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Trinucleon photoeffect to isospin 3/2, using coupled hyperspherical harmonics*

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We extend the Fabre-Levinger calculation of the ^{3}He photoeffect ($E1$ transition to final isospin 3/2) by including an additional grand orbital ($L = 3$) in the final state wave function. The coupled differential equations between $L = 1$ and $L = 3$ for the final state are given. We solve for two model potentials, and find the Blatt-Biedenharn eigenphase shifts and the mixing parameter, and also the total cross section for photodisintegration. We find reasonable agreement (generally within 10%) with the cross section found for one hyperspherical harmonic. The convergence speed of the hyperspherical harmonic expansion for the outgoing channel is also examined by comparison with sum rules for the integrated cross section: We find rapid convergence for both models.

[NUCLEAR REACTIONS Photodisintegration of trinucleon system, coupled hyperspherical harmonics.]

I. INTRODUCTION

The trinucleon photoeffect has been calculated by Gunn-Irving[1], Barbour-Phillips[2], Gibson-Lehman[3], and Fabre-Levinger[4]. The first attempt to use hyperspherical harmonics (h.h.) for the trinucleon photoeffect was made by Delves[5]. Unfortunately his calculation gives poor agreement with sum rules.[4,6] Experiments on three-body break-up were reported by Fetisov et al.[7], Gorbunov[8], Berman et al.[9] and Gerstenberg et al.[10]

The first successful application of h.h. to El transitions to isospin 3/2 states of the trinucleon was made by Fabre and Levinger[4], (FL). Recently Myers et al.[11] used sum rules to test the accuracy of the use of a single h.h. for a spin-independent potential of Wigner character: they find agreement within 6% for the integrated cross section.

We perform our calculation of the wave functions using an expansion in h.h. We keep only a finite number of terms of the expansion, giving us the same number of coupled equations, which are integrated numerically. The convergence of this method has been tested for the wave function of the trinucleon ground state; we are able to find quite accurate solutions.[12,13] We must investigate the convergence of the wave function for the final state.

The accuracy of the final state wave function is tested by comparison[11] of two different calculations of moments of the photoeffect cross section. These moments can be calculated numerically from the cross section $\sigma_T(E_\gamma)$ found using the final state wave function. The moments can also be determined by sum rules[6] which are expressed entirely in terms of the trinucleon ground state wave function, and of the potential and its assumed exchange mixture. We extend the comparison made by Myers et al. to the use of two coupled h.h. for the final state wave function, and to the use of a spin-dependent potential.

We note that if we neglect the tensor force and use a potential with a soft core which gives correct values of the binding energy and the form factors of the trinucleon[13], the first partial wave with grand orbital L=0 contains about 98% of the total wave function, while the mixed symmetry wave L=2 contains about 1%. Electric dipole transitions obey the selection rule $\Delta L = \pm 1$. Then transitions to a final L=1 come from the dominant L=0 or L=2, and constitute roughly 98% of the total cross section. About 1% of the cross section comes from an initial wave L=2, leading to a final wave, L=3. This rapid convergence of the ground state wave function, combined with the dipole selection rule, explains the good agreement with the Thomas-Reiche-Kuhn sum rule for the integrated cross section found by Myers et al.[11], using only a single partial wave, L=1, in the final state.

We have also shown[14] that the forms of the trinucleon partial waves L=0 or 2 are insensitive to the assumed potential, when we impose the constraints of the binding energy and r.m.s. radius. The total cross section is then expected to be sensitive only to the details of the wave function L=1 for the final state. The shape of this wave is modified by the coupling between waves of odd grand orbital, L=1, 3, 5, ... At low energy the effect of the coupling is predominately between waves of small grand orbital. In this paper we study the effect of the coupling of the wave L=3 on the wave L=1; and we find that the change caused by this coupling is small.[15]

This result implies a rapid convergence of our h.h. expansion, in accord with the good agreement found between two different calculations of the moments of the cross section.

We use two different soft core central potentials: the spin-independent Volkov potential[16] and the spin-dependent V^X potential.[13] These potentials were specified for even parity of the two-nucleon system. In Section IV we present these potentials for even parity states, and specify our assumptions for their exchange character. The Volkov potential gives[11,12] a triton energy of -8.46 MeV and an rms radius of 1.73 fm. The V^X potential gives[13] the ^{3}He energy as -7.74 MeV, and also gives excellent agreement with experiment on the ^{3}He and ^{3}H form factors, up to squared momentum transfers of 8 fm^{-2}.

Comparison of measured bremsstrahlung weighted cross sections for two-body and three-body break-up in the ^{3}He photoeffect with sum rule calculations of the bremsstrahlung weighted cross sections for specified isospin final states has shown[17] that almost all of the observed three-body break-up is due to isospin 3/2 final states. For some reason, not yet completely understood, the isospin 1/2 state has a small branching ratio for three-body break-up. We therefore follow FL in comparing our calculations for the V^X potential for isospin 3/2 final states with experimental measurements on three-body break-up.

In the next section we use the group theoretic classification of symmetries of the three-body system to derive an expression for the photoeffect cross section. In Section III we expand the continuum wave function in hyperspherical harmonics. We find the coupled differential equations for two hyperradial functions, with grand orbitals one and three, respectively. We also find the differential and total photoeffect cross sections in terms of four overlap integrals. In Section IV we give numerical results for the Volkov potential (spin-independent, of assumed Wigner character) and the V^X potential (spin-dependent, zero potential in odd parity two-body states). In Section V we conclude that expansions in h.h. give rapid convergence for the cases considered; we find fair agreement with other calculations for isospin 3/2 and good agreement with Gorbunov's experimental results for three-body break-up.

II TRINUCLEON PHOTOEFFECT TO ISOSPIN 3/2 STATE

There are four independent spin-isospin functions for a system of three nucleons in a state with total spin S and total isospin T given by S = T = 1/2, with given projections S_z and T_z. Our notation follows.

A: completely antisymmetric
S: completely symmetric
A': mixed symmetry; antisymmetric for the pair (1,2)
S': mixed symmetry, symmetric for the pair (1,2)

For total spin and isospin given by S = 1/2, T = 3/2, with given projections there are two independent functions of mixed symmetry.

A": antisymmetric for the pair (1,2)
S": symmetric for the pair (1,2)

Appendix A gives a summary of spin-isospin wave functions, and gives expressions for the six wave functions listed above.

We use Jacobi variables[4]

$$\vec{\xi}_1 = \vec{x}_1 - \vec{x}_2;\ \vec{\xi}_2 = 3^{1/2}(\vec{x}_3 - \vec{X}) \qquad (1)$$

Here $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are the nucleon coordinates, and $\vec{X}$ is the coordinate of the center of mass. We designate the canonically conjugate wave vectors by $\vec{k}_1$ and $\vec{k}_2$. We use these two pairs of three-dimensional vectors to define two vectors in six-dimensional space: $\vec{\xi}(\vec{\xi}_1,\vec{\xi}_2)$ and $\vec{k}(\vec{k}_1,\vec{k}_2)$.

If we neglect the tensor force, the ground state wave function of the trinucleon can be written as

$$\Psi_G(s,t,\vec{\xi}) = A\psi_G^S(\vec{\xi}) + 2^{-1/2}[A'\ \psi_G^+(\vec{\xi}) + S'\ \psi_G^-(\vec{\xi})] + S\psi_G^A(\vec{\xi}). \qquad (2)$$

Here s denotes the spin variables, and t denotes isospin variables. $\psi_G^S(\vec{\xi})$ is completely symmetric for the exchange of any pair of space coordinates, while $\psi_G^A(\vec{\xi})$ is completely antisymmetric for space exchange. $\psi_G^+(\vec{\xi})$ is symmetric while $\psi_G^-(\vec{\xi})$ is antisymmetric for exchange of the pair (1,2). The mixed symmetry states $\psi_G^\pm$ and the completely antisymmetric state ψ_G^A disappear when the even spin singlet and triplet potentials are the same (for instance for an interaction of Wigner character). In general the wave ψ_G^A has a negligible weight in the trinucleon ground state.

The final state, with quantum numbers T = 3/2, S = 1/2, 1^- is written

$$\Psi_F^{(-)}(s,t,\vec{k},\vec{\xi}) = [A''\psi_F^+(\vec{k},\vec{\xi}) - S''\psi_F^-(\vec{k},\vec{\xi})]/2 \qquad (3)$$

The (-) superscript on Ψ_F designates an incoming wave. ψ_F^+ is a wave symmetric on exchange of (1,2), while ψ_F^- is antisymmetric. These wave functions depend on the momenta of the particles in the continuum.

Following FL, the differential cross section for electric dipole photodisintegration is given by

$$d\sigma = (2\pi/\hbar c)|\langle\Psi_F^-|D|\Psi_G\rangle|^2\ \rho_E. \qquad (4)$$

The density of final states per unit energy is

$$\rho_E = d^6k/(2\pi)^6 dE. \qquad (5)$$

The energy $E = (\hbar^2/M)(k_1{}^2 + k_2{}^2)$, and D is the dipole operator.

The matrix element for E-1 transitions is[4]

$$\langle\Psi_G|D|\Psi_F^-\rangle = i\, e(2T_z)(2\pi E_\gamma)^{1/2}(4^{-1}3^{-1/2})$$

$$\{\langle\Psi_G^S|\xi_{2z}\,\psi_F^+ + \xi_{1z}\,\psi_F^-\rangle + 2^{-1/2}[\langle\psi_G^+|\xi_{2z}\psi_F^+$$

$$-\xi_{1z}\psi_F^-\rangle - \langle\psi_G^-|\xi_{2z}\psi_F^- + \xi_{1z}\psi_F^+\rangle]\}\ . \qquad (6)$$

Here e is the magnitude of the electron charge, and T_z is the third component of isospin of the trinucleon. We take the polarization of the photon along the z-axis. Only the first term, involving $\psi_G^S(\vec{\xi})$, contributes for a Wigner force.

III. Expansion in Hyperspherical Harmonics

We now discuss the calculation of the wave functions Ψ_G and $\Psi_F^{(-)}$ using the hyperspherical formalism. The vectors $\vec{\xi}$ and $\vec{k}$ are replaced by the coordinates (Ω,ξ) and (Ω_k,k) in six-dimensional space. The five angles Ω are chosen as (ω_1,ω_2,ϕ), where ω_i are each of the two angles of the vector $\vec{\xi}_i$ in spherical polar coordinates, and ϕ is defined (in disagreement with some of our earlier work[4,11]) as

$$\tan\phi = \xi_1/\xi_2\ . \qquad (7)$$

The hyperradius ξ in six-dimensional space is

$$\xi = (\xi_1{}^2 + \xi_2{}^2)^{1/2}\ . \qquad (8)$$

With this choice of coordinates, the kinetic energy operator becomes

$$T = -\frac{\hbar^2}{2M}\sum_{i=1}^{3}\nabla^2_{x_i}$$

$$= -\frac{\hbar^2}{6M}\nabla^2_x -$$

$$\frac{\hbar^2}{M}[\nabla^2_{\xi_1} + \nabla^2_{\xi_2}]$$

$$\nabla^2_{\xi_1} + \nabla^2_{\xi_2} = \frac{\partial^2}{\partial\xi^2} + \frac{5}{\xi}\frac{\partial}{\partial\xi} + \frac{\Lambda^2(\Omega)}{\xi^2} \qquad (9)$$

$$\Lambda^2(\Omega) = \frac{\partial^2}{\partial\phi^2} + 4\cot 2\phi\,\frac{\partial}{\partial\phi}$$

$$-\ \ell_2{}^2/\cos^2\phi - \ell_1{}^2/\sin^2\phi\ .$$

$\ell_1{}^2(\omega_1)$ and $\ell_2{}^2(\omega_2)$ are the usual squared angular momentum operator for particle 1 and 2 respectively. The eigenfunctions $H_{[L]}(\Omega)$ of the angular operator Λ^2 are solutions of

$$[\Lambda^2(\Omega) + L(L+4)]H_{[L]}(\Omega) = 0$$

We call these functions hyperspherical harmonics, or h.h. (Simonov[18] calls them K-harmonics; while Fano[19] refers to "Macek coordinates".) The normalized h.h. are given by[20,21,22]

$$H_{[L]}(\Omega) = H^{m_1,m_2}_{L,\ell_1,\ell_2}(\Omega)$$

$$= Y^{m_1}_{\ell_1}(\omega_1)\,Y^{m_2}_{\ell_2}(\omega_2)\ {}^{(2)}P_L^{\ell_2,\ell_1}(\phi) \qquad (10)$$

$${}^{(2)}P_L^{\ell_2,\ell_1}(\phi) = \left|\frac{2(L+2)n\,(L-n+1)!}{\Gamma(n+\ell_1+\frac{3}{2})\Gamma(n+\ell_2+\frac{3}{2})}\right|^{1/2}$$

$$\times(\sin\phi)^{\ell_1}(\cos\phi)^{\ell_2}$$

$$\times P_n^{\ell_1+\frac{1}{2},\ell_2+\frac{1}{2}}(\cos 2\phi)\ ; \qquad (11)$$

$L = 2n+\ell_1+\ell_2$

In Eq. (11), $P_n^{\alpha,\beta}$ is a Jacobi polynomial; the quantum number n is a non-negative integer. L is called the grand orbital. The subscript [L] in Eq. (10) represents the five quantum numbers $(\ell_1,m_1;\ell_2,m_2;L)$. The h.h. have a parity $(-1)^L$ for inversion of the vector $\vec{\xi}$. We couple together spherical harmonics to give a designated total orbital angular momentum ℓ, and projection m, as follows

$$Y_{[L]}(\Omega) = Y^{\ell,m}_{L,\ell_1,\ell_2}(\Omega) = \sum_{m_1,m_2}$$

$$\times\langle\ell_1,m_1;\ell_2,m_2|\ell,m\rangle\, H^{m_1,m_2}_{L,\ell_1,\ell_2}(\Omega)\ . \qquad (12)$$

$\langle\ell_1,m_1;\ell_2,m_2|\ell,m\rangle$ is a Clebsch-Gordan coefficient. We use the set of basis functions, Eq. (12) in this paper. A plane wave is expanded in h.h. as follows

$$\exp(i\vec{k}\cdot\vec{\xi}) = (2\pi)^3\sum_{[L]=0}^{+\infty} i^L H^*_{[L]}(\Omega_k)$$

$$\times H_{[L]}(\Omega)\,J_{L+2}(k\xi)/(k\xi)^2\ . \qquad (13)$$

The sum is taken over all quantum numbers [L], for L varying from 0 to infinity; and J_ν is a Bessel function.

The volume element in six-dimensional space is

$$d^6\xi = d^3\xi_1\,d^3\xi_2 = d\Omega\,\xi^5\,d\xi$$

$$d\Omega = d\omega_1\,d\omega_2\,(\sin\phi\cos\phi)^2\,d\phi \qquad (14)$$

We rewrite Eq. (5) for the density of states.

$$\rho(E) = \frac{1}{2}\frac{Mk^4}{\hbar^2}\frac{d\Omega_k}{(2\pi)^6}\ ;$$

$$E = (\hbar^2/M)(k_1{}^2 + k_2{}^2) = (\hbar^2/M)k^2. \qquad (15)$$

The h.h. $Y_{[L]}(\Omega)$ does not have a definite symmetry for spatial exchange except for exchange of the (1,2) pair, for which the parity is $(-1)^{\ell_1}$.

We have shown in an earlier paper[21] how to construct from the complete set $Y_{[L]}(\Omega)$ another set having the desired symmetry properties for spatial exchange of nucleon pairs. We use the following principle. Let $B_L(s,t,\Omega)$ be an h.h., which includes spin s and isospin t, antisymmetric for complete exchange of any two fermions. The minimum grand orbital L_m is defined by the condition that the interaction cannot generate any other h.h. with grand orbital $L \leq L_m$.

We define an optimal subset by the condition that for each element $B_{L_m+2K}(s,t,\Omega)$ of this basis we have, for K a positive integer,

$$\int B^*_{L_m+2K}(s,t,\Omega)V(\xi,\Omega)B_{L_m}(s,t,\Omega)d\Omega \neq 0 \tag{16}$$

It is understood that for each value of the grand orbital $L_m + 2K$ we take the minimum number of orthogonal h.h. We call $B_{L_m}(s,t,\Omega)$ a fundamental h.h.

The ground state of the trinucleon has L_m=0. Then,

$$B_0(s,t,\Omega) = A(s,t)\, Y_{[0]}(\Omega);$$

$$Y_{[0]}(\Omega) = \pi^{-3/2} . \tag{17}$$

The ground state is expanded in the following form[21]

$$\Psi_G(s,t,\vec{\xi}) = \sum_{K=0}^{+\infty} \{AP^0_{2K}(\Omega)\, \psi^S_{2K}(\xi) + 2^{-1/2}[A'P^+_{2K}(\Omega) + S'P^-_{2K}(\Omega)]\psi^M_{2K}(\xi)\}. \tag{18}$$

The expansion only includes h.h. for even grand orbital L = 2K since we have a state of even parity.

$P^0_{2K}(\Omega)$ is a normalized h.h. of grand orbital 2K, with total angular momentum ℓ = 0, completely symmetric for spatial exchange of any pair of nucleons. On the other hand, $P^+_{2K}(\Omega)$ has mixed symmetry and is even for (1,2) exchange, while $P^-_{2K}(\Omega)$ is odd. For the final 1^- state with isospin 3/2, we have $L_m = \ell = 1$. We choose the polarization along the z-axis (m = 0), giving us

$$B_1(s,t,\Omega) = \frac{1}{\sqrt{2}}\,[A''P^+_1(\Omega) - S''P^-_1(\Omega)],$$

$$P_1{}^+(\Omega) = Y^{10}_{101}(\Omega) = 4(2/\pi)^{1/2}\cos\phi\; Y_0{}^0(\omega_1)\; Y_1{}^0(\omega_2),$$

$$P^-_1(\Omega) = Y^{10}_{110}(\Omega) = 4(2/\pi)^{1/2}\sin\phi\; Y_1{}^0(\omega_1)\; Y_0{}^0(\omega_2). \tag{19}$$

For a potential of Wigner character, the elements of the optimal subset are given by

$$\int B^*_{2K+1}(s,t,\Omega)V(\xi,\Omega)B_1(s,t,\Omega)d\Omega \neq 0$$

with the potential

$$V(\xi,\Omega) = \pi^{3/2} \sum_{K=0}^{+\infty} a_{2K}P^0_{2K}(\xi)V_{2K}(\Omega). \tag{20}$$

The constants a_{2K} are given by Beiner and Fabre[12]: $a_0 = 1$, $a_2 = 0$, $a_4 = 3^{1/2}$, $a_6 = -8^{1/2}$, etc. The hypermultipoles of a two-body potential $V(r_{ij})$ are defined as

$$V_{2K}(\xi) = \frac{8}{\pi^{1/2}}\,\frac{K!}{\Gamma(K+\frac{3}{2})}\int_0^1 V(u\xi)\, P_K^{\frac{1}{2},\frac{1}{2}}(1-2u^2)(1-u^2)^{1/2}u^2du. \tag{21}$$

(Also see FL for the case of a Gaussian two-body potential.) For K = 1, the state $B_3(s,t,\Omega)$ is the projection on L = 3 of the product $P^0_4(\Omega)B_1(s,t,\Omega)$. We find

$$B_3(s,t,\Omega) = 2^{-1/2}[A''P^+_3(\Omega)-S''P^-_3(\Omega)]$$

with

$$P^+_3(\Omega) = 3^{-1/2}\, Y^{10}_{301}(\Omega) - (2/3)^{1/2}\, Y^{10}_{321}(\Omega),$$

$$P^-_3(\Omega) = -3^{-1/2}\, Y^{10}_{310}(\Omega)-(2/3)^{1/2}\, Y^{10}_{312}(\Omega). \tag{22}$$

These functions B_{2K+1} are normalized totally antisymmetric functions of total spin 1/2, total isospin 3/2, total angular momentum ℓ = 1, (m = 0) for grand orbital 2K + 1. We expand the final wave function:

$$\Psi^-_F = \frac{i}{2}\,(2\pi)^3 \sum_{K=0}^{+\infty} (-1)^K\, B_{2K+1}(s,t,\Omega) \times \psi_{2K+1}(\Omega_k,k;\xi). \tag{23}$$

Here the partial wave $\psi_{2K+1}(\Omega_k,k;\xi)$ is the hyperradial wave function which depends on six-vector $\vec{k}$, expressed in hyperspherical representation as (Ω_k,k). We use Eqs. (4), (14), (15), and (18), and the properties of the members of the optimal subset B_{2K+1}. Limiting the expansion to K = 0 and K = 1, the differential cross section becomes

$$d\sigma^{3/2} = \frac{\pi^2}{36}\left(\frac{e^2}{\hbar c}\right)\frac{M}{\hbar^2}\, E_\gamma k^4\; d\Omega_k\,|\langle\psi^S_0 - \frac{1}{\sqrt{2}}\,\psi^M_2|\xi|\psi_1\rangle - \langle\psi^S_4 - \frac{1}{\sqrt{2}}\,\psi^M_2|\xi|\psi_3\rangle|^2. \tag{24}$$

Here the ground state hyperradial functions $\psi^S_0(\xi)$ and $\psi^S_4(\xi)$ are the partial waves for the symmetrical state, and $\psi^M_2(\xi)$ is the partial wave for the mixed symmetry state.

We call Ψ_B the wave function for a continuum state with no nucleon-nucleon interaction (a plane wave in FL). We expand this free wave function in the optimal subset:

$$\Psi_B^- = \sum_{K=0}^{+\infty} B_{2K+1}(s,t,\Omega)|\Psi_B\rangle B_{2K+1}(s,t,\Omega)$$
$$= 2^{1/2}\frac{(2\pi)^3}{(k\xi)^2}\sum_{K=0}^{+\infty}(-1)^K J_{2K+3}(k\xi)$$
$$[P_{2K+1}^+(\Omega_k) - P_{2K+1}^-(\Omega_k)]B_{2K+1}(s,t,\Omega) \qquad (25)$$

That is, in Born approximation we have

$$\psi_{2K+1}^B = [P_{2K+1}^+(\Omega_k) - P_{2K+1}^-(\Omega_k)]J_{2K+3}(k\xi)/(k\xi)^2 . \qquad (26)$$

The Bessel functions J_ν can be expressed using Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$, which are outgoing and incoming waves respectively.

The partial waves ψ_{2K+1} for the problem with nucleon-nucleon interaction in the final state are a solution of a system of coupled equations obeying the following conditions: First, the functions ψ_{2K+1} are regular at the origin; second, their amplitudes are determined so that at very large ξ they correspond to an incoming normalized plane wave Ψ_B^-. The system of coupled equations is written

$$\langle B_{2K+1}|T + \sum_{i,j>i} V(r_{ij}) - E|\Psi_F^-\rangle = 0$$
$$K = 0,\ 1,\ \ldots .$$

For a central potential the final state 1^-, isospin 3/2 limits the interaction to a spin-singlet even, $V^{(1+)}$ and a triplet odd, $V^{(3-)}$. Define

$$V^+ = (V^{(1+)} + V^{(3-)})/2,$$
$$V^- = (V^{(1+)} - V^{(3-)})/2 . \qquad (27)$$

For a Wigner potential, V^- is zero.

A calculation[22,23] gives the coupled differential equations, for grand orbitals 1 and 3 as follows:

$$\{T_1+3[V_0^+(\xi)+V_2^-(\xi)]-E\}\psi_1 - 3[V_2^-(\xi)+V_4^+(\xi)]\psi_3 = 0$$
$$\{T_3+3[V_0^+(\xi) + \tfrac{1}{3}V_2^-(\xi) - \tfrac{8}{3}V_6^-(\xi)]-E\},$$
$$\psi_3 - 3[V_2^-(\xi) + V_4^+(\xi)]\psi_1 = 0$$
$$T_L = -\frac{\hbar^2}{M}\left(\frac{\partial^2}{\partial\xi^2} + \frac{5}{\xi}\frac{\partial}{\partial\xi} - L(L+4)/\xi^2\right) . \qquad (28)$$

(In our past papers[15] we have made some errors in the signs in Eq. (28), particularly for the $V_2^-(\xi)$ terms. This sign error was corrected in our 1976 paper[4] for a single h.h.) This system of coupled equations has two independent solutions $(\psi_1^\alpha,\psi_3^\alpha)$ and $(\psi_1^\beta,\psi_3^\beta)$ corresponding to Blatt-Biedenharn[24] phases δ_α and δ_β. For example, the α solution has the asymptotic behavior, for grand orbital either 1 or 3

$$\psi_L^\alpha(k,\xi) \underset{\xi\to\infty}{\sim} [\cos\delta_\alpha J_{L+2}(k\xi) - \sin\delta_\alpha N_{L+2}(k\xi)]/(k\xi)^2 \qquad (29)$$

N_ν is the Neumann function. (Our normalization differs from FL by a factor of $(2/\pi)^{1/2}k^{-5/2}$). The general solution describing an incoming wave is[25]

$$\Psi_F^- = \frac{i}{\sqrt{2}}(2\pi)^3\{A_\alpha e^{-i\delta_\alpha}[\cos\varepsilon B_1(\Omega)\psi_1^\alpha(k,\xi)+\sin\varepsilon B_3(\Omega)\psi_3^\alpha(k,\xi)]$$
$$+ A_\beta e^{-i\delta_\beta}[-\sin\varepsilon B_1(\Omega)\psi_1^\beta(k,\xi)+\cos\varepsilon B_3(\Omega)\psi_3^\beta(k,\xi)]\} \qquad (30)$$

Here ε is the mixing coefficient. The functions $A_\alpha(\Omega_k)$ and $A_\beta(\Omega_k)$ are obtained by comparison with the partial wave expansion (25) of the incoming plane wave Ψ_B^-:

$$A_\alpha(\Omega_k) = [P_1^+(\Omega_k) - P_1^-(\Omega_k)]\cos\varepsilon - [P_3^+(\Omega_k) - P_3^-(\Omega_k)]\sin\varepsilon,$$
$$A_\beta(\Omega_k) = [P_1^+(\Omega_k) - P_1^-(\Omega_k)]\sin\varepsilon + [P_3^+(\Omega_k) - P_3^-(\Omega_k)]\cos\varepsilon. \qquad (31)$$

These functions are orthogonal, and normalized to 2:

$$\int A_\alpha A_\beta d\Omega_k = 0; \qquad \int (A_\alpha)^2 d\Omega_k = \int (A_\beta)^2 d\Omega_k = 2 . \qquad (32)$$

We now substitute (30) in Eq. (24) to find the differential cross section

$$d\sigma^{3/2} = (\pi^2/36)(e^2/\hbar c)(M/\hbar^2)(E_\gamma k^4)d\Omega_k|A_\alpha(\Omega_k)[R_{01}^\alpha\cos\varepsilon - R_{43}^\alpha\sin\varepsilon]$$
$$- \exp(i\delta)A_\beta(\Omega_k)[R_{01}^\beta\sin\varepsilon + R_{43}^\beta\cos\varepsilon]|^2; \ \delta = \delta_\alpha - \delta_\beta . \qquad (33)$$

The four overlap integrals $R_{L,L'}^a$ are designated by the grand orbital L of the symmetric part of the initial state and L' of the final state; the superscript states whether we use the α or β solution.

$$R_{01}^\alpha = \langle\psi_0^S - 2^{-1/2}\psi_2^M|\xi|\psi_1^\alpha\rangle; \ R_{01}^\beta = \langle\psi_0^S - 2^{-1/2}\psi_2^M|\xi|\psi_2^\beta\rangle,$$
$$R_{43}^\alpha = \langle\psi_4^S - 2^{-1/2}\psi_2^M|\xi|\psi_3^\alpha\rangle; \ R_{43}^\beta = \langle\psi_4^S - 2^{-1/2}\psi_2^M|\xi|\psi_3^\beta\rangle. \qquad (34)$$

We find the total cross section by integrating Eq. (33) over the 5 angles contained in Ω_k. The total cross section σ_T is independent of the phase difference δ because of orthogonality of $A_\alpha(\Omega_K)$ and $A_\beta(\Omega_K)$ given in Eq. (32).

$$\sigma_T = (\pi^2/18)(e^2/\hbar c)(M/\hbar^2)(E_\gamma k^4)\{(\cos\varepsilon\, R^\alpha_{01} - \sin\varepsilon\, R^\alpha_{43})^2 + (\sin\varepsilon\, R^\beta_{01} + \cos\varepsilon\, R^\beta_{43})^2\}. \quad (35)$$

Setting $\varepsilon = 0$ and $\psi_3 = 0$ (and noting that R^α_{01} corresponds to $(2/\pi)^{1/2} k^{-5/2} r_{if}$ of FL) we reproduce the equation given earlier by FL for σ_T for a single uncoupled partial wave, with grand orbital one.

IV. Numerical Results

We use two potentials, the spin-independent Volkov potential[16] and the spin-dependent central potential[13,22] V^X, which was adjusted to agree with experimental values of the trinucleon energy, and the form factors of ^3H and ^3He, at least for small momentum transfers. For the first potential, we use the ^3H ground state wave function calculated by Beiner and Fabre[12]; for the second potential the ^3He wave function calculated by Ballot et al.[13] We modify the original Volkov potential by assuming[11] pure Wigner exchange. We assume a Serber mixture for the V^X potential: namely that the potential is zero for states of odd parity.

We use the standard Runge-Kutta numerical method to solve the coupled differential equations (28). We follow the analysis due to Blatt-Biedenharn[24], Hulthen-Sugawara[25], and Drechsel-Maximon[26] in order to find initial conditions that give the α or β eigenphase solutions. (See Fang[23] for details.) The numerical results for the Volkov spin independent potential are compiled in Table I and the total cross section of $\sigma_T(E_\gamma)$ of Eq. (35) for ^3H photoeffect is plotted vs E_γ in Fig. 1. Note that δ_1 (for a single h.h.) is close to δ_α and somewhat smaller.

In Fig. 1 we include results by Myers[11] for Born-approximation (dash-dot) and for a single h.h. (dashed) with grand orbital one, as well as our new results (solid) for 2 h.h. We see that the broad peak in Born approximation is narrowed and pulled to lower energy by the strong attraction using a single h.h. Use of two coupled h.h. corresponds to an additional attraction, giving some additional narrowing and shift to lower energy. However, the use of a second h.h. gives a relatively small change in the total cross section, in general less than 10%.

Our numerical accuracy in calculation of the total cross section can be checked by comparison with the sum rule for the bremsstrahlung-weighted cross section σ_{-1}, which is proportional[6,11] to the mean-square value of the hyperradius ξ for the ground state wave function. In Table II the sum rule value of 1.42 mb and the values of 1.43 mb for Born approximation and for a single, uncoupled h.h. are taken from Myers[11]. Our value of 1.40 mb, found by Gauss-Gegenbauer quadrature[11,27] of the cross sections in Table 1, is in agreement with the other two values, within the numerical accuracy of our calculation.

The speed of convergence of the h.h. expansion is tested by the value of the integrated cross section σ_0. The Thomas-Reiche-Kuhn value of 19.9 MeV mb holds for any potential that commutes with the dipole operator. The value of 47.6 MeV mb for Born approximation is in serious disagreement with the sum rule; while the value[11] of 21.1 MeV mb for a single h.h. is only 6% above the sum rule value. Our present use of coupled h.h. gives 20.5 MeV mb, only 3% above the TRK value!

Table I
Phase Shifts, Overlap Integrals and Cross Sections for Volkov Potential

E_γ(MeV)	tan δ_1	tan δ_α	tan δ_β	tan ε	R^α_{01}(fm)	R^α_{43}(fm)	R^β_{01}(fm)	R^β_{43}(fm)	σ_T(mb)
8.83	0.26	0.28	0.05	-0.40	175.	0.40	29.	0.0	0.011
9.51	0.54	0.66	0.11	-0.14	214.	-6.0	36.	0.0	0.26
10.61	1.23	2.11	0.17	-0.03	251.	-6.5	30.7	1.2	1.73
12.33	20.5	-3.83	0.25	0.06	171.	-3.7	21.3	2.5	3.02
15.02	-1.63	-1.18	0.33	0.16	68.8	-1.0	12.3	3.4	1.69
19.38	-0.94	-0.81	0.47	0.23	22.1	-0.11	5.78	3.13	0.6[illegible]
26.99	-0.91	-0.85	0.68	0.27	5.56	-0.03	2.30	1.92	0.16
41.79	-1.23	-1.03	1.07	0.32	0.90	0.00	0.62	0.62	0.035
76.92	-2.93	-2.10	1.86	0.39	0.032	-0.011	0.011	-0.0054	0.21×10^{-3}
201.72	2.60	5.43	1.72	0.85	-0.0010	-0.0073	-0.0058	-0.0088	0.26×10^{-3}

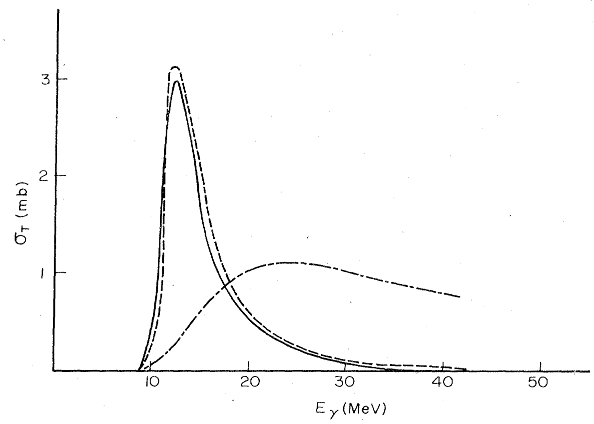

Fig. 1. Cross section σ_T in mb for ^3H, E1 transitions to isospin 3/2 state vs photon energy E_γ in MeV, for Volkov potential. The dash-dot line shows Born approximation; the dashed line shows a single h.h. (both from Myers[11]); the solid line shows two coupled h.h. from Table I.

The sum rule for the moment σ_1 is proportional to the ground state expectation value of the kinetic energy; the approximate value of 290 MeV^2 mb in the Table is from Myers[11]. We also reproduce Myers' values of 2100 MeV^2 mb for Born approximation and 338 MeV^2 mb for an uncoupled h.h. Our present value of 342 MeV^2 mb is 18% above the approximate sum rule value. This discrepancy (much larger than the 3% above for the integrated cross section) is due in part to (i) approximations in the sum rule value, (ii) approximations in Gauss-Gegenbauer quadrature, and (iii) sensitivity of σ_1 to value of $\sigma(E_\gamma)$ at high photon energies, where the truncation at grand orbital 3 is likely a poor approximation. (Note that the highest energy in Table I contributes substantially to σ_1. Figure 1 shows that the curve for 2 h.h. is lower than that for 1 h.h. up to 40 MeV; but the former curve is higher at 201.7 MeV.)

We use the same numerical techniques to find the eigenphase solutions of Eq. (28) for the continuum wave function with the spin-dependent V^x potential. The phase parameters δ_α, δ_β and ε are given for ten photon energies in Table III along with tan δ_1 for a single h.h.[4] We combine our continuum eigenphase solutions with Ballot's numerical results for the ground state wavefunction of ^3He to find the four overlap integrals, Eq. (34) and the total cross section, Eq. (35). In Fig. 2 we compare three cross sections all using Ballot's ground state wave function for the V^x potential: i) Born approximation[4], using a free partial wave with grand orbital one, shown dash-dot; ii) a single h.h. for the V^x potential[4] shown dashed and iii) the present work using 2 h.h. shown as a solid line. Figure 2 shows the same general features as Fig. 1 for the spin-independent case: the very broad peak in Born approximation is narrowed and shifted towards lower energy when we use a single h.h. for an attractive potential. There is a smaller shift when we use a second h.h. Cross sections for 2 h.h. agree with those for 1 h.h. to better than 10% for photon-energies of less than 100 MeV.

We use our 10 cross sections for 2 h.h. to give the moments σ_{-1}, and σ_0 presented in Table IV. We compare with the moments in Born approximation and with a single h.h.[4] The three values of σ_{-1} agree, within round-off errors, as they must. They also agree with the sum rule value,[4] based on the rms radius. We evaluate the integrated cross section σ_0 using the O'Connell-Prats[17] approximation for a symmetric ground state. We truncate the ground state wave function at the lowest partial wave, $u_0(\xi)$. This truncates the hypermultipole expansion of $\xi_1^2 V(\xi_1)$ at $V_2(\xi)$. Our value σ_0 = 29.5 MeV mb for 2 h.h. is 2% above the sum rule value σ_0 = 28.8 MeV mb.

V. Conclusions and Discussion

In this paper we extend the calculations of FL[4] and Myers[11] by including another hyperspherical harmonic in the expansion of the continuum wave function.

We find excellent convergence in the model calculation of a Volkov potential

Table II

Comparison with Sum Rules, Volkov Potential

Moment	Born[a]	Uncoupled	Coupled	Sum Rule[a]
σ_{-1}	1.43 mb	1.43 mb	1.40 mb	1.43 mb
σ_0	47.6 MeV mb	21.1 MeV mb	20.5 MeV mb	19.9 MeV mb
σ_1	2100. MeV^2 mb	338 MeV^2 mb	342 MeV^2 mb	290 MeV^2 mb (approx.)

a. See Myers[11].

Table III
Phase Shifts, Overlap Integrals and Cross Section for V^X Potential

E_γ(MeV)	tan δ_1	tan δ_α	tan δ_β	tan ε	R^α_{01}(fm)	R^α_{43}(fm)	R^β_{01}(fm)	R^β_{43}(fm)	σ_T(mb)
8.13	0.128	0.143	0.0082	-0.20	59.3	-0.6	15.2	-0.06	0.00212
8.81	0.248	0.277	0.0151	-0.160	81.3	-0.90	14.4	0.21	0.0347
9.91	0.384	0.440	0.0257	-0.114	85.6	-1.01	13.1	0.67	0.186
11.63	0.544	0.640	0.0362	-0.080	72.8	-0.94	10.4	1.19	0.515
14.32	0.736	0.876	0.0494	-0.0443	49.8	-0.62	7.35	1.52	0.854
18.68	0.960	1.14	0.065	-0.0074	27.4	-0.265	4.43	1.38	0.941
26.29	1.142	1.30	0.094	0.036	11.6	-0.040	2.22	0.760	0.689
41.09	1.151	1.254	0.141	0.100	3.58	0.0136	0.770	0.174	0.327
76.22	0.880	0.965	0.223	0.247	0.658	-0.0113	0.0706	-0.0722	0.0834
201.00	0.370	0.590	0.223	0.830	0.0168	-0.0191	-0.0131	-0.0193	0.00500

assumed to have pure Wigner character. First, Fig. 1 shows that the values of the total photoeffect cross section at a specified energy show rapid convergence when we consider Born approximation, h.h. and two coupled h.h. Second, Table II shows a corresponding rapid convergence in values of the moments σ_0 and σ_1. Third, Table II also shows that the value of σ_0 has almost converged to the Thomas-Reiche-Kuhn value, missing by only 3% if we use 2 coupled h.h.

Convergence is equally rapid for our use of the spin-dependent V^X potential, zero in two nucleon states of odd parity: see Fig. 2 for the rapid convergence of the total cross section at a given energy. The agreement to 2% between the integrated cross section for 2 h.h. and an approximate sum rule value is as satisfying as the excellent agreement noted above for the Volkov potential.

Since this V^X potential is in rough agreement with potentials chosen by other theorists[2,3] and may well be similar to the potential in the real world, a brief comparison with calculations and experiment is in order. Figure 3 shows the Gibson-Lehman[3] calculation for final isospin 3/2 states as a dashed curve, and our cross sections for 2 h.h. as a solid curve. Considering the different potentials used, the overall agreement between the calculations is satisfactory. The figure also shows three different sets of measurements on three-body break-up, which include a small contribution from isospin 1/2 states: (i) Gorbunov[8] as x's with standard errors, (ii) Gerstenberg[10] as squares, and (iii) Berman[9] as circles. The experimental results disagree with each other above 20 MeV. Our calculation agrees with Gorbunov almost within experimental accuracy up to 50 MeV. We can also compare our σ_{-1} and σ_0 values for 2 h.h. with

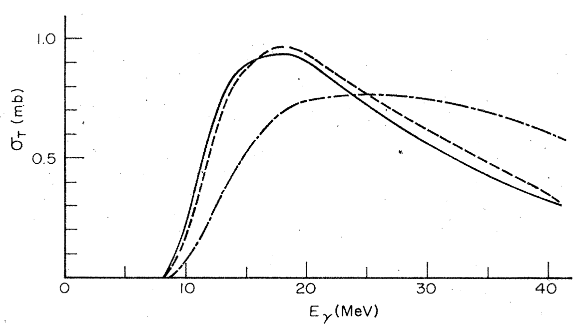

Fig. 2. Cross section σ_T in mb for ^{3}He, E1 transitions to isospin 3/2 state vs photon energy E_γ in MeV, for V^X potential. The dash-dot line shows Born approximation; the dashed line shows a single h.h. (both from Fabre[4]); the solid line shows two coupled h.h. from Table III.

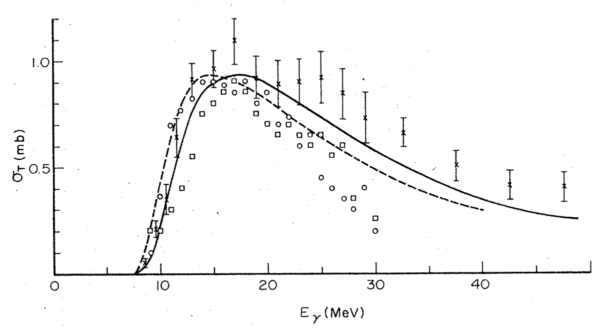

Fig. 3. Cross section σ in mb for ^{3}He photoeffect vs photon energy E_γ in MeV. The x's show experiments on three-body breakup by Gorbunov[8], the circles show Berman's results[9] and the squares show Gerstenberg's[10]. The dashed curve shows calculations by Gibson[3], the solid curve is from Table III for h.h. (Both calculations are for isospin 3/2 final state.)

Table IV

Moments for V^x Potentials

Moment	Born[a]	1 h.h.[a]	2 h.h.	Sum Rule
σ_{-1}	1.14 mb	1.12 mb	1.11 mb	1.1 mb
σ_0	38.9 MeV mb	29.0 MeV mb	29.5 MeV mb	28.8 MeV mb

a. See Fabre.[4]

Gorbunov's values for three-body break-up. Our σ_{-1} value is 20% below the experimental 1.42 ± 0.07 mb; and our σ_0 value is 32% below Gorbunov's 43.6 ± 2.7 MeV mb.

Comparison with other calculations and with experiments could be extended in five ways. First, we should evaluate the sum rule for σ_1 for a spin-dependent force for comparison with our values in Table III, and we should evaluate σ_0 with fewer approximations. Second, we should use the h.h. expansion for other choices of the two-nucleon potential, including coulomb and tensor forces.[28] Third, we should use Eq. (33) for the differential cross sections to make comparisons with the energy distributions (which amounts to the dependence on the angle Φ_k in momentum space) found in other calculations and in experiment. We should include E2 transitions and their effect on angular distributions. Fourth, we should evaluate the ground state Stark effect to determine the trinucleon polarizability and hence[6] σ_{-2}. Fifth, we should study E1 transitions to isospin 1/2 states.[29]

We can also apply h.h. expansions to several different problems, as follows. (i) Use of 3 or more h.h. for the continuum wave function. (ii) Further calculations of three-body to three-body scattering[22] and of corresponding virial coefficients[30]. (iii) Photon absorption at low energy by ^{12}C leading to three alphas, by treating this nucleus as a 3-α system[31]. (iv) The low energy photoeffect of ^6Li (as a N-N-α system) and of ^9Be (as a N-α-α) system). (v) The decay of ω meson into three pions. (vi) Photopion production from the deuteron, using good continuum wave functions for the π-N-N final state. (vii) Photopion production from ^3H and ^3He, using good continuum wave function for the 3-neutron and 3-proton systems, respectively. (viii) Analysis of the classical three-body problem by finding the classical limit[32] of the quantum mechanical h.h. expansion.

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Appendix A

Let us consider three mathematical entities (1), (2), and (3) that obey the orthonormal rule:

$$\langle i|j\rangle = \delta_{ij} .$$

We can write three orthonormal combinations as follows:

Table V

Spin-Isospin Wave Functions

Notation	Combinations	Total Spin S	Total Isospin T	Symmetry
A	$(1/\sqrt{2})[(+,-)-(-,+)]$	1/2	1/2	a
S	$(1/\sqrt{2})[(+,+)-(-,-)]$	1/2	1/2	s
A'	$(1/\sqrt{2})[(+,-)+(-,+)]$	1/2	1/2	m_-
S'	$(1/\sqrt{2})[(+,+)+(-,-)]$	1/2	1/2	m_+
S''	(0,+)	1/2	3/2	m_+
A''	(0,-)	1/2	3/2	m_-
S'''	(+,0)	3/2	1/2	m_+
A'''	(-,0)	3/2	1/2	m_-
$S^{(iv)}$	(0,0)	3/2	3/2	s

$$(0) = (1/\sqrt{3})[(1) + (2) + (3)],$$

$$(+) = (1/\sqrt{6})[2(3) - (1) - (2)],$$

$$(-) = (1/\sqrt{2})[(1) - (2)]. \qquad (I)$$

When we describe a spin state of a system of three nucleons for a specified projection of total spin S_z, $|i\rangle$ gives the spin state. For example for $S_z = 1/2$, $|i\rangle = \beta_i\alpha_j\alpha_k$, where α designates spin up and β designated spin down. Of course the same notation holds for isospin. For example, for $T_z = 1/2$, $|i\rangle = n_i p_j p_k$, where p is a proton with positive isospin, and n is a neutron. Then (0) stands for a spin state that is completely symmetric for exchange of any pair, and has S = 3/2; or (0) stands for a completely symmetric isospin state with T = 3/2. The states designated (+) or (-) have mixed symmetry and S = 1/2 (or T = 1/2). The former is symmetric for exchange of entities (1) and (2); the latter is antisymmetric for this exchange.

Using Eq. (I) we form the nine possible combinations of spin and isospin. We designate a combination of isospin (a) and spin (b) by the symbol (a,b). The overall symmetry for the spin-isospin wave function is given in the right column: "a" is a completely antisymmetric, s is completely symmetric; m_+ has mixed symmetry, symmetric or antisymmetric for interchange of 1 and 2.

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1. J. C. Gunn and J. Irving, Phil. Mag., 42, 1353 (1951).
2. I. M. Barbour and A. C. Phillips, Phys. Rev. C 1, 165 (1970).
3. B. F. Gibson and D. R. Lehman, Phys. Rev. C 13, 477 (1976).
4. M. Fabre de la Ripelle and J. S. Levinger, Nuov. Cim. 25A, 555 (1975), and Nuov. Cim. Letters 16, 413 (1976).
5. L. M. Delves, Nucl. Phys. 20, 268 (1962).
6. J. S. Levinger, Nuclear Photo-Disintegration (Oxford Univ. Press) 1961.
7. V. N. Fetisov, A. M. Gorbunov and V. T. Varfolomeev, Nucl. Phys. 71, 305 (1965).
8. A. N. Gorbunov, Photonuclear and Photomesic Processes, Proc. of the P. N. Lebedev Physics Institute, 1974 (Nauka, Moscow), Vol. 71, p. 1 (in Russian); Photonuclear and Photomesic Processes, Trudy, (Editor, D. V. Skobel'tsyn), Consultants Bureau, New York, 1976, Vol. 71, p. 1 (transl.).
9. B. L. Berman, S. C. Fultz and P. Yergin, Phys. Rev. C 10, 2221 (1974).
10. H. M. Gerstenberg and J. S. O'Connell, Phys. Rev. 144, 834 (1966).
11. Karen J. Myers, K. K. Fang and J. S. Levinger, Phys. Rev. C 15, 1215 (1977).
12. M. Beiner and M. Fabre de la Ripelle, Nuov. Cim. Letters 1, 584 (1971).
13. J. L. Ballot, M. Beiner and M. Fabre de la Ripelle, Proceedings of the Symposium on Present Status and Novel Developments in Many-Body Problems, Roma, 1972, edited by V. Calogero and C. Ciofi degli Atti (Bologna, 1974), p. 565.
14. J. L. Ballot and M. Fabre de la Ripelle, Few Body Dynamics, edited by A. N. Mitra et al., North-Holland, New York, 1976, p. 146.
15. K. K. Fang, J. S. Levinger and M. Fabre de la Ripelle, Few Body Dynamics, edited by A. N. Mitra et al., North-Holland, New York, 1976, p. 358.
16. A. Volkov, Nucl. Phys. 74, 33 (1965).
17. J. S. O'Connell and F. Prats, Phys. Rev. 184, 1007 (1969).
18. Yu. A. Simonov, Sov. J. Nucl. Phys. (transl.) 3, 461 (1966).
19. U. Fano, Physics Today 29, 32 (1976).
20. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (McGraw-Hill, New York, 1953), Vol. II, pp. 1730 ff.
21. M. Fabre de la Ripelle, Proceedings of the International School on Nuclear Theory Physics, Predeal, 1969.
22. K. K. Fang, Phys. Rev. C 15, 1204 (1977).
23. K. K. Fang, Ph.D. dissertation, Rensselaer Polytechnic Institute, 1976, unpublished.
24. J. Blatt and L. Biedenharn, Phys. Rev. 86, 399 (1952).
25. L. Hulthen and M. Sugawara, Handbook of Physics (Springer-Verlag, New York, 1957), Vol. 39.
26. D. Drechsel and L. Maximon, Ann. Phys. (N. Y.) 49, 403 (1968).
27. T. Brady, M. Fuda, E. Harms, J. S. Levinger and R. Stagat, Phys. Rev. 186, 1069 (1969).
28. V. F. Demin, Yu. F. Pokrovsky and V. D. Efros, Phys. Lett. 44B, 227 (1973); and J. L. Ballot, private communication.
29. J. Revai and J. Raynal, Nuov. Cim. Letters 9, 461 (1974); J. S. Levinger Bull. Amer. Phys. Soc. 22, 612 (1977).
30. S. Y. Larsen and P. L. Mascheroni, Phys. Rev. A 2, 1018 (1970).
31. M. Vallieres, H. T. Coelho and T. K. Das, Nuclear Phys. A271, 95 (1976).
32. L. S. Brown, Amer. J. Phys. 41, 525 (1973).