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Three-body model of ^{9}Be and ^{12}C nuclei

Ahmed Osman*
International Centre for Theoretical Physics, Trieste, Italy
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The structure of light nuclei is considered using the α cluster model. On the basis of α clusters, the model structures of the nuclei ^{9}Be and ^{12}C are proposed as composed of three particles. The bound state problems for these nuclei are solved using separable potentials, where the Faddeev equations are given. Three different types of the nonlocal separable potentials are used, i.e., potentials of the Yamaguchi, Gaussian, and Tabakin forms. The Gaussian and Tabakin potentials contain both attraction and repulsion. Solving the Faddeev equations, numerical calculations are performed for the resulting integral equations. The present theoretically calculated values of binding energies for these nuclei are in good agreement with the experimental data. The very close agreement between theoretical and experimental values of the binding energies indicates that the repulsive forces of the nuclear potential are very important and should be taken into account.

[NUCLEAR STRUCTURE ^{9}Be, ^{12}C; three-body model. Calculated binding energies.]

I. INTRODUCTION

The α cluster model[1,2] explains with some success the structure of light nuclei. According to the α cluster model, the α particles will be treated as rigid entities without internal structure. Since the α particles are very tightly bound with an unusual stable structure, the structures of the ^{9}Be and ^{12}C nuclei are described in the present work using the α cluster model. The structure of the bound state of the ^{9}Be nucleus is taken as composed of two α particles and a neutron, while the ^{12}C nucleus is considered as composed of three α particles. With the present structure, each of the considered nuclei is described as constructed of three particles. In the present work we follow the Faddeev and Lovelace formulations. Faddeev presents an exact solution for the three-body problem. In this solution, Faddeev has shown that a well-behaved set of three-body equations involve the two-body T matrix rather than the potential. In this approach, the T matrix plays a central role. So, in the three-body Faddeev equation the two-body T matrix plays the part of a potential in the two-body Lippmann-Schwinger equation.

The bound state of light nuclei with three-particle construction could be well explained as a three-body problem. The set of coupled integral equations from the three-body theory developed by Faddeev[3,4] and Lovelace[5] has been applied for light nuclei by Harrington,[6] Wong,[7] and Osman.[8-10] Another method suggested by Mitra[11] and Eyges[12] known as the wave function method is employed and applied by Leung and Park.[13] Using separable potentials with attraction and repulsion for the α-α pairs, Hebach and Henneberg[14] found two 0^{+} states in the neighborhood of the three-particles threshold in studying the ^{12}C nucleus.

In the present calculations we solve the Faddeev equations using nonlocal but separable (NLS) potentials for the two-body interactions. The nucleon-α interaction used consists of both central and spin-orbit terms. The spin-orbit term is small and is taken in a way to fit the corresponding experimental nucleon-α phase shifts. For this purpose, Mitra, Bhasin, and Bhakar[15] introduced a suitable separable form for the nucleon-α interaction. This interaction was used by Osman[10] to extract the binding energy for the bound state ^{6}Li nucleus. The α particles are considered as rigid entities without any internal structure and so interacting with each other via a potential determined by α-α scattering experiments. In the present calculations, the Faddeev equations are solved using nonlocal separable α-α interactions obtained by fitting the S-wave α-α scattering length and effective range. We use three types of nonlocal separable potentials. The different forms of the NLS potentials considered are the Yamaguchi,[16] the Gaussian, and the Tabakin[17] potentials. The Yamaguchi potential is taken as completely attractive, while the Gaussian and Tabakin potentials contain both attraction and repulsion.

In Sec. II we introduce the different forms of the nonlocal separable potentials used. The three-body integral equations are considered in Sec. III. Numerical calculations and results are given in Sec. IV. Section V is devoted to discussion and conclusions.

II. NONLOCAL SEPARABLE POTENTIALS

Separable potentials have been widely used in calculating the three-nucleon binding energy, radius, Coulomb energy, form factor, and also the neutron-deuteron scattering cross section. These potentials are widely used because their use ensures great simplicity in the analysis of the three-body problem. In the present work we use nonlocal separable potentials for the two-body interactions.

For the nucleon-α interaction we use a form which consists of a central term and a spin-orbit term. The spin-orbit term in the nucleon-α interaction is taken to be small and in a way that it fits the corresponding experimental nucleon-α phase shifts. For this purpose, in the present calculations we use a suitable form for the nucleon-α interaction, which is introduced in Ref. 15. The form of the nucleon-α interaction used together with the values of the different parameters are given explicitly in Refs. 10 and 15. The P-wave nucleon-α interaction used is of the form

$$V_{n\alpha}(\vec{q},\vec{q}') = -\frac{2\pi\lambda_\alpha}{m_R}\, g_1(q)\, g_1(q') \times \sum_M Y_1^{M*}(\hat{q})[1+(\vec{L}\cdot\vec{\sigma}/t')]Y_1^M(\hat{q}'),$$

where

$$g_1(q) = q(q^2+f^2)^{-1}$$

with

$$\lambda_\alpha^{-1} = 11.312 \text{ fm}, \quad f^{-1} = 1.028 \text{ fm}, \quad t' = 21.6,$$

and m_R is the reduced mass of the nucleon-α system given by

$$m_R = mm_\alpha/(m+m_\alpha),$$

where m_α is the mass of the α particle.

Since, in the α cluster model the α particles are treated as rigid entities without any internal structure, they (the α particles) are interacting with each other via a potential determined by α-α scattering experiments. This simplification and choosing the form of the potentials as separable are of great help in solving the three-body problem exactly. The S-wave α-α interactions used here are chosen to fit the scattering length and effective range, which are determined experimentally with Coulomb effects removed. These potentials are suggested in a way that both attraction and repulsion can be given. For separable potentials between α particles, each of mass m_α, we have

$$V_{\alpha\alpha}(p,p') = -(\hbar^2/m_\alpha)(1/2\pi^2)f(p)f(p'). \quad (1)$$

From the nonlocal separable α-α potential given by Eq. (1), the two-body α-α binding energy $\hbar^2\mathcal{K}^2/m_\alpha$ is obtained from the relation

$$1 = \frac{2}{\pi}\int_0^\infty dp\, \frac{p^2 f^2(p)}{p^2+\mathcal{K}^2}. \quad (2)$$

Also, the S-wave phase shifts are obtained from the well known relation

$$\tan\delta(k) = kf^2(k)\Big/\left(1-\frac{2}{\pi}P\int_0^\infty dp\, \frac{p^2 f^2(p)}{p^2-k^2}\right). \quad (3)$$

In this work we use three types of nonlocal separable α-α potentials. These are the Yamaguchi[16] potential, the Gaussian potential, and the Tabakin[17] potential. The Gaussian and Tabakin potentials give both attraction and repulsion. For the purpose of comparison, the Yamaguchi potential is considered, and it gives only attraction. The Yamaguchi potential (which implies the Yukawa-type spatial dependence) has the form

$$f_Y(p) = \alpha/(p^2+b^2). \quad (4)$$

Also, we use S-wave nonlocal separable Gaussian interaction, which contains both attraction and repulsion, of the form

$$f_G(p) = \alpha(p_c^2-p^2)(a+bp^2)\exp(-dp^2). \quad (5)$$

The value of p_c can easily be determined since the S-wave α-α phase shift changes sign as a function of the bombarding energy. The third form for the α-α interaction we use is the Tabakin potential. This contains both attraction and repulsion and is a variant of the Yamaguchi potential and probably has a better theoretical foundation. The Tabakin potential used has the form

$$f_T(p) = \alpha(p_c^2-p^2)[(p^2+d^2)/(p^2+b^2)](p^4+a^4)^{-1}. \quad (6)$$

The values of the parameters for the Yamaguchi potential are those suggested by Harrington.[6] The

TABLE I. Values of the nonlocal separable α-α potential parameters.

α-α potential form	a	b	d	α^2	Ground-state energy of ^{8}Be (MeV)
Yamaguchi		0.736 fm^{-1}		2.36 fm^{-6}	−2.92
Gaussian	1.0 fm^{-2}	0.05	0.5 fm^2	4.432 fm^2	−3.43
Tabakin	1.9 fm^{-1}	1.3 fm^{-1}	5.0 fm^{-1}	3.1865 fm^{-6}	−3.62

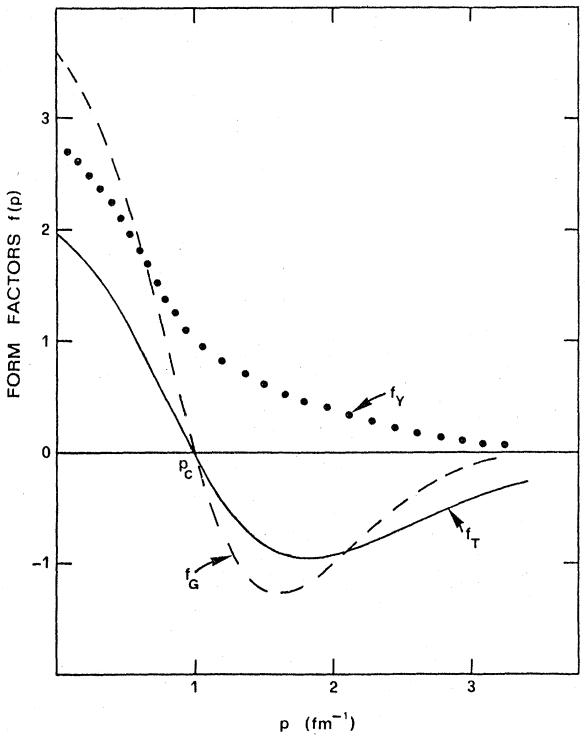

FIG. 1. The potential form factors $f(p)$. The curves f_Y, f_G, and f_T refer to the Yamaguchi, the Gaussian, and the Tabakin potentials, respectively.

values of the Gaussian potential and the Tabakain potential parameters which are used to fit the S-wave α-α nuclear scattering phase shifts are listed in Table I. The values of the α-α nuclear binding energies $\hbar^2\mathfrak{K}^2/m_\alpha$ which are obtained using the different sets of parameters for the different forms of the α-α interaction are also shown in Table I. The energies listed in Table I are only the nuclear binding energies. To obtain values for the actual binding energies, Coulomb energy must be added. The approximate value for the Coulomb energy can be obtained from a formula[18] for a uniform spherical charge distribution as

$$E_C(^AZ) = (0.584\ \text{MeV})Z(Z-1)A^{-1/3}. \tag{7}$$

According to this formula, the value of the Coulomb energy for the ^{8}Be nucleus is $E_C(^8\text{Be}) - 2E_C(^4\text{He}) = 2.03$ MeV. Adding this value of Coulomb energy to the values of the ground-state energies for the ^{8}Be nucleus which are listed in Table I, can lead to producing a stable ^{8}Be bound state. Thus, the theoretical calculations produce a stable bound state for ^{8}Be, while the observed ground state is unstable by about 0.1 MeV. This difference does not mean that the value of Coulomb repulsion between the two α particles is unreasonable, but this difference is a little large and it may be because the wave functions resulting from the potentials used are too large at small distances.

The corresponding shapes with the typical forms of the different potential factors $f(p)$ as given by Eqs. (4), (5), and (6) are shown in Fig. 1.

III. THREE-BODY INTEGRAL EQUATIONS

Faddeev[3,4] obtained a well-behaved set of three-body equations involving the two-body T matrix rather than the potential. This set of integral equations is obtained by rearranging the Lippmann-Schwinger equation.

Using the notations introduced by Lovelace, the two-body T matrix is given by

$$T_l(p,q;E) = V_l(p,q) + \frac{1}{\pi}\int dk^2\, \frac{V_l(p,k)kT_l(k,q;E)}{k^2-E}. \tag{8}$$

If $\varphi_{nl}(p;E)$ are eigenfunctions satisfying

$$\lambda_{nl}(E)\varphi_{nl}(p;E) = \frac{1}{\pi}\int dk^2\, \frac{V_l(p,k)k\varphi_{nl}(k;E)}{k^2-E}, \tag{9}$$

then these eigenfunctions form a complete orthonormal set for negative values of E, and can be obtained by standard numerical methods. The two-body T matrix is thus given by

$$T_l(p,q;E) = \sum \frac{\lambda_{nl}(E)}{1-\lambda_{nl}(E)}\,\varphi_{nl}(p;E)\varphi_{nl}(q;E). \tag{10}$$

In the case of the three-body problem the two-body T matrix in the three-body Faddeev equations plays the part of the potential in the two-body Lippmann-Schwinger equation. In this case, the set of Faddeev coupled integral equations is given as

$$\Psi_l(p,q;Z) = \Phi_l(p,q;Z) + 2\sum_{l'}\int dq'^2\int dp'^2\, \frac{[(2l+1)(2l'+1)]^{1/2}P_l(x)P_{l'}(x')T_l(p,P;Z-q^2)}{\sqrt{3}\pi q(p'^2+q'^2-Z)}\,\Psi_{l'}(p',q';Z). \tag{11}$$

In these equations, $\Psi_l(p,q;Z)$ is the three-body T matrix element for a final state of two particles in a relative l state and the third particle in an l state relative to the center of mass of the first two. The inhomogeneous term $\Phi_l(p,q;Z)$ is the symmetrized T matrix with one noninteracting particle. Z is the total energy of the three particles. $T_l(p,P;Z-q^2)$ is the two-body T matrix. Also,

$$P = p'^2 + q'^2 - q^2 , \tag{12}$$

$$x = [(q'^2 - q^2) + 3(q^2 - p'^2)]/(2\sqrt{3}\,Pq) , \tag{13}$$

$$x' = (-4q^2 + 3p'^2 + q'^2)/(2\sqrt{3}\,p'q') . \tag{14}$$

Introducing the separable expansion of T_l as given by Eq. (10) into the Faddeev equation given by Eq. (11), one obtains the following expression for Ψ_l:

$$\Psi_l(p,q;Z) = \Phi_l(p,q;Z) + \sum_n \frac{\lambda_{nl}(Z - q^2)}{1 - \lambda_{nl}(Z - q^2)}\, \varphi_{nl}(p;Z - q^2)\chi_{nl}(q;Z) , \tag{15}$$

with the function $\chi_{nl}(q;Z)$ satisfying the integral equations

$$\chi_{nl}(q;Z) = \zeta_{nl}(q;Z) + \sum_{n',l'} \int dq'^2 K_{nl,n'l'}(q,q';Z)\chi_{n'l'}(q';Z) , \tag{16}$$

where

$$\zeta_{nl}(q;Z) = 2\sum_{l'} \int dq'^2 \int dp'^2\, \frac{[(2l+1)(2l'+1)]^{1/2} P_l(x)P_{l'}(x')}{\sqrt{3}\,\pi q(p'^2 + q'^2 - Z)}\, \varphi_{nl}(P;Z - q^2)\Phi_{l'}(p',q';Z) \tag{17}$$

and

$$K_{nl,n'l'}(q,q';Z) = 2\int dp'^2\, \frac{[(2l+1)(2l'+1)]^{1/2} P_l(x)P_{l'}(x')}{\sqrt{3}\,\pi q(p'^2 + q'^2 - Z)}\, \frac{\lambda_{n'l'}(Z - q'^2)}{1 - \lambda_{n'l'}(Z - q'^2)}\, \varphi_{nl}(P;Z - q^2)\varphi_{n'l'}(p';Z - q'^2) . \tag{18}$$

If the sum over n' and l' is kept as a finite number of terms, then Eq. (16) becomes an ordinary Fredholm equation. In this case it can be solved by numerical methods. So, one can find the energy Z for which the eigenvalue of the kernel is unity. Therefore the homogeneous equation possesses a solution. This value of energy would then correspond to a bound state of the three-body system.

IV. NUMERICAL CALCULATIONS AND RESULTS

The three-body binding energies are obtained from the numerical solution of the integral equations (15)–(18). These equations have certain features which simplify their solution. With the above-mentioned two-body potentials for both the nucleon-α and α-α interactions, the three-body ground-state energies are calculated to a very high accuracy, using only a very limited number of summation terms. Due to these features of the potentials used, the well-behaved Schmidt-Hilbert theory[19] of integral equations is applied. With these general features, the integral equations (15)–(18) are solved numerically. In these numerical calculations a 24-point Gaussian integration is used. Appropriate weights and abscissas are chosen for the rapid convergence at infinity, such as to convert the integral equation into a matrix eigenvalue equation. The eigenvalues of the matrix are given as a function of Z. These values of Z for which a matrix eigenvalue takes the value 1, are the three-body bound-state energies. In the present calculations, contributions to the eigenvalues λ_{nl} for higher values of n (i.e., $n \geq 4$) are very small. Also, the contributions to λ_{nl} for $l \geq 5$ are neglected.

Using this method, the Faddeev-Lovelace formalism is applied to the cases of the nuclei ^{9}Be and ^{12}C. The ^{9}Be nucleus is taken as composed of two α particles and a neutron. The ^{12}C nucleus is considered as a three-α-particle system. By the above-mentioned method, numerical calculations are performed to obtain the nuclear binding energies. The results obtained for the three-body ground-state energies of the ^{9}Be and ^{12}C nuclei using the different α-α potential forms are summarized in Table II.

Our present theoretically calculated values of the binding energies can be compared with the experimental values.[20] The experimental values[20] of the ground-state energies are −1.57 and −7.28 MeV for the ^{9}Be and ^{12}C nuclei, respectively. Our theoretical values for energies listed in Table II are only the nuclear ground-state energies. Thus to obtain the actual values of the binding energies, we must remember to add the Coulomb energy resulting from Coulomb repulsion among the α particles. These Coulomb energies are approximately calculated using the formula given by Eq. (7). According to this formula, the Coulomb energy for the ^{9}Be nucleus is $E_C(^9\text{Be}) - 2E_C(^4\text{He}) = 2.59$ MeV. For the case of ^{12}C nucleus, the Coulomb energy is $E_C(^{12}\text{C}) - 3E_C(^4\text{He}) = 5.44$ MeV. Thus adding these values for the Coulomb energies to the values

TABLE II. Calculated nuclear binding energies.

α-α potential form	Ground-state energy of ^{9}Be (MeV)	Ground-state energy of ^{12}C (MeV)
Yamaguchi	−4.52	−12.93
Gaussian	−4.01	−11.16
Tabakin	−4.28	−12.78

listed in Table II means that our theoretically calculated values for the ground-state energies for both the ^{9}Be and ^{12}C nuclei are quite reasonable and in good agreement with the experimentally observed values.

V. DISCUSSION AND CONCLUSIONS

In the present calculations, we have used single separable potentials which can give both attraction and repulsion. This property is a general characteristic or nonlocal potentials. A very simple Gaussian form for nucleon-nucleon potential with attraction and repulsion was used by us[21] in three-body calculations. But in the present case, with the calculations considered in the present paper, for the type of the present nonlocal seperable potentials, one cannot tell if the interaction is purely attractive or purely repulsive merely by knowing the overall sign. In other words, many wrong conclusions can be drawn if one looks only at a single factor $f(p)$ and not both. These unfamiliar features are shared by and exist for all separable potentials. One must examine and consider the detailed structure of the nonlocality. The single separable potentials used here already incorporate the attraction and repulsion, and this reduces the total number of separable terms needed to reproduce the two-nucleon data. This reduction in the number of separable terms simplifies considerably our three-body calculations.

In the present calculations we proposed a three-body model for both the nuclei ^{9}Be and ^{12}C on the α cluster model. The α particles are taken as rigid entities. We have solved the Faddeev equations numerically for separable interactions. We have employed three types of the potential, i.e., Yamaguchi, Gaussian, and Tabakin. Throughout this work we are only concerned with the nuclear binding energies. Accepting the rough estimate of the Coulomb energies and adding their approximate values to the nuclear values, we obtained reasonable values for the ground-state energies which are not far from the observed experimental values. However, the Coulomb forces could be treated accurately by treating the pure Coulomb T matrix in the integral equations. Inclusion of Coulomb forces in the three-body calculations is found[8] to give more accurate values for the binding energies.

We expect that considering higher wave forces will give additional attraction to produce the excited states. This involves solving sets of coupled integral equations with complicated kernels, but it is rather straightforward numerical calculation. Moreover, the potentials used already include the size effect of an α particle in that the repulsive part of the potentials originates from the exclusion principle operating between two composite α particles. Thus, the only effect that should be explicitly included in these potentials is the Coulomb repulsion.

Our present calculations lead us to a consistent picture of the ^{9}Be and ^{12}C nuclei. From these calculations the important physical observables, i.e., the energies, can be deduced from the theory which employs the two-body data only. Thus the models presented here provide a good description of the composition of the nuclei ^{9}Be and ^{12}C. Also, it is shown that it serves as a useful proving ground for methods of solving the three-body problem. In the present calculations we used three separable approximations to the α-α interaction. A glance at Tables I and II shows that the Tabakin potential is preferable, in view of the fact that it provides the best account of the two-body scattering as well as the three-body observables. So, from the agreement between the present theoretical calculations for the binding energies and the experimental values we can draw some interesting points. It is found that for the same two-body potential, the two-α system is unbound, while the three-α system is bound. Since an increase in the attractive parts of the potential yields the correct binding energies, the effect of the two-body forces is more attractive in the three-body system than in the two-body system, due to the closed channel contribution. The α cluster description for the nuclei is only one of the structures, and it must be treated as an approximation. So, it is better to sum over all possible three-particle constructions. Thus the structure of the nuclei will be quite well represented by the coupled three-body systems. Also, in the present calculations, the Faddeev equations are solved with two-body T matrix neglecting the interaction between the α particles and the constituents of the second channel, but keeping the complete two-body T matrix. This indicates that the inelastic processes tend to increase the binding energies of the constituent particles closer to the experimental values of the ground-state energies of the nuclei.

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*Permanent address: Physics Department, Faculty of Science, Cairo University, Cairo, Egypt.

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