpha(E_0)|^2$ is the same as the ratio for the decay probabilities for α particles of energies E and E_0 and is given by[8]

$$\frac{|N_\alpha(E)|^2}{|N_\alpha(E_0)|^2} \approx 10^{1.61(Z-2)(1/\sqrt{W_0}-1/\sqrt{W})}. \tag{14}$$

W_0 and W are the kinetic energies of the α particles of total energies E_0 and E (in units of MeV). The Coulomb factor $C_r(E, E_0)$ does not vary quickly enough to compensate for this and thus:

$$\mathfrak{M} = \frac{2e^2}{(2E)^{1/2}} N_\alpha(E_0)\frac{[F(Z,\epsilon_+)F(Z,\epsilon_-)]^{1/2}}{V_0^{3/2}} \times C_i(E_0, E_0)C_r(E, E_0)\frac{\bar{u}_-(\vec{p}_-)\gamma_4 u_+(\vec{p}_+)}{(\epsilon_+ + \epsilon_-)(\vec{t}^2-\omega^2)}. \tag{15}$$

For typical values of $Z=84$ and $W_0=6$ MeV the ratio of the first and second term in the square brackets in Eq. (13) is less than 5×10^{-3}.

III. TRANSITION PROBABILITY FOR IPP

The differential transition probability per second for the IPP process in α decay is given by

$$d\tilde{w} = \frac{1}{T}\sum_{\mu\nu}|M|^2\frac{V_0{}^3}{(2\pi)^9}\, d\vec{g}\, d\vec{p}_+ d\vec{p}_-, \tag{16}$$

where $\sum_{\mu\nu}$ is the sum over the spin states of the electron-positron pair.

Combining Eqs. (4), (15), and (16), integrating, and using standard trace techniques to sum over the spin states, we obtain the differential angular distribution for created pairs dw:

$$dw = \frac{\alpha^2}{\pi^3}|N_\alpha(E_0)|^2 F(Z,\epsilon_+)F(Z,\epsilon_-)C_i{}^2(E_0,E_0) \times C_r{}^2(E,E_0)\frac{g p_+ p_-}{(\epsilon_+ + \epsilon_-)^2}\frac{\epsilon_+\epsilon_- + \vec{p}_+\vec{p}_- - 1}{(\epsilon_+\epsilon_- + 1 - \vec{p}_+\vec{p}_-)^2} \times d(-\cos\theta)\, d\epsilon_+ d\epsilon_-, \tag{17}$$

where θ is the angle between $\vec{p}_+$ and $\vec{p}_-$ and g is, in a nonrelativistic form, given by $g=(2M_\alpha)^{1/2}(W_0-\epsilon_+-\epsilon_-)^{1/2}$. To calculate the total number of pairs we integrate Eq. (17) over the appropriate angular and energy ranges. The angular integral is straightforward:

$$I(\epsilon_+,\epsilon_-)=\int_0^\pi \frac{\epsilon_+\epsilon_-+\vec{p}_+\vec{p}_--1}{(\epsilon_+\epsilon_-+1-\vec{p}_+\vec{p}_-)^2}\,d(-\cos\theta) \tag{18a}$$

and I is given by

$$I(\epsilon_+,\epsilon_-)=\frac{4\epsilon_+\epsilon_-}{(\epsilon_++\epsilon_-)^2}+\frac{1}{p_+p_-}\ln\frac{\epsilon_+\epsilon_-+1-p_+p_-}{\epsilon_+\epsilon_-+1+p_+p_-}\,. \tag{18b}$$

Then the differential probability for pair creation in the energy interval $d\epsilon_+d\epsilon_-, dw(\epsilon_+,\epsilon_-)$ is

$$\begin{aligned} dw(\epsilon_+,\epsilon_-)=&\frac{\alpha^2}{\pi^3}|N_\alpha(E_0)|^2F(Z,\epsilon_+)F(Z,\epsilon_-)\\ &\times C_i^2(E_0,E_0)C_r^2(E,E_0)\frac{gp_+p_-}{(\epsilon_++\epsilon_-)^2}\\ &\times I(\epsilon_+,\epsilon_-)\,d\epsilon_+d\epsilon_-\,. \end{aligned} \tag{19}$$

The number of emitted pairs in α decay produced per second, B_α^{+-}, is the energy integral of Eq. (19):

$$B_\alpha^{+-}=\frac{\alpha^2}{\pi^3}|N_\alpha(E_0)|^2C_i^2(E_0,E_0)V(W_0)\,, \tag{20}$$

where

$$V(W_0)=\int_1^{W_0-1}d\epsilon_+p_+F(Z,\epsilon_+)K(\epsilon_+) \tag{21}$$

and

$$K(\epsilon_+)=\int_1^{W_0-\epsilon_+}d\epsilon_-F(Z,\epsilon_-)p_-C_r^2(E,E_0)\frac{gI(\epsilon_+,\epsilon_-)}{(\epsilon_++\epsilon_-)^2}\,. \tag{22}$$

The number of emitted α particles per second in α decay is

$$B_\alpha=\frac{(2M_\alpha)^{1/2}W_0^{\;1/2}}{2\pi}|N_\alpha(E_0)|^2\,. \tag{23}$$

Then the number of pairs created per α disintegration is given by

$$\frac{B_\alpha^{+-}}{B_\alpha}=\frac{2\alpha^2}{\pi^2}C_i^2(E_0,E_0)\frac{V(W_0)}{W_0^{\;1/2}}\,. \tag{24}$$

The necessary numerical integrations of Eqs. (21) and (22) can be performed to give predictions for specific α transitions.

IV. DISCUSSION

The only experimental value available for comparison is that obtained with ^{241}Am. This decay is dominated by the emission of 5.486 MeV (86%) and 5.445 MeV α particles (13%). Our theoretical prediction in this case is 2.3×10^{-9} pairs per disintegration, a value which is in excellent agreement with the experimental value of $(3.1\pm0.6)\times10^{-9}$ pairs per disintegration. In Table I numerical predictions are given for different Z and energy values. Sources which are available to experimental investigation have been chosen in several cases and new experimental measurements would be of obvious interest.

TABLE I. Number of pairs produced per α disintegration calculated for different Z and energy values.

Z	W_0 (MeV)	B_α^{+-}/B_α ($\times10^{-9}$)
95	5.224	1.8
	5.439	2.2
	5.482	2.4
	6.01	7.2
89	4.949	0.98
	5.54	4.5
	5.818	8.1
84	5.22	3.4
	5.68	9.4
	6.38	32.5

In our calculations of the transition probability for IPP in α decay the contribution of the mechanism which proceeds via nuclear intermediate states is neglected. This approximation is valid for low-energy α transitions. However, the transition amplitude for this mechanism depends strongly on the transition energy. For $Z=84$ and $W_0=6$ MeV the contribution of this mechanism to the total IPP decay rate is less than 0.5%, while for $W_0=10$ MeV and for the same nucleus this contribution is of the order of 5%. Therefore the possible effects of nuclear forces could be observed in the case of higher-energy α emitters.

APPENDIX

The classical turning point R_0 for an s-wave α particle with kinetic energy W_0 in the Coulomb field of $(Z-2)$ protons is

$$R_0=\frac{2(Z-2)\alpha}{W_0} \tag{A1}$$

and

$$\rho_0=R_0p_0=4(Z-2)\alpha\left(\frac{M_\alpha}{2W_0}\right)^{1/2}\,.$$

The values of the regular f and irregular g Coulomb wave functions in this field for an s-wave α particle with kinetic energy W at the point R_0 defined by (A1) are

$$f(R_0)=\frac{F(\rho_0)}{\rho_0}\,,\quad g(R_0)=\frac{G(\rho_0)}{\rho_0}\,, \tag{A2}$$

where the functions F and G have been given by Biedenharn *et al.*[9]:

$$\left\{\begin{matrix} F(\rho_0) \\ G(\rho_0)/\sqrt{3} \end{matrix}\right\} = \frac{\pi^{1/2}}{3}\rho^{1/6}\left\{ y^{1/2}[I_{-1/3}(z) \mp I_{1/3}(z)] + \rho_0^{-2/3}\frac{y^3}{5}[I_{-4/3}(z) \mp I_{4/3}(z)] \right.$$
$$+ \rho^{-4/3}\left[-\frac{3y^4-9y}{35}[I_{2/3}(z) \mp I_{-2/3}(z)] + \left(\frac{y^{11/2}}{50}+\frac{9y^{5/2}}{70}\right)[I_{-1/3}(z) \mp I_{1/3}(z)]\right]$$
$$\left. + \rho^{-4/3}\left[\frac{y^4+3y}{7}[I_{2/3}(z) \mp I_{-2/3}(z)] - \frac{3y^{5/2}}{14}[I_{-1/3}(z) \mp I_{1/3}(z)]\right] + \cdots \right\}, \tag{A3}$$

where $I_{n/3}(z)$ are the modified Bessel functions of the first kind and order $\pm\frac{1}{3}, \pm\frac{2}{3}, \ldots$ and

$$z = \tfrac{2}{3}y^{3/2},$$
$$y = \frac{\rho - \rho_0}{\rho_0^{1/3}}, \quad \rho \geq \rho_0,$$
$$\rho = 4(Z-2)\alpha\left(\frac{M_\alpha}{2W}\right)^{1/2}.$$

For large values of the parameter $\eta = 2(Z-2)/v$ the wave functions F_L and G_L for $L \neq 0$ are quite independent of L because the dependence is $(1 + [L^2(L+1)^2]/(96\eta^4))$. Thus the functions (A3) are good enough in our case for all relevant outgoing waves. Normalizing the free wave amplitudes of the outgoing α particle on the Coulomb function values at the point R_0 we have for the coefficients in (10):

$$C_i^2(E_0, E_0) = G^2(\rho_0), \qquad C_r^2(E, E_0) = F^2(\rho_0). \tag{A4}$$

The normalization factors of the plane wave functions of the electron and positron on the Coulomb values at the point $r = 0$ are

$$F(Z, \epsilon_\pm) = \frac{2\pi\eta_\pm}{1 - e^{-2\pi\eta_\pm}}, \tag{A5}$$

where

$$\eta_\pm = \pm Z\alpha\frac{\epsilon_\pm}{(\epsilon_\pm^2 - 1)^{1/2}}.$$

[1]N. Arley and C. Möller, K. Dan. Vidensk. Selsk., Mat.-Phys. Medd. 15, 90 (1938).
[2]L. Tisza, Phys. Z. Sowjetunion 11, 245 (1937).
[3]J. S. Greenberg and M. Deutsch, Phys. Rev. 102, 415 (1956).
[4]K. Huang, Phys. Rev. 102, 422 (1956).
[5]K. C. Richards and M. E. Rose, Phys. Lett. 25B, 501 (1967).
[6]D. F. Hubbard and M. E. Rose, Nucl. Phys. 84, 337 (1966).
[7]A. Ljubičić and B. A. Logan, Phys. Rev. C 7, 1541 (1973).
[8]E. Segrè, *Nuclei and Particles* (Benjamin, Reading, Mass., 1965), p. 278.
[9]L. C. Biedenharn, R. L. Gluckstern, M. H. Hull, and G. Breit, Phys. Rev. 97, 542 (1955).