Three-nucleon results for one-boson-exchange potential

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(Received 30 June 1977)

The binding energy of ^{3}H, percentage S-, S'-, and D-state probabilities and charge form factor of ^{3}He are calculated using the Adhikari-Sloan separable expansion to the Holinde and Machleidt one-boson-exchange potential. The results show that the Adhikari-Sloan expansion has good convergence for the binding energy, and the lowest order Adhikari-Sloan expansion considered gives excellent results for the form factor.

[NUCLEAR STRUCTURE ^{3}H binding energy, ^{3}He charge form factor, Faddeev approach, separable expansion to realistic N-N interaction.]

I. INTRODUCTION

In a previous publication, here referred to as I,[1] we compared the use of the unitary pole expansion (UPE)[2] and the Adhikari-Sloan separable expansion (ASE)[3] in calculating trinucleon bound state observables for the Reid soft core potential (RSC).[4] In I, we found that the ASE in general gave better results for lower rank separable expansion than the UPE. In particular, it was not necessary to resort to T-matrix perturbation theory to achieve convergence in the binding energy of ^{3}H (E_T), and the lowest rank ASE considered gave far better results for the charge form factor of ^{3}He (CFF) than a higher rank UPE.

In the present communication, we present trinucleon results for the one-boson-exchange potential (OBEP) of Holinde and Machleidt (HM),[5] using the ASE with the hope of showing: (i) The convergence for the binding energy of ^{3}H and CFF is as good as, if not better than, was the case for the RSC. (ii) Although with OBEP we improve the agreement with experiment for E_T,[6] the minimum in the CFF moves to higher momentum transfer, and away from the experimental results.[7-9] (iii) The success of the low rank ASE for the one-boson-exchange potentials, which are commonly used in relativistic two-body equations such as the Blankenbecler-Sugar (BS) equation,[10] suggests that such an expansion can be used to construct separable nucleon-nucleon amplitudes which satisfy relativistic two-body unitarity, and at the same time give a good representation of the experimental data. Such amplitudes can then be employed in relativistic three-body calculations such as π-d scattering at medium energies,[11,12] or the determination of relativistic effects in E_T and CFF.[13]

We can write the OBEP for the exchange of scalar (s), pseudoscalar (ps), and vector (v) bosons as[14]

$$V_{\mathrm{OBEP}}(\vec{q}',\vec{q}) = \sum_{\alpha=\mathrm{s,ps,v}} V_\alpha(\vec{q}',\vec{q}) , \tag{1}$$

where V_α is the Born amplitude resulting from the exchange of one boson α from a nucleon with initial center of mass momentum $\vec{q}$, to another nucleon with final center of mass momentum $-\vec{q}'$. The problem with OBEP arises from the asymptotic behavior of the amplitude $V_\alpha(\vec{q}',\vec{q})$, which displays polynomial divergence in q and/or q'. The worst of these divergences occurs in the case of vector boson exchange, where, for example with q' fixed, the tensor coupling component of the vector meson amplitude diverges as q^2 for q tending to infinity. To overcome this problem the coupling constants of the bosons are multiplied by strong interaction form factors $F_\alpha(\vec{q}',\vec{q})$, which die off sufficiently fast in q and q' to ensure that the potential will be well behaved asymptotically once these form factors are included. In practice this regularization technique reduces to formation of the final potential from a linear combination of the exchange amplitudes calculated with the boson masses, and with several large cutoff masses. For q and q' small, the effect of the additional terms involving the cutoff masses is negligible; however, since these additional terms exhibit the same asymptotic behavior as the terms involving the boson masses, as q and/or q' become large they cancel the divergences of the boson amplitudes, producing a potential with good asymptotic behavior.

This delicate cancellation between the amplitude with the boson masses and that with the cutoff masses leads to difficulties in constructing the UPE for OBEP. We recall that the UPE is formed from the eigenvectors of the kernel of the Lippman-Schwinger equation,[2] some of which contribute only attraction to the potential, and others only repulsion. In the case of OBEP, nearly all the repulsion comes from the exchange of vector bosons (the ρ, ω, and

ϕ in HM), and hence the repulsive eigenvectors depend mainly on the vector part of the potential. Unfortunately the worst divergences occur in the vector boson amplitudes, and thus, in numerically calculating the vector boson part of the potential by the method described above, enormous cancellations occur, even at moderately large values of q and q'. These cancellations make the accurate computation of the repulsive eigenvector impossible. In fact, on calculation, most of them tend to be approximately constant, or widely oscillatory at quite large momentum values. We could overcome this problem by resorting to analytic cancellation which makes the calculation of the kernel highly time consuming and renders the UPE an impractical approach to the construction of the separable expansion for the two-body amplitude. However, since this complication arises only in determining the repulsive eigenvectors, we can still construct the unitary pole approximation (UPA), which involves only the attractive eigenvector corresponding to the deuteron wave function. Of course, since the ASE does not involve the calculation of the eigenvectors of the kernel it can be constructed without any numerical problem arising from the cancellation in the vector meson's contribution to the potential. Furthermore, these cancellations do not affect the accuracy of other quantities calculated with the HM potential such as the scattering phase shifts or, most importantly, the deuteron wave function (we obtain $E_D = -2.2250$ MeV and $P_D = 5.746\%$ in perfect agreement with HM).

II. RESULTS AND CONCLUSIONS

Since the main aim of the present calculation is to show that the ASE can be constructed with great success for OBEP, we will restrict our three-body calculations to the use of the nonrelativistic Faddeev equation. In this way we avoid some of the problems involved in using correct relativistic three-body equations.[15] Since the potential in Eq. (1), after regularization, was used in conjunction with the Lippmann-Schwinger equation with relativistic unitarity to fit the experimental data, we use the prescription of minimal relativity[16] to construct a potential of the form

$$V(\vec{q}',\vec{q}) = \left(\frac{M}{E_{q'}}\right)^{1/2} V_{\mathrm{OBEP}}(\vec{q}',\vec{q}) \left(\frac{M}{E_q}\right)^{1/2} \tag{2}$$

with $E_q = (q^2 + M^2)^{1/2}$ and M the nucleon mass, to be used in the nonrelativistic Lippmann-Schwinger equation.

With the above stated problem in constructing the UPA for OBEP, we will restrict our three-nucleon results to the use of the UPA, ASE(1,2), and ASE(4,6), where we imply by ASE(M_1, M_2) a rank $M_1(M_2)$ $^1S_0(^3S_1 - {}^3D_1)$ Adhikari-Sloan separable expansion. To solve the Faddeev equation for the trinucleon bound state we need to know the two-body T matrix for the energies $E = -E_T - 3\lambda q^2/4$ ($\lambda = \hbar^2/m$) with $0 \leq q < \infty$, and E_T the binding energy of the trinucleon. To take optimum advantage of the special features of the ASE, we chose our expansion such that the two-body T matrix $\langle p'l' | T^{\mathrm{ASE}}(E_n) | l_n p_n \rangle$ is exact for all p' and l', and $M_1(M_2)$ energies E_n, momenta p_n, and angular momenta l_n, from the set of (E_n, p_n, l_n) are given by

$$\begin{aligned} &E_n = E_0 - 3\lambda q_n^2/4 \quad (n = 1, \ldots N_Q) \\ &p_n^2 = -E_n/\lambda, \quad l_n = 0, 2 \end{aligned} \tag{3}$$

with $E_0 = -7.0$ MeV, and q_n the quadrature points used to convert the homogenous Faddeev integral equation to a set of algebraic equations. The explicit values of (E_n, l_n) used to construct the ASE for the $^1S_0$ and $^3S_1 - {}^3D_1$ channels of the HM potential are the same as those used for the RSC potential and are given in Table III of I,[1]

We present in Table I the trinucleon results for the binding energy of ^{3}H (E_T), the percentage S-, S'-, and D-state probability [$P(S)$, $P(S')$, and $P(D)$] of the ^{3}H wave function, the rms charge radius of ^{3}He $\langle r_{\mathrm{ch}}^2 \rangle^{1/2}$, the position of the minimum in CFF (K_{min}^2), and the ratio of the experimental to the calculated CFF at the maximum (i.e., $K^2 = 20$ fm^{-2}) R for the OBEP of Holinde and Machleidt (HM).[5] For comparison, we have also included the results of Brandenburg, Sauer, and Machleidt (BSM),[7] obtained by solving the two dimensional Faddeev equations in momentum space for the same potential.

To examine the accuracy of our results for the

TABLE I. Three-nucleon results for the OBEP of Holinde and Machleidt (Ref. 5) using the UPA, ASE(1,2), and ASE(2,4). Also included are the results of Brandenburg *et al.* (Ref. 7).

Method	E_T (MeV)	$P(S)$ (%)	$P(S')$ (%)	$P(D)$ (%)	$\langle r_{\mathrm{ch}}^2 \rangle^{1/2}$ (fm)	K_{min}^2 (fm^{-2})	R
UPA	7.503	90.64	1.19	8.12	2.035	16.0	5.5
ASE(1,2)	7.415	90.21	1.25	8.46	2.054	15.3	4.7
ASE(4,6)	7.344	90.33	1.25	8.35	2.059	15.3	4.6
BSM	7.47 ± 0.1	91.57	1.10	7.31	1.97	15.48 ± 0.3	

binding energy we have performed the following tests: (i) To examine the convergence of the ASE we have calculated E_T with ASE(2,4) with the resultant binding energy of 7.330 MeV. Determining the rate of convergence $|E_T(1,2) - E_T(2,4)| = 0.085$ MeV and $|E_T(2,4) - E_T(4,6)| = 0.014$ MeV and comparing it with the corresponding quantities for the RSC of 0.101 and 0.011 MeV, respectively, we expect our results for HM to be of comparable accuracy to the RSC[1] and better than ±0.02 MeV. (ii) To study the sensitivity of E_T to the choice of E_n, the energies at which the off-shell T matrix is exact, we have taken E_0 to be −7.5 MeV instead of −7.0 MeV. With this choice of E_0, the corresponding binding energy is 7.348 MeV using the ASE(4, 6), a change of 0.004 MeV from the value in Table I. As a test of the number of energies E_n—for which the off-shell T matrix is exact—needed for a given rank expansion to maintain accuracy, we have calculated E_T with only two of the four basis energies for the singlet expansion, and three of the six basis energies for the triplet expansion in the ASE(4,6) T matrix. Despite the fact that we have not taken full advantage of the special feature of the ASE, we obtain a binding energy of 7.352 MeV with ASE(4,6), a change of 0.008 MeV from the result in Table I.

Combining the possible sources of error, i.e., finite rank expansion, choice of energies E_n, and finite quadrature points, we expect an error in E_T using ASE(4,6) of less than 0.03 MeV.

In I[1] we showed that although we have restricted the two-body interaction to the $^1S_0$ and $^3S_1 - {}^3D_1$ channels, the antisymmetry leads to an infinite series for the partial wave expansion of the wave function. The criterion used for truncating the series for the wave function was determined by the normalization $\langle\Psi|\Psi\rangle$ which we could determine in closed form [see Eq. (45) of I]. For the RSC potential the restriction $l_\alpha + L_\alpha \leq 10$ on the partial wave expansion was sufficient to give 99.63% of the normalization, where l_α (L_α) is the orbital angular momentum of the pair ($\beta\gamma$) (spectator α). For the HM potential this restriction using [UPA, ASE(1, 2), ASE(4,6)] gives, respectively (99.71, 99.71, 99.70)% of $\langle\Psi|\Psi\rangle$. Calculating $\langle\Psi|\Psi\rangle$ using ASE(4,6) and all terms with $l_\alpha + L_\alpha \leq 20$ gives us 0.9991, an increase of 0.21 at the cost of nearly doubling the number of terms in the partial wave series. We have thus restricted our expansion to $l_\alpha + L_\alpha$ with the belief that, as in the RSC calculation, the additional terms would not change our results appreciably.

In Table II we present the percentage probability for the Blatt-Derrick components of the trinucleon wave function with $l_\alpha + L_\alpha \leq 10$. Since the P-wave nucleon-nucleon interaction is not included in our calculation, we have not tabulated the P-wave probabilities. In the table, L and S refer to the total orbital angular momentum and spin in L-S coupling. The column labeled P gives the symmetry under permutation of the spin-isospin part of the wave function. Thus A denotes the totally antisymmetric component, and ± denotes the two mixed symmetry components. As we observe from the table there is good agreement between the two ASE results, and quite good agreement between the UPA and ASE results. We also note that there are relatively large contributions to the wave function from channels with large L_α (e.g., channels 16–18). The same situation was true for the RSC potential. This large contribution from high partial waves implies that one needs to be careful in truncating the expansion for the wave function, and can effect the results for the S-, S', and D-state probabilities as we will see.

TABLE II. Percentage probabilities of the Blatt-Derrick components of the trinucleon wave function for the HM potential.

Channel	L	S	l_α	L_α	P	UPA	ASE(1,2)	ASE(4,6)
1	0	$\frac{1}{2}$	0	0	A	89.60	89.12	89.18
2	0	$\frac{1}{2}$	0	0	−	0.55	0.57	0.57
3	0	$\frac{1}{2}$	1	1	+	0.53	0.56	0.56
4	0	$\frac{1}{2}$	2	2	A	0.88	0.92	0.98
5	0	$\frac{1}{2}$	2	2	−	0.043	0.047	0.048
6	0	$\frac{1}{2}$	3	3	+	0.046	0.050	0.050
7	0	$\frac{1}{2}$	4	4	A	0.16	0.17	0.17
8	0	$\frac{1}{2}$	4	4	−	0.0081	0.0085	0.0084
9	0	$\frac{1}{2}$	5	5	+	0.014	0.015	0.014
10	2	$\frac{3}{2}$	0	2	−	1.00	1.03	1.01
11	2	$\frac{3}{2}$	1	1	+	2.42	2.49	2.42
12	2	$\frac{3}{2}$	1	3	+	0.89	0.97	0.96
13	2	$\frac{3}{2}$	2	0	−	2.71	2.84	2.80
14	2	$\frac{3}{2}$	2	2	−	0.16	0.16	0.16
15	2	$\frac{3}{2}$	3	1	+	0.30	0.32	0.34
16	2	$\frac{3}{2}$	2	4	−	0.13	0.13	0.14
17	2	$\frac{3}{2}$	3	3	+	0.12	0.12	0.12
18	2	$\frac{3}{2}$	3	5	+	0.18	0.18	0.19
19	2	$\frac{3}{2}$	4	2	−	0.050	0.053	0.054
20	2	$\frac{3}{2}$	4	4	−	0.016	0.016	0.016
21	2	$\frac{3}{2}$	4	6	−	0.029	0.030	0.030
22	2	$\frac{3}{2}$	5	5	+	0.023	0.025	0.023
23	2	$\frac{3}{2}$	5	3	+	0.076	0.081	0.083
24	2	$\frac{3}{2}$	6	4	−	0.013	0.014	0.014

In Table I we present the S-, S', and D-state probabilities of the trinucleon wave function using the different separable expansions with $l_\alpha + L_\alpha \leq 10$. Included also are the results of BSM.[7] By comparing the results for the different separable expansions, we observe the general close agreement particularly for the ASE(1,2) and ASE(4,6), which is a strong indication of the convergence of the ASE. As a test of our restriction $l_\alpha + L_\alpha \leq 10$, we have calculated the probabilities using the ASE(4,6) with $l_\alpha + L_\alpha \leq 20$, the resultant values (90, 2, 1.3, 8.4) are in excellent agreement with those in Table I, and a confirmation of the fact that the restriction $l_\alpha + L_\alpha \leq 10$ will not introduce significant error into our results. A comparison of our results for the probabilities with those of BSM shows a discrepancy of ~1% for $P(S)$ and $P(D)$. For the RSC potential a discrepancy of the same magnitude was present between our results using the ASE, and the solution of the two dimensional Faddeev equation in momentum space.[1] This discrepancy was mainly due to the fact that we had included more terms in our partial wave expansion which add to the D-state component of the wave function and effectively reduce the S-state probability. A similar effect is present here for the HM potential.

Having established the convergence of the ASE for the trinucleon wave function, we present in Fig. 1 the CFF of ^{3}He for the different expansions. Also included are the experimental results of McCarthy *et al.*[17] The most notable feature of these results is that the ASE(1,2) and ASE(2,4) give indistinguishable results. This means that as far as the CFF is concerned we need not go beyond the ASE(1,2). This is a major simplification in the trinucleon wave function needed to calculate the contribution from meson exchange currents and three-body forces. At the same time this simpler wave function that reproduces the experimental CFF data for $K^2 < 8$ fm^{-2} can be used to construct pion optical potentials and to study such reactions as $d(p,\pi)t$.

Finally, we compare in Table I our results with those of BSM for the rms charge radius and the position of the minimum in the CFF. The general agreement is an indication of the success of the ASE of low rank for OBEP.

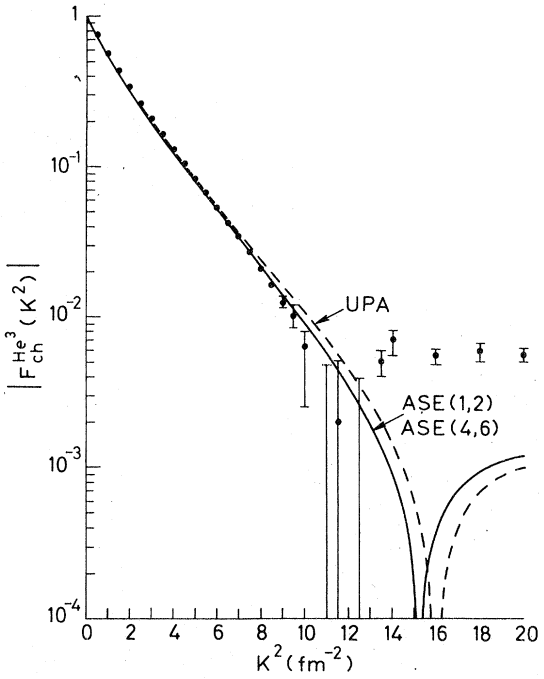

FIG. 1. The charge form factor of ^{3}He using the UPA, ASE(1,2), and ASE(2,4). Also included are the experimental results of McCarthy *et al.* (Ref. 15).

In conclusion we note that the ASE for $^1S_0$ and $^3S_1 - {}^3D_1$ channels of the HM OBEP converges for the binding energy of ^{3}H and the charge form factor of ^{3}He without perturbation theory. The ASE(1,2) gives a simple yet accurate representation of the OBEP trinucleon wave function for other studies. Furthermore, the method can be used to construct separable amplitudes for use in relativistic three-body calculations. Finally we confirm previous results[7-9] on the fact that as one improves the binding energy of ^{3}H the fit to the experimental CFF of ^{3}He gets worse.

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