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Basis for quasi-two-body scaling

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Recent measurements of high energy inclusive cross sections in ^{6}Li, close to the elastic limit, allow for experimental verification of the "coherent recoil" postulate of "quasi-two-body scaling." Theoretical arguments for the dominance of coherent recoil up to internal nucleon momenta $k \simeq 1.2$ GeV/c are presented, along with a new "derivation" of the relationship between quasi-two-body scaling and an "effective" ground state momentum distribution. A possible relationship between inclusive and elastic cross sections is explored.

[NUCLEAR REACTIONS Quasi-two-body scaling, "effective" momentum distributions.]

I. INTRODUCTION

In a recent series of pioneering papers[1-3] Amado and Woloshyn have used a simple exactly soluble one dimensional boson model with δ-function interactions between particle pairs to investigate general relationships governing the ground state momentum distribution, $n(k)$, at high momentum, $k \geqslant 0.2$ GeV/c, in nuclei. While this model is admittedly only a rough approximation to the actual nuclear ground state, it makes a first attempt to *delimit the regimes* that should occur in nature and which one can hope to distinguish experimentally. Above the very low momentum region, $k \leqslant 0.2$ GeV/c (where a Fermi gas picture can be employed, and which does not concern us in this work), Amado and Woloshyn predict two distinct regions. In the first of these regions the internal momentum k of a nucleon is balanced by being distributed approximately equally among the $A-1$ remaining nucleons. In this region, which we term the "coherent recoil" region, the model predicts an exponential falloff: $n(k) \cong \exp(-k/k_0)$ (although a realistic value for k_0 cannot be predicted from this model). Above a momentum k_t, which we denote as the *transition momentum*, and which is approximately independent of A, the high momentum of a particular nucleon is balanced by sharing the momentum k among a small number of nucleons, $n = 1, 2, \ldots$. Finally, in the asymptotic limit, the model reproduces the general result that $n(k) \cong 1/k^4$. In this asymptotic region $n(k)$ is dominated by two particle spatial correlations alone so that the balancing momentum is carried off by a single nucleon. In the one-dimensional model studied this region sets in at $k_a \cong 3k_t$. Above k_t, $n(k)$ falls off in an inverse polynomial fashion, slower than the exponential falloff in the coherent recoil region. The simple theory provides no estimate for k_t. Hopefully, a complete theory of the behavior of $n(k)$ would be expected to show the presence of these regions and to provide suitable predictions for k_0 and k_t.

We shall not use the proposals of Amado and Woloshyn in this work except as a general guide to the interpretation of high momentum phenomena. Rather we shall study the latest data on inclusive cross sections, in kinematic regions that are capable of measuring very high momenta, in order to determine the mechanisms that are at work. We shall find, in fact, that the data show that, in the final state, the recoils are indeed "coherent" up to momenta of 1.2 GeV/c, and we shall show, in a simple manner, that contributions of few-body ($n = 1, 2, 3$) recoils are suppressed by a kinematic mechanism that is *characteristic of inclusive measurements*.

To do so it is necessary to describe in some detail the method of "quasi-two-body scaling" recently introduced in a brief letter,[4] which provides a new organization of a large class of inclusive cross sections which bear on the subject of high nucleon momenta within nuclei. We concentrate our attention on the inclusive process, $p + A \to p + X$.

In this process we define the (momentum, energy) of the incident proton as (p, E_p), that of the detected proton as (q, E_q), that of one of the final state nucleons (later to be identified as the scattered proton) as $(p', E_{p'})$, and the momentum and energy of the remaining $A-1$ nucleons as $(\sum \vec{k}_i, \sum E_i)$. Thus we have

$$\vec{p} = \vec{q} + \vec{p}\,' + \sum \vec{k}_i \equiv \vec{q} + \vec{p}\,' - \vec{k}, \tag{1}$$

$$E_p = E_q + E_{p'} + \left(\sum E_i - m_A\right) \equiv E_q + E_{p'} - E_k. \tag{2}$$

The first point to be stressed is that quasi-two-body scaling (QTBS) is meant only to describe inclusive cross sections in kinematic regions not accessible in interactions of protons with free *stationary* protons. We restrict our treatment to kinematic regions that require at least $k \cong 0.2$ GeV/c for the process to be permitted by energy-momentum conservation.

The first postulate of QTBS is that the inclusive cross sections fall off rapidly with k, the final momentum of the $A-1$ nucleons, so that the minimum permissible final momentum, designated $k_{\min}$, dominates the cross section. Under these circumstances

$$|\vec{k}_{\min}| = |\vec{p} - \vec{q}| - |\vec{p}'| , \tag{3}$$

the dominant configuration arising with $\vec{k}_{\min}$ parallel to $\vec{p}'$.

The second postulate of QTBS is that the $A-1$ nucleons recoil in a coherent manner, i.e., that the $A-1$ nucleons recoil either as a bound nucleus in the ground state or as a nucleus bound or unbound in a state of low excitation, but that no *fast* nucleons, other than the one with $\vec{p}'$ are emitted. Thus QTBS describes a final state that *resembles* the kinematic configuration of an initial state where a proton shares its momentum k with the other $A-1$ nucleons. With this assumption we can write:

$$p' = [(E_p + m_p - T_{k_{\min}} - \bar{\epsilon} - E_q)^2 - m_p{}^2]^{1/2}, \tag{4}$$

where

$$T_{k_{\min}} = (k_{\min}{}^2 + m_{A-1}{}^2)^{1/2} - m_{A-1}$$

is the kinetic energy of recoil of the $A-1$ nucleus and $\bar{\epsilon}$ is the mean excitation energy plus the mass difference, $(m_{A-1} + m_p) - m_A$.

Equations (3) and (4) completely define $k_{\min}(p, q, \theta, A)$ where θ is the laboratory angle of the detected proton:

$$k_{\min} = (p^2 + q^2 - 2pq\cos\theta)^{1/2} - [(E_p + m_p - T_{k_{\min}} - \bar{\epsilon} - E_q)^2 - m_p{}^2]^{1/2}. \tag{5}$$

The third assumption of QTBS is that

$$\frac{d\sigma}{d^3q} = \frac{C(p, k_{\min}) G(k_{\min})}{|\vec{p} - \vec{q}|} , \tag{6}$$

where $G(k_{\min})/|\vec{p} - \vec{q}|$ is the "probability" of obtaining a recoiling $A-1$ nucleus of momentum $k_{\min}$ and $C(p, k_{\min})$ describes the p and $k_{\min}$ dependence of the cross section for the inclusive process $p + A \to p + p' +$ (recoiling $A-1$ nucleus). $C(p, k_{\min})$ is assumed to vary with $k_{\min}$ much more slowly than $G(k_{\min})$. Here, $d\sigma/d^3q$ is the inclusive cross section $(d^3\sigma/q^2 dq d\Omega_q)$.

If the inclusive process is indeed dominated by coherent recoil and if we assume that the recoiling $A-1$ nucleus is essentially a spectator, we can rephrase the preceding assumptions in an intuitively more appealing way. $(+k, +E_k)$ is now the virtual momentum and energy of the target nucleon. The incident proton of momentum $\vec{p}$ is scattered to the momentum $\vec{p}'$ lifting the target nucleon onto the mass shell with momentum $\vec{q}$. Requiring that the momentum transfer $\vec{k} \to \vec{q}$ $[t_x = (E_k - E_q)^2 - (\vec{k} - \vec{q})^2$, with $E_k \equiv m_p - \bar{\epsilon} - T_k]$ be much smaller than the momentum transfer $\vec{p} \to \vec{q}$ clearly ensures that the detected particle is the target nucleon. $C(p, k_{\min})$ is now proportional to the cross section for scattering of the incident proton by the target nucleon at the appropriate value of t and s, $[s = (E_p + E_k)^2 - (\vec{p} + \vec{k}_{\min})^2]$, while $G(k_{\min})/|\vec{p} - \vec{q}|$ is the probability of finding a nucleon in the nucleus of momentum $k_{\min}$. In fact, in an oversimplified view which neglects all final state interactions,[5] one can give an explicit expression for $G(k_{\min})$

$$\int_{k_{\min}}^{k_{\max}} n(k) k dk = G(k_{\min}) - G(k_{\max}) \cong G(k_{\min}) \tag{7}$$

and can obtain an explicit expression[4] for $C(p, k_{\min})$

$$C(p, k_{\min}) = \frac{s(s - 4m^2)}{32\pi^2 p m E_q} \frac{d\sigma(k_{\min} \to q)}{dt} , \tag{8}$$

where $d\sigma/dt$ is the elastic p-p scattering cross section.

Amado and Woloshyn have shown[6] in an elegant fashion how final state interactions destroy this simple dependence of $d\sigma/d^3q$ on $n(k)$. However, we shall return in Sec. IV to a discussion of this picture and to a way to modify it to retain the benefits of scaling by replacing $n(k)$ by an "effective momentum distribution," $n_{\text{eff}}(k)$. Meanwhile, we treat Eq. (6) in a completely phenomenological way. [Thus in Ref. 4 we plotted $|\vec{p} - \vec{q}| d\sigma/d^3q$ vs $k_{\min}$ to extract a first estimate of $G(k_{\min})$. For this phenomenological treatment the origin and numerical value of $C(p, k_{\min})$ is not important, and C was set equal to unity.]

At this point it is instructive to examine some of the features of Eq. (5). Two kinematic conditions are worthy of special comment. In the relativistic limit, $E_p \gg m_p$, and for nonrelativistic detected particles observed at 180° we have $k_{\min} \cong q$ so that the magnitude of the observed momentum is equal to the magnitude of the virtual momentum of the struck nucleon. In this case the momentum transfer to the struck particle is quite small. At 90° the relationship is quite different, i.e., for relativistic incident particles, $k \cong q^2/2m_p$. For the purposes of this work the most important feature of Eq. (5) is, however, contained in the dependence of $k_{\min}$ on the *mass* of the recoiling nucleus. This appears in the kinetic energy of recoil,

$$T_{k_{\min}} = (k_{\min}{}^2 + m_r{}^2)^{1/2} - m_r,$$

where m_r is the mass of the recoil. In the case of coherent recoil $m_r = m_{A-1}$, whereas in the extreme asymptotic region $m_r = m_p$. It is in this term that the A dependence of QTBS enters since it is the assumption of QTBS that $m_r = m_{A-1}$. However, it is important to recognize that the relationship of Eq. (5) between q and k with

$$T_{k_{\min}} = (k_{\min}{}^2 + m_r{}^2)^{1/2} - m_r$$

is valid for any m_r.

The body of this work is now contained in the following sections. In Sec. II we shall examine recently acquired data that demonstrate conclusively that up to 1.2 GeV/c the inclusive cross sections are dominated by coherent recoil, thus verifying one of the assumptions of QTBS. In Sec. III we shall examine the kinematics of the data that have been studied. We shall show that for most of the data studied in Ref. 4 there is *always enough energy* to observe recoils in the region where the momentum k is shared by a few ($n = 2, 3, \ldots$) nucleons and that the failure to do so is a consequence of the $k_{\min}(m_r)$ relationship of Eq. (5). Indeed, we shall show that measurement of *inclusive* cross sections, which depend on this relationship, are insensitive to the presence of two-particle spatial correlations whose contributions are depressed relative to the contributions from coherent recoils. In Sec. IV we consider the description of the cross sections in terms of an "effective momentum distribution" and show that when coherent recoil dominates and final state interactions are not neglected the observed cross sections can be treated as the product of the measured p-p cross section and an "effective" momentum distribution, which is a functional of the ground state wave function and is only very roughly proportional to the actual momentum distribution. Finally, in Sec. V, we show how the dominance of coherent recoils and QTBS suggests circumstances under which the t dependence of the nuclear elastic scattering cross sections can be obtained directly from the inelastic scattering structure function $G(k_{\min})$.

II. EXPERIMENTAL VERIFICATION OF THE COHERENT RECOIL ASSUMPTION

In the first[4] study of QTBS it was observed that the coherent recoil assumption gave a remarkably consistent organization of the existing data. A straightforward application of Eq. (6) was made at this stage, setting $C = 1$ as a first approximation. For a large range of incident proton momenta (0.6 to 5.9 GeV/c and angles (93° to 180°) $G(k_{\min})$ was found to be a "universal" function of $k_{\min}$, $\cong \exp(-k_{\min}/k_0)$. Most striking was the fact that, although the differential cross sections per nucleon for Be, C, Cu, and Ta targets were quite different in their q dependence and magnitude, $G(k_{\min})$ was identical in both shape and magnitude *provided* that the coherence assumption, $m_r = m_{A-1}$ was made. Because of these results the study of QTBS was extended in A down to ^{6}Li and extended in angle to 158° and 100° in a recent LAMPF experiment.[7] Once again the validity of the coherent recoil postulate was verified with high accuracy. (Studies at these angles highlighted the importance of using the experimental values of $C(p, k_{\min})$ i.e. $C \neq 1$ to show the universality of $G(k_{\min})$.)

In addition, two high sensitivity measurements now provide a much more direct test of the coherent recoil postulate. The first significant measurement was that of Brody *et al.*[8] in the reactions $(p, \alpha) + {}^6\text{Li} \to p + X$. These measurements were made for incident protons with 0.6 GeV kinetic energy and for incident α particles with 0.72 GeV kinetic energy, the ejected protons being observed at 180°. These experiments were the first to show that the empirically observed structure functions $G(k_{\min})$ followed the same exponential form up to values of k *close to the elastic scattering limit*, i.e., close to the kinematic region where *energy conservation requires the recoil to be coherent.* Because of the significance of this result, in the first experiment carried out with the high resolution spectrometer (HRS) at LAMPF, a more careful comparison[7] of ^{6}Li with heavier nuclei was made, this time studying $p + A \to p + X$ for 0.80 GeV kinetic energy and $\theta = 158°$ and 100°.

This experiment not only verified the quasi-two-body hypothesis in more detail over a large range of the variables $k_{\min}(p, q, \theta, A)$ but in particular showed that $G(k_{\min})$ was independent of A and that for ^{6}Li the function was identical over the whole range from $k = 0.2$ GeV/c to $k = 1.5$ GeV/c, the latter momentum corresponding to an energy within a few MeV of the elastic limit in ^{6}Li. In these experiments $d^3\sigma/d^3q$ varied over a range greater than 10^6:1.

In the 100° measurements at 0.8 GeV the HRS again had the sensitivity to study the region near the elastic endpoint where once more *energy conservation forces the recoil to be coherent.* Since this is just the kinematic postulate of quasi-two-body scaling, we are, without any assumption about the final state, actually measuring $G(k)$ in this region. We observe that this is the same functional form found at low k where the coherence is not at all forced by energy conservation. Thus the data suggest that the recoil is also coherent at low k. Of course, in this low region such an assumption is quite plausible since the incoming projectile, at the expense of only a small energy transfer, nudges the target proton of momentum $+\vec{k}$ onto the mass shell, the recoiling nucleus acting as a "coherent spectator." Since between these two extremes $G(k)$ is essentially unchanged, *we conclude that the inclusive cross sections are dominated by coherent recoils at all observed k* so far studied.

The fact that the shape of $G(k)$ for both ^{6}Li and ^{181}Ta are identical is another important factor in this argument. At a momentum q, where the recoiling $(A-1=5)$ nucleus carries off all the remaining available energy as center of mass kinetic energy, the recoiling $(A-1=180)$ Ta nucleus carries off negligible kinetic energy so that the forward going particle of momentum p' carries off 400 MeV of kinetic energy. Thus at this q there is unquestionably sufficient energy for 1, 2, or 3 *additional* fast nucleons to be emitted in the final state. But if this were occurring we should presumably detect a different form for $G(k_{\min})$ in Ta, whereas such a form would be eliminated by energy conservation in ^{6}Li. Yet $G(k_{\min})$ is identical for ^{6}Li and Ta.

The argument can also be appreciated by noting that if, in Ta^{181}, the recoil were entirely taken off by 5 nucleons (as must be true for Li^6 near the elastic limit) the shape of the cross-sections for Li^6 and Ta^{181} would be the same. Yet they are very different, falling off very differently with q in this high momentum region.

Thus, we conclude that coherent recoil dominates the data up to $k \cong 1.2$ GeV/c.

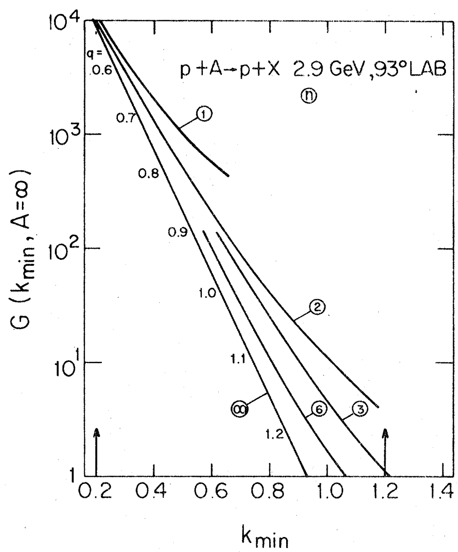

FIG. 1. Inhibition of few-body recoils (2.9 GeV, 93°). The left hand curve shows plot of $G(k)$, vs $k_{\min}$, for $n=\infty$. For each value of q the values of k for $m_r = nm_p$ are plotted to the right. Vertical distance from curve n to curve $n=\infty$ shows inhibition of n body recoils relative to $n=\infty$. Arrows on abscissa show the range of k covered by the data.

III. KINEMATIC SUPPRESSION OF THE EFFECT OF TWO-PARTICLE CORRELATIONS IN INCLUSIVE CROSS SECTIONS

We now demonstrate that the coherence follows from a kinematic suppression of low mass recoils that is an immediate consequence of Eq. (5). We see from this equation that $T_{k_{\min}}$ increases with decreasing m_r (and with increasing $k_{\min}$) forcing $k_{\min}$ to increase relative to the value at $m_r = m_{A-1}$, very large changes occurring for $m_r = nm_p$, $n=1,2,3$. Figure 1 shows the changes in $k_{\min}$ with m_r for one of the kinematic regions (2.9 GeV, 93°) covered in the data of Ref. 4. We have chosen to illustrate Eq. (5) in the following graphical way. The straight line shows a plot of $G(k_{\min})$ vs $k_{\min}$ for $k_0 \cong 90$ MeV/c for an infinitely heavy nucleus ($\bar{\epsilon}$ is set equal to zero for all calculations). The value of q corresponding to $k_{\min}$ is marked at selected points. To the right of each point is plotted the value of $k_{\min}$ for that q obtained from Eq. (5) for $m_r = nm_p$, $n=1,2,\ldots$. Arrows on the abscissa indicate regions of $k_{\min}$ covered by the data in Ref. (4). Note that up to $k_{\min} = .64$ GeV/c $m_r = 1$ recoils are kinematically permitted while $m_r = 2$ recoils are permitted up to the maximum value measured. Thus the assertion of Amado and Woloshyn[9] that coherent recoil arises because single proton recoils are energetically forbidden is only partly correct. Recoils of $n_r = 2, 3\ldots$ fragments are not described by coherent recoil yet they are kinematically allowed. However, the point to be noted is that since $k_{\min}$ is higher for low mass recoils *for the same q* and since on *any* theory $G(k_{\min})$ is expected to fall rapidly with $k_{\min}$, the contribution of light recoils is depressed. A measure of this depression in the region $k < k_t$ is the vertical distance on this plot from the curve for $m = m_r$ to that for $m = m_{A-1}$.

This appears to be the simple kinematic reason for the success of the coherence assumption of QTBS.

There is, of course, another simple way to account for the observation of predominantly coherent recoils independent of the kinematic suppression. As we have previously noted, there is *no theoretical estimate* at present for k_t. However, from all the data studied it is clear that if k_t were $\cong 1.2$ GeV/c the two-body spatial correlation contribution to $n(k)$ itself would be small and the dependence of $n(k)$ on k would be exponential. Of course, both effects, the kinematic suppression and a high transition momentum, favor coherent recoils and both may be present.

We summarize this section with the remark that in any inclusive measurement, where q rather than k is observed, the relationship between them is basically determined from Eqs. (1) and (2). Thus, for any composite system where the cross sections fall off with the momentum transfer to the recoiling system we expect a suppression of contributions from two-body spatial correlations. Only by measuring the momentum of the forward going proton, p', in coincidence with q, can we obtain a measure k *directly* and avoid the kinematic suppression.

IV. RELATIONSHIP BETWEEN QTBS AND THE NUCLEON MOMENTUM DISTRIBUTION

QTBS can be "derived" from the postulate that the inclusive cross sections in nuclei are the average of the "p-p cross sections" over the ground state momentum distribution. In particular Amado and Woloshyn,[5] using the Born approximation, neglecting both shadowing effects and final state interactions, and applying closure to evaluate the sums over final states, have given an expression for $d\sigma/d^3q$ which was in part the "justification" of quasi-two-body scaling. In their treatment $G(k)$ is related to the ground state momentum distribution $n(k)$ by Eq. (7).

On the other hand, they have also shown[6] how final state interactions destroy the simple dependence of $d\sigma/d^3q$ on $n(k)$. They have also realized[9] that the closure approximation used in their derivation fails for high values of k where the momentum distribution is dominated by the high energy low mass recoils so that this energy cannot be neglected in the energy conserving δ function.

More recently, two diverse studies of QTBS have been carried out which make use of explicit forms for $C(p,k_{\min})$ and do not set $C=1$ as was done in the early work.[4] Frankel and Woloshyn[10] have recently analyzed previously unpublished data[11] in the reaction $p+A\to p+X$ with incident protons of kinetic energy 0.73 GeV for a large range of laboratory angles, 90°–150°, and for many nuclei, Be to Th. They used the formula for $C(p,k_{\min})$ derived by Woloshyn [Eq. (8)] and used the measured p-p on shell cross sections at the appropriate values of s and t. Plots of $|\vec{p}-\vec{q}|(d\sigma/d^3q)/C$ gave the same function $G(k_{\min})$ for the whole range of variables. Frankel and Frati[12] have also applied QTBS to the production of antiprotons in the reaction $p+A\to\bar{p}+X$ at energies far below the threshold for antiproton production on stationary protons. For this reaction $k_{\min}$ is an entirely different function of q than that appearing in Eq. (5) and $C(p,k_{\min})$, obtained from the $\bar{p}$ production cross section above threshold, is a very different function than that employed in p-p scattering. *Nevertheless*, QTBS gives the same universal function $G(k_{\min})$.

How then can we understand the success of QTBS in view of the difficulties with the Born approximation, final state interactions, and closure? Is it possible to retain the simple and useful picture that the cross sections are averages of the measured p-p cross section over some effective ground state momentum distribution, a picture supported by the data?

In what follows we make use of the fact that the data, in the regions so far amenable to experimental investigation appear to be dominated by coherent recoil. With this simplification, Professor H. Primakoff and the author have examined anew the origin of QTBS and obtained a "derivation" that, in spite of the crude approximations, throws light on the factorization of Eq. (6) and the physical significance of $G(k_{\min})$. In this derivation, one approximates, in view of the dominance of coherent recoil, the initial wave function $\Phi_i(\vec{r}_1,\ldots,\vec{r}_A)$ by $\varphi_i(\vec{r}_1)\psi_i(r_2,\ldots,\vec{r}_A)$ where $\vec{r}_1$ is the coordinate of the struck nucleon and similarly approximates the final wave function by (see Ref. 13)

$$\varphi_f(\vec{r}_1)\exp\left(\frac{-i\vec{q}\cdot(\vec{r}_2+\cdots+\vec{r}_A)}{(A-1)}\right)\psi_f^*(\vec{r}_2,\ldots,\vec{r}_A).$$

The matrix element for the process then becomes

$$\int[e^{i\vec{p}'\cdot\vec{r}}\Phi_f(\vec{r}_1,\ldots,\vec{r}_A)]^*T[\Phi_i(\vec{r}_1,\ldots,\vec{r}_A)e^{i\vec{p}\cdot\vec{r}}]\cong\int\left[e^{i\vec{p}'\cdot\vec{r}}\varphi_f(\vec{r}_1)\exp\left(\frac{-i\vec{q}\cdot(\vec{r}_2+\cdots+\vec{r}_A)}{(A-1)}\right)\psi_f(\vec{r}_2,\ldots,\vec{r}_A)\right]^*$$
$$\times T[\psi_i(\vec{r}_2,\ldots,\vec{r}_A)\varphi_i(\vec{r}_1)e^{i\vec{p}\cdot\vec{r}}], \qquad (9)$$

where

$$T(\vec{r}_1,\ldots,\vec{r}_A;\vec{r})=H^{\rm int}(\vec{r}_1,\ldots,\vec{r}_A;\vec{r})+H^{\rm int}(\vec{r}_1,\ldots,\vec{r}_A;\vec{r})\,[E-H(\vec{r}_1,\ldots,\vec{r}_A;\vec{r})+i\epsilon]^{-1}H^{\rm int}(\vec{r}_1,\ldots,\vec{r}_A;\vec{r})$$

with

$$H(\vec{r}_1,\ldots,\vec{r}_A;\vec{r})=H^0(\vec{r}_1,\ldots,\vec{r}_A)+H^0(\vec{r})+H^{\rm int}(\vec{r}_1,\ldots,\vec{r}_A;\vec{r}),$$
$$E=E_i^0+E_p.$$

Expanding $T(r_1, \ldots, \vec{r}_A; \vec{r}) = T(\vec{r}_1 - \vec{r}) + T(\vec{r}_2 - \vec{r}) + \cdots + T(r_1, r_2, \vec{r} + \cdots$, i.e., expanding T into sums of $(2, 3, \ldots)$-body operators, we retain only $T(\vec{r}_1 - \vec{r})$ thereby employing the impulse approximation.[14] Further, our expression for the final state wave function of the detected proton

$$\varphi_f(\vec{r}_1) = e^{i\vec{q}\cdot\vec{r}_1}[1 + N_f(\vec{q}\cdot\vec{r}_1)]; \quad N_f \to 0 \text{ as } \vec{r}_1 \to \infty \tag{10}$$

exhibits explicitly the effect on the outgoing (subsequently detected) proton of final state interactions arising from the recoiling $A-1$ nucleons.

We proceed to replace $T(\vec{r}_1 - \vec{r})$ by its Fourier transform

$$\int \exp[i\Delta\cdot(\vec{r}_1 - \vec{r})]T(\Delta)d\vec{\Delta}$$

and carry out the integration over $d\vec{r}$ and $d\vec{\Delta}$. The resulting squared matrix element summed over the final states of the recoiling $A-1$ nucleus is then

$$|T(\vec{p}-\vec{p}\,')\int e^{i(\vec{p}-\vec{p}\,')\cdot r_1}e^{-i\vec{q}\cdot\vec{r}_1}[1 + N_f^*(\vec{q}\cdot\vec{r}_1)]\varphi_i(\vec{r}_1)|^2 \sum_f |\psi_f^*(\vec{r}_2, \ldots, \vec{r}_A)\exp[i\vec{q}\cdot(\vec{r}_2 + \cdots + \vec{r}_A)/(A-1)]\psi_i(\vec{r}_2, \ldots, \vec{r}_A)|^2$$

$$\cong |T(\vec{p}-\vec{p}\,')\int e^{i(\vec{p}-\vec{p}\,')\cdot\vec{r}_1}e^{-i\vec{q}\cdot\vec{r}_1}[1 + N_f^*(\vec{q}\cdot\vec{r}_1)]\varphi_i(\vec{r}_1)|^2 . \tag{11}$$

The most significant part of this result is that $T(\vec{p}-\vec{p}\,')$ can be factored out of the matrix element without further approximations so that $\sum|\text{M.E.}|^2$ is indeed proportional to the *measured* cross section of an incident proton on a free stationary proton.

It is important to stress that we have used the closure approximation, which, as Amado and Woloshyn have point out,[9] may be a poor approximation if the matrix elements are dominated by configurations of high excitation. However, our basic *assumption* of coherent recoil and its support from the data rules out these high excitations so that in the regime studied here the closure approximation is in fact expected to be valid and a good approximation.

The integral over $\vec{r}$ in Eq. (11) explicitly exhibits the Amado and Woloshyn result[6] that in the limit $\vec{p} = \vec{p}\,'$ the matrix element vanishes identically because of the orthogonality of the initial and final state wave functions $\varphi_i(\vec{r}_1)$ and $\varphi_f(\vec{r}_1)$. Writing this integral in terms of $+\vec{k}$, the momentum of the struck nucleon, we have

$$\int e^{i\vec{k}\cdot\vec{r}_1}[1 + N_f^*((\vec{p}-\vec{p}\,'-\vec{k})\cdot\vec{r}_1)]\varphi_i(\vec{r}_1)d\vec{r}_1 . \tag{12}$$

So that for $n_f = 0$ the square of this integral is just $|\phi_i(k)|^2 = n(k)$. We therefore see that it is the presence of n_f which destroys the simple relationship between $d\sigma/d^3q$ and $n(k)$. *However, if we define the square of this integral as* $n_{eff}(k)$ then for momentum transfers $\vec{p}-\vec{p}\,'$ less than $\vec{k}$, so that $\vec{q} \cong \vec{k}$, the derivation of QTBS can be carried out as before. Thus, instead of the Born-approximation result

$$\frac{d\sigma}{d^3q} = \frac{C_{\text{Born}}(p, k_{\min})}{|\vec{p}-\vec{q}|}\int_{k_{\min}}^{k_{\max}}\left[\int e^{-i\vec{k}\cdot\vec{r}}\varphi_i(r)d\vec{r}\right]^2 kdk$$

$$= \frac{C_{\text{Born}}(p, k_{\min})}{|\vec{p}-\vec{q}|}\int_{k_{\min}}^{k_{\max}} n(k)kdk$$

$$\cong C_{\text{Born}}(p, k)\frac{G(k_{\min})}{|\vec{p}-\vec{q}|} \tag{13}$$

we obtain

$$\frac{d\sigma}{d^3q} = \frac{C(p, k_{\min})}{|\vec{p}-\vec{q}|}\int_{k_{\min}}^{k_{\max}}\{e^{ik\cdot r}[1 + N_f^*(-k\cdot r)]J \times \varphi_i(r)d\vec{r}\}^2 kdk$$

$$= \frac{C(p, k_{\min})}{|\vec{p}-\vec{q}|}\int_{k_{\min}}^{k_{\max}} n_{eff}(k)kdk$$

$$= \frac{C(p, k_{\min})}{(\vec{p}-\vec{q})}[G_{eff}(k_{\min}) - G_{eff}(k_{\max})]$$

$$\cong C(p, k_{\min})\frac{G_{eff}(k_{\min})}{|\vec{p}-\vec{q}|} . \tag{14}$$

Here $C(p, k_{\min})$ is proportional to the *measured* p-p cross section and $G_{eff}(k_{\min}) - G_{eff}(k_{\max})$ is a modified form that depends on $n_{eff}(k)$ in the same way that $G(k_{\min}) - G(k_{\max})$ depends on $n(k)$. If $n_{eff}(k)$ is also a rapidly falling function of k, $G(k_{\max}) \ll G(k_{\min})$ and *the original form of QTBS is retained.*

It is worth remarking that Eq. (14) relates the differential cross section to the *ground state wave function* and to the final state interactions *both* of which must come from the solution of the appropriate many-body problem with the true Hamiltonian. That the cross section cannot be related directly to the *ground state momentum distribution* $n(k)$ via the Born approximation, as shown by Amado and Woloshyn,[6] is not a real loss.

It is now interesting to note that for large k the main contributions to the integral $\int \cdots d\vec{r}$ in Eq. (14) come from $r \lesssim k^{-1} < R$ (the nuclear radius) so that $n_f(-k \cdot r)$ can be replaced by $n_f(0)\{1 - (\vec{k} \cdot \vec{r}) \times [(d/dx) \ln n_f(x)]_{x=0}\}$. This yields as a crude approximation

$$n_{\text{eff}}(k) \cong |[1 + N_f(0)]\varphi_i(k) + iN_f(0) \times [(d/dx) \ln n_f(x)]_{x=0} \vec{k} \cdot \vec{\nabla}_{\vec{k}} \varphi_i(k)|^2 . \quad (15)$$

Thus, if $n(k) = |\phi(k)|^2$ decreases exponentially with k, $n_{\text{eff}}(k)$ will also decrease roughly exponentially with k and, in view of Eq. (14), so will $G_{\text{eff}}(k_{\min})$ with $k_{\min}$.

In summary, by assuming that factorization of the initial and final nuclear wave functions approximately represents the observed phenomenon of coherent recoil and by using the impulse approximation we can derive the basic expression for the cross section as being proportional to the measured p-p cross section and a structure function $G(k_{\min})$ where now $G(k_{\min})$ is a functional of $n_{\text{eff}}(k_{\min})$ which is *very* roughly proportional to $n(k_{\min})$. In the spirit of these crude approximations it is not difficult to modify the calculation to see that QTBS should be valid for incident and/or ejected particles other than protons.

V. INCLUSIVE AND ELASTIC SCATTERING

In view of the importance of "coherence" to an understanding of inelastic structure functions and of the fact that energy conservation forces coherence just below the elastic limit, it is tempting, if only qualitatively, to see whether the structure function near the elastic limit is related to the *elastic* scattering cross section and if this can be elucidated using the factorization of QTBS.

Once again, we turn to the kinematics of quasi-two-body scaling: In the reaction $\vec{p} \to \vec{p}\,' + \vec{q} + \vec{k}$, as the momentum of the detected proton $\vec{q}$ increases (approaching its value for elastic scattering from the nucleus), the final momentum $\vec{p}\,'$ gets smaller and smaller. Just at the edge of the continuum the forward going proton of momentum $\vec{p}\,'$ is traveling with zero velocity relative to the recoiling $(A-1)$ nucleus of momentum $\vec{k}$. In this configuration, there is a finite probability, S, of transitions of the proton to the vacant ground state (or excited states) left by the ejected target nucleon. The capture probability $S(v_r)$ can depend only on the *relative* velocity, v_r, of the recoiling nucleus and the forward going proton. It cannot depend on the sign of v_r. [We can estimate S as the average of $S(v_r)$ around $v_r = 0$, but for simplicity, we chose $\bar{S}(v_r) = S(0)$]. What is essential for our argument, however, is that S depends only on the relative velocity v_r and not on the laboratory angle or velocity of the recoiling nucleus. Hence S is t independent.

The configuration of QTBS for $v_r = 0$ is just

$$|\vec{p} - \vec{q}| \cong \frac{-\vec{k}_i}{A-1} - \vec{k}_i \cong -\left(\frac{A}{A-1}\right)\vec{k}_i .$$

Thus, near the elastic limit, we have

$$k_i \cong \frac{A-1}{A} |\vec{p} - \vec{q}| . \quad (16)$$

We now assume that the elastic scattering cross section is proportional to the probability that the incoming nucleon scatters from a nucleon of momentum $-\vec{k}_i$ and to the *t-independent* capture probability S so that

$$\frac{d\sigma}{dt} \cong C(p, k_i) G\left(\frac{A}{A-1} |\vec{p} - \vec{q}|\right) S , \quad (17)$$

where C is the elastic p-p scattering matrix element at k_i, Eq. (8).

We hasten to point out that this exchange mechanism is expected to be valid only at rather high momenta and backward angles where again the exchange momentum transfer $\vec{k} - \vec{q}$ is less than the direct momentum transfer $\vec{p} - \vec{q}$. For example, for $A = \infty$, $\theta = 180°$ $t(\text{exchange}) = 2m_p^2 - 2m_p E_p$ while $t(\text{direct}) = (-2p)^2$. For relativistic projectiles $E_p \gg m_p$ $t(\text{exchange})/t(\text{direct}) = m_p/2p \ll 1$ so that this requirement would apply. The present elastic scattering data at energies below 1 GeV are still not in this asymptotic region. Nevertheless, we expect that the falloff of $d\sigma/dt$ will reflect the scale of k_0.

More data are needed in the inelastic (inclusive) region to very low states in the continuum, i.e., near the elastic limit and on elastic cross sections at high momenta to explore the connections between the structure function and the elastic form factors.

VI. CONCLUSIONS

From a study of the data describing the A dependence of the structure function $G(k)$ and especially from that portion near the elastic limit in ^{6}Li we have shown that the inclusive cross sections appear to be dominated by almost coherent recoil of the $A-1$ residual nucleus. This is accounted for by noticing that the connection between the recoil momentum k and the observed momentum q in the inclusive processes provides a kinematic suppression of few-body recoils and also by the additional possibility that the transition momentum, k_t, marking the separation between the coherent recoil region and the few-body recoil region is as high as 1.2 GeV/c. Finally, we have outlined a new derivation of quasi-two-body scaling in terms of an effective momentum distribution that incorporates final state interactions, and have suggested a very high momentum connection between the structure function and elastic scattering cross sections.

VII. ACKNOWLEDGMENTS

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[13]For simplicity we do not antisymmetrize $\varphi_i(\vec{r}_1)\psi_i \times (\vec{r}_2, \ldots, \vec{r}_A)$ and

$$e^{i\vec{q}\cdot\vec{r}_1}\varphi_f(\vec{r}_1)\,e\left(\frac{-i\vec{q}\cdot\vec{r}_2+\cdots+\vec{r}_A}{(A-1)}\right)\psi_f(\vec{r}_2, \ldots, \vec{r}_A).$$

[14]Note that $T(\vec{r}_2-\vec{r})+T(\vec{r}_3-\vec{r})+\ldots+T(\vec{r}_4-\vec{r})$ do not contribute because of the orthogonality of $\varphi_i(\vec{r}_1)$ and $\varphi_f(\vec{r}_1)$.