Consequences of wave function orthogonality for medium energy nuclear reactions*

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In the usual models of high-energy bound-state to continuum transitions no account is taken of the orthogonality of the bound and continuum wave functions. This orthogonality induces considerable cancellations in the overlap integrals expressing the transition amplitudes for reactions such as $(e,e'p)$, (γ,p), and (π,N), which are simply not included in the distorted-wave Born-approximation calculations which to date remain the only computationally feasible heirarchy of approximations. The object of this paper is to present a new formulation of the bound-state to continuum transition problem, based upon flux conservation, in which the orthogonality of wave functions is taken into account *ab initio*. The new formulation, while exact if exact wave functions are used, offers the possibility of using approximate wave functions for the continuum states without doing violence to the cancellations induced by orthogonality. The method is applied to single-particle states obeying the Schrödinger and Dirac equations, as well as to a coupled-channel model in which absorptive processes can be described in a fully consistent manner. Several types of absorption vertex are considered, and in the (π,N) case the equivalence of pseudoscalar and pseudovector πNN coupling is seen to follow directly from wave function orthogonality.

NUCLEAR REACTIONS Orthogonality constraints on bound-continuum transitions. Applied to Schrödinger, Dirac, and coupled-channel wave functions, describing $(e,e'p)$, (γ,p) and (π,N) reactions on nuclei at medium energies. New form of DWBA. Equivalence of pseudoscalar and pseudovector coupling in (π,N). Relativistic corrections vs high momentum components in (γ,p).

I. INTRODUCTION

Reactions such as $A(\pi,N)B$ or $A(\gamma,N)B$ are interesting because of the possibility that they can shed light on the high Fourier components of nuclear single-particle wave functions at momenta considerably larger than the Fermi momentum $k_F \sim 250$ MeV/c. The amplitudes for such reactions have the form

$$\int d^3x \langle B;\vec{p}\,|j(\vec{x})\,|A\rangle e^{i\vec{k}\cdot\vec{x}}, \tag{1}$$

where the state $|A\rangle$ is usually the ground state of the target, and the state $|B;\vec{p}\rangle$ is a continuum state of the nuclear Hamiltonian asymptotic to a nucleon with momentum $\vec{p}$ and the state $|B\rangle$ of the residual nucleus, and with incoming wave boundary conditions. The conditions under which these reactions are performed at medium energies require the momentum p to be large compared with k, by virtue of energy conservation; that is,

$$P=(2MW+W^2)^{1/2}, \tag{2}$$

where

$$W=\omega(k)-b \tag{3}$$

is the energy $\omega(k)$ introduced by the absorbed particle, less the nucleon binding energy b. On the other hand, the momentum k is generally not small on the scale of the initial nuclear state $|A\rangle$, and hence cannot be neglected, e.g., in the spirit of the long-wavelength approximation.[1] The problem is therefore to evaluate (1) with p large, but without neglecting k, or at least to determine the leading contributions in a feasible scheme of approximation.

To make the above remarks more explicit, let us consider the simplest possible case, where a single nucleon bound in a potential is ejected into a continuum state of the same potential. The amplitude is then

$$A_{\vec{p}}(\vec{k};\mathcal{O})=\int d^3x\,\phi_{\vec{p}}^{(-)}(\vec{x})\mathcal{O}\psi(\vec{x})\,e^{i\vec{k}\cdot\vec{x}}, \tag{4}$$

where $\mathcal{O}$ is some operator on nucleon position and spin coordinates. If the absorbed particle is of scalar character, then $\mathcal{O}=1$; as Amado and Woloshyn have recently pointed out,[2] $A_{\vec{p}}(\vec{k};1)$ must vanish in the limit of small k, by virtue of the orthogonality of the bound and scattering wave functions. That is, even though the wavelength of the final nucleon is very small, we cannot simply replace the state $\phi_{\vec{p}}^{(-)*}(\vec{x})$ by $(e^{-i\vec{p}\cdot\vec{x}}/(2\pi)^{3/2})$, thereby obtaining the Fourier transform of the bound state,

$$A_{\vec{p}}^{(0)}(\vec{k};1)=\tilde{\psi}(\vec{p}-\vec{k}), \tag{5}$$

since the rescattering part of $\phi_{\vec{p}}^{(-)*}(\vec{x})$ will contribute something of order $-\tilde{\psi}(\vec{p})$ to $A_{\vec{p}}(\vec{k};1)$, because of the orthogonality. Similarly, we should expect rescattering corrections to be important in determining the large-p, fixed-k behavior of Eq. (4), or Eq. (1), even when the operator $\mathcal{O}$ [or the integral $\int d^3x j(\vec{x})$] is not a c number.

The object of this paper is to set forth a new method for systematically approximating matrix elements such as Eq. (1) or Eq. (4), without doing violence to the cancellations imposed by the orthogonality of the nuclear states. The method is based on the coordinate-space representation of the dynamical equations of the nuclear wave functions, and is related to the method recently employed by Amado[3] to determine the high momentum behavior of Hartree-Fock wave functions. In Sec. II, I shall treat the simplest case, in which a single particle moves in a static, real potential, according to the nonrelativistic Schrödinger equation. The absorbed particle will be taken to have either scalar or vector (photon) character, and I shall exhibit explicitly the result reported by Fink, Hebach, and Kummel,[4] that wave-function orthogonality suppresses strongly spin-flip (magnetic) photoabsorption relative to the convective part of the amplitude. In Sec. III, we shall consider the same sort of thing applied to the Dirac equation, with a view toward obtaining useful large-p approximations for $(e, e'p)$, (π^+, p), and (γ, p) reactions. An interesting point found in this connection is that the recently reported result of Miller and Weber,[5] in which they found that the (p, π^+) amplitude depends strongly on whether pseudovector or pseudoscalar πNN coupling is used, may be an artifact of the use of (nonorthogonal) plane waves to describe the continuum. That is, Friar's recent result on the near equivalence of the two types of coupling, obtained through judicious application of commutation relations,[6] is reproduced here as a consequence of wave-function orthogonality. Finally, Sec. IV is dedicated to the problem of inelasticity. A simple coupled-channel model which contains many of the ingredients of more realistic systems is investigated in detail.

II. NONRELATIVISTIC SINGLE-PARTICLE DYNAMICS

A. Scalar probe

We begin with the amplitude

$$A_{\vec{p}}(\vec{k}; 1) = \int d^3x \, \phi_{\vec{p}}^{(-)*}(\vec{x}) \psi(\vec{x}) e^{i\vec{k}\cdot\vec{x}}, \tag{6}$$

where $\phi_{\vec{p}}^{(-)}(\vec{x})$ and $\psi(\vec{x})$ are solutions of the Schrödinger equations

$$[\nabla^2 + p^2 - 2MV(\vec{x})]\phi_{\vec{p}}^{(-)*}(\vec{x}) = 0, \tag{7}$$

$$[\nabla^2 - \kappa^2 - 2MV(\vec{x})]\psi(\vec{x}) = 0. \tag{8}$$

The orthogonality of the functions ϕ and ψ is shown as follows: Multiply Eq. (7) on the left by $\psi(\vec{x})$ and Eq. (8) by $\phi_{\vec{p}}^{(-)*}(\vec{x})$ and subtract, thereby obtaining the relation

$$\phi_{\vec{p}}^{(-)*}(\vec{x})\psi(\vec{x}) = -(p^2 + \kappa^2)^{-1} \nabla \cdot \vec{J}(\vec{x}), \tag{9}$$

where

$$\vec{J}(\vec{x}) = \psi(\vec{x}) \nabla \phi_{\vec{p}}^{(-)*}(\vec{x}) - \phi_{\vec{p}}^{(-)*}(\vec{x}) \nabla \psi(\vec{x}). \tag{10}$$

Clearly, since $\psi(\vec{x})$ vanishes as $(e^{-\kappa x}/x)$ for large x, the surface integral of $\vec{J}(\vec{x})$ over a sphere of arbitrarily large radius, centered at the origin, vanishes, and we may therefore apply Gauss's theorem to write

$$\begin{aligned}\int d^3x \, \phi_{\vec{p}}^{(-)*}(\vec{x})\psi(\vec{x}) &= -(p^2+\kappa^2)^{-1} \int d^3x \nabla \cdot \vec{J}(\vec{x}) \\ &= -(p^2+\kappa^2)^{-1} \lim_{R\to\infty} R^2 \int d\hat{x}\, \hat{x} \cdot \vec{J}(\hat{x}R) \\ &= 0,\end{aligned} \tag{11}$$

which expresses the orthogonality.

To obtain $A_{\vec{p}}(\vec{k}; 1)$, we multiply Eq. (9) by $e^{i\vec{k}\cdot\vec{x}}$ before integrating, so that

$$A_{\vec{p}}(\vec{k}; 1) = -(p^2+\kappa^2)^{-1} \int d^3x \, e^{i\vec{k}\cdot\vec{x}} \nabla \cdot \vec{J}(\vec{x}), \tag{12}$$

and we apply the identity

$$e^{i\vec{k}\cdot\vec{x}} \nabla \cdot \vec{J}(\vec{x}) \equiv \nabla \cdot [e^{i\vec{k}\cdot\vec{x}} \vec{J}(\vec{x})] - i\vec{k} \cdot \vec{J}(\vec{x}) e^{i\vec{k}\cdot\vec{x}},$$

together with Gausss' theorem, to find

$$A_{\vec{p}}(\vec{k}; 1) = (p^2+\kappa^2)^{-1} \int d^3x \, e^{i\vec{k}\cdot\vec{x}} i\vec{k} \cdot \vec{J}(\vec{x}). \tag{13}$$

We see that the right-hand side of Eq. (13) vanishes as $k \to 0$, as it should. To make Eq. (13) useful for practical calculations, we integrate once more by parts and add and subtract $i\vec{p}$ to ∇, obtaining, at last,

$$\begin{aligned}A_{\vec{p}}(\vec{k}; 1) = {} & 2i/[(\vec{p}-\vec{k})^2 + \kappa^2] \\ & \times \int d^3x \, e^{i\vec{k}\cdot\vec{x}} \vec{k} \cdot [(\nabla + i\vec{p})\phi_{\vec{p}}^{(-)*}(\vec{x})]\psi(\vec{x}).\end{aligned} \tag{14}$$

The virtue of the operator $\nabla + i\vec{p}$ is that it projects off the plane-wave part of $\phi_{\vec{p}}^{(-)*}(\vec{x})$. The motivation for integrating by parts to remove $\nabla\psi(\vec{r})$ is that it is the bound-state wave function whose structure we hope to elucidate through the reaction amplitude $A_{\vec{p}}(\vec{k}; 1)$; generally, we have a much better idea of the gradient of the scattering state $\phi_{\vec{p}}^{(-)}(\vec{x})$, by virtue of its large momentum, than of the gradient of the bound state. For example, the approximation

$$\nabla \phi_{\vec{p}}^{(-)*}(\vec{x}) \simeq -i\hat{p}(p^2 - 2MV(\vec{x}))^{1/2} \phi_{\vec{p}}^{(-)*}(\vec{x}) \tag{15}$$

(the "local WKB" approximation) seems *ab initio* more reasonable than, say,

$$\nabla\psi(\vec{x}) \simeq -\hat{x}\kappa\psi(\vec{x}); \tag{16}$$

yet something like Eq. (16) is the best we can do in the absence of detailed knowledge of $\psi(\vec{x})$. If we take Eq. (15) seriously, we obtain for (14)

$$A_{\vec{p}}(\vec{k};1) \simeq \frac{-2M\vec{k}\cdot\hat{p}/p}{(\vec{p}-\vec{k})^2+\kappa^2} \times \int d^3x\, e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{p}}^{(-)*}(\vec{x})V(\vec{x})\psi(\vec{x}). \quad (17)$$

Equation (17) is suitable for further approximation; however, it is interesting to note that if $\phi_{\vec{p}}^{(-)*}(\vec{x})$ is now replaced by a plane wave, we find in the limit of large p

$$A_{\vec{p}}(\vec{k};1) \sim \frac{\vec{k}\cdot\hat{p}}{p}\,\tilde{\psi}(\vec{p}-\vec{k}), \quad (18)$$

which, for potentials of reasonable shape, expresses the leading large-p behavior of $A_{\vec{p}}(\vec{k};1)$. We see that the orthogonality has greatly reduced the amplitude from the plane-wave Born approximation, Eq. (5), in fact by the factor $(k^2/2M\omega(k))^{1/2}$.

B. Nonrelativistic (γ,p) reaction

The nonrelativistic amplitude for (γ,p), under the same assumptions obtaining in II A, is proportional to

$$A_{\vec{p}}\left(\vec{k};\frac{\hat{e}\cdot(\overleftarrow{\nabla}-\overrightarrow{\nabla})}{2M}+\mu_p\frac{\vec{\sigma}\cdot\vec{k}\times\hat{\epsilon}}{2M}\right) = \int d^3x\,\phi_{\vec{p}}^{(-)*}(\vec{x})e^{i\vec{k}\cdot\vec{x}}\left[\frac{\hat{\epsilon}\cdot(\overleftarrow{\nabla}-\overrightarrow{\nabla})}{2M} + \mu_p\frac{\vec{\sigma}\cdot\vec{k}\times\hat{\epsilon}}{2M}\right]\psi(\vec{x}), \quad (19)$$

where the photon polarization $\hat{\epsilon}$ is transverse, $\hat{\epsilon}\cdot\vec{k}=0$. The convective amplitude is of course not affected by orthogonality *per se*, since [as in Eq. (13)] the gradient coupling ensures that the radial integrals will involve different partial waves, which are not in general orthogonal. (That is, the orthogonality of different partial waves is taken care of by their angular dependence, and not by their radial dependence.) We may thus write, in the same spirit as was used in going from Eq. (14) to Eq. (17),

$$A_{\vec{p}}\left(\vec{k};\hat{\epsilon}\cdot\frac{(\overleftarrow{\nabla}-\overrightarrow{\nabla})}{2M}\right) \simeq i\,\frac{\hat{\epsilon}\cdot\hat{p}}{p}\,\frac{p^2+k^2+\kappa^2}{(\vec{p}-\vec{k})^2+\kappa^2} \times \int d^3x\,e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{p}}^{(-)*}(\vec{x})V(\vec{x})\psi(\vec{x}). \quad (20)$$

On the other hand, since the operator $\vec{\sigma}\cdot\vec{k}\times\hat{\epsilon}$ does not mix orbital angular momenta, we may expect some cancellation in the magnetic photoabsorption term, resulting from radial wave-function orthogonality. We find, by means of the same manipulations as employed previously, that

$$A_{\vec{p}}\left(\vec{k};\mu_p\frac{\vec{\sigma}\cdot\vec{k}\times\hat{\epsilon}}{2M}\right) \simeq -\frac{\mu_p}{(\vec{p}-\vec{k})^2+\kappa^2}\int d^3x\,e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{p}}^{(-)*}(\vec{x})\left(\frac{\vec{k}\cdot\hat{p}}{p}V(\vec{x})\vec{\sigma}\cdot\vec{k}\times\hat{\epsilon}-[V(\vec{x}),\vec{\sigma}\cdot\vec{k}\times\hat{\epsilon}]\right)\psi(\vec{x}). \quad (21)$$

Equation (21) makes clear that a considerable cancellation takes place, compared with the plane-wave approximation. The ratio of magnetic to convective (γ,p) cross sections, treating Eq. (19) as though $\phi_{\vec{p}}^{(-)}$ may be replaced with a plane wave, is

$$\frac{d\sigma^M}{d\sigma^E} \simeq \frac{(\mu_p k/p)^2}{\sin^2\theta}. \quad (22)$$

On the other hand, when the orthogonality of the wave functions is taken into account, the ratio (22) is reduced by factors such as $(k/p)^2$ or $(V_{s.o.}/V_c)^2$, where $V_{s.o.}$ is the strength of the nucleon spin-orbit potential and V_c is the strength of the central potential. The net result is to reestablish the convective term as the dominant contribution to (γ,p) at medium energies, particularly in the 1s or 1p shell nuclei, in agreement both with experiment[7] and with the calculations of Fink *et al.*[4]

III. SINGLE-PARTICLE STATES SATISFYING THE DIRAC EQUATION

A. Scalar probe

We suppose the bound and continuum states to be eigenstates of the Dirac Hamiltonian

$$H=-i\vec{\alpha}\cdot\nabla+\beta[M+U(r)]+V(r), \quad (23)$$

where we have assumed a central potential which is of mixed character under Lorentz transformation: $\beta U(r)$ is a scalar, whereas $V(r)$ is the fourth component of a 4 vector. Then we shall be interested in amplitudes of the form

$$B_{\vec{p}}(\vec{k};\mathcal{O}) = \int d^3x\,e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{p}}^{(-)*}(\vec{x})\mathcal{O}\psi(\vec{x}), \quad (24)$$

where $\mathcal{O}$ is a 4×4 matrix and ψ and ϕ are Dirac spinors. The amplitude $B_{\vec{p}}(\vec{k};1)$ is obtained by noting that $(H\psi=E_0\psi)$

$$-i\nabla\cdot\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\vec{\alpha}\psi(\vec{x}) = (E_p-E_0)\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\psi(\vec{x}), \quad (25)$$

so that

$$B_{\vec{p}}(\vec{k};1) = (E_p-E_0)^{-1}\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\vec{k}\cdot\vec{\alpha}\psi(\vec{x})\,e^{i\vec{k}\cdot\vec{x}}. \quad (26)$$

Clearly, $B_{\vec{p}}(0;1)$ vanishes identically. To proceed further, we must attempt to eliminate the off-diagonal matrix $\vec{k}\cdot\vec{\alpha}$. Writing

$$i\nabla\phi_{\vec{p}}^{(-)\dagger}\cdot\vec{\alpha}\vec{\alpha}\cdot\vec{k}+\phi_{\vec{p}}^{(-)\dagger}[\beta(M+U)+V-E_p]\vec{\alpha}\cdot\vec{k}=0 \quad (27)$$

and

$$-i\vec{\alpha}\cdot\vec{k}\vec{\alpha}\cdot\nabla\psi+\vec{\alpha}\cdot\vec{k}[\beta(M+U)+V-E_0]\psi=0, \tag{28}$$

we find

$$(E_p+E_0-2V)\phi_{\vec{p}}^{(-)\dagger}\vec{\alpha}\cdot\vec{k}\psi=i\phi_{\vec{p}}^{(-)\dagger}\vec{k}\cdot(\overleftarrow{\nabla}-\overrightarrow{\nabla})\psi-\nabla\cdot[\phi_{\vec{p}}^{(-)\dagger}\vec{k}\times\vec{\sigma}\psi], \tag{29}$$

so that

$$\int d^3x\, e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\vec{\alpha}\cdot\vec{k}\psi(\vec{x})=i\int d^3x\left[\frac{e^{i\vec{k}\cdot\vec{x}}}{E_p+E_0-2V(\vec{x})}\right.$$
$$\left.\times\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\vec{k}\cdot(\overleftarrow{\nabla}-\overrightarrow{\nabla})\psi(\vec{x})+\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\vec{k}\times\vec{\sigma}\psi(\vec{x})\cdot\nabla\left(\frac{e^{i\vec{k}\cdot\vec{x}}}{E_p+E_0-2V(\vec{x})}\right)\right]. \tag{30}$$

Substituting Eq. (30) into Eq. (26), we see immediately how $B_{\vec{p}}(\vec{k};1)$ reduces to $A_{\vec{p}}(\vec{k};1)$ upon nonrelativistic reduction.

B. Photoabsorption

The relativistic photoabsorption amplitude is proportional to $B_{\vec{p}}(\vec{k};\vec{\alpha}\cdot\hat{\epsilon})$ (in the absence of anomalous magnetic moments), which we see from Eqs. (28)–(30), is given by

$$B_{\vec{p}}(\vec{k};\vec{\alpha}\cdot\hat{\epsilon})=i\int d^3x\frac{e^{i\vec{k}\cdot\vec{x}}}{E_p+E_0-2V(\vec{x})}\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\hat{\epsilon}\cdot(\overleftarrow{\nabla}-\overrightarrow{\nabla})\psi(\vec{x})+\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\hat{\epsilon}\times\vec{\sigma}\psi(\vec{x})\cdot\nabla\left(\frac{e^{i\vec{k}\cdot\vec{x}}}{E_p+E_0-2V(\vec{x})}\right). \tag{31}$$

In the nonrelativistic limit, $E_p+E_0\to 2M, M\gg|V|$, Eq. (31) reduces to Eq. (19) except for the anomalous magnetic moment of the proton, which must be put in by hand. (See, e.g., Friar.[8])

C. Pion absorption (π^+,p) or (π^0,p)

The two kinds of πNN coupling in common use are related by the equivalence theorem,[6,9] up to a point: Let us compare pseudoscalar (PS) and pseudovector (PV) coupling, for which the amplitudes (assuming the nuclear potential lacks a symmetry term) are proportional to

$$\text{PS:}\ G\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\beta\gamma^5\psi(\vec{x})\chi_{\vec{k}}^{(+)}(\vec{x}) \tag{32}$$

and

$$\text{PV:}\ \frac{G}{2M}\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\beta\gamma^5\beta[-i\vec{\alpha}\cdot\nabla_r-\omega(k)]\psi(\vec{x})\chi_{\vec{k}}^{(+)}(\vec{x}), \tag{33}$$

where $\chi_{\vec{k}}^{(+)}(\vec{x})$ is the coordinate-space pion wave function. Using the fact that energy conservation requires

$$\omega(k)+E_0=E_p, \tag{34}$$

we find, upon repeated application of the Dirac equations for $\phi_{\vec{p}}^{(-)}$ and ψ, that the amplitudes may be cast in the forms

$$\text{PS:}\ iG\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\beta\vec{\sigma}\psi(\vec{x})\cdot\nabla\left(\frac{\chi_{\vec{k}}^{(+)}(\vec{x})}{E_p+E_0-2V(\vec{x})}\right), \tag{35}$$

$$\text{PV:}\ iG\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\times\beta\vec{\sigma}\psi(\vec{x})\cdot\nabla\left(\frac{(1+U(\vec{x})/M)\chi_{\vec{k}}^{(+)}(\vec{x})}{E_p+E_0-2V(\vec{x})}\right). \tag{36}$$

Equations (35) and (36) reproduce Friar's result.[6]

It is perhaps worth remarking here that it is unreasonable to approximate the pion wave function $\chi_{\vec{k}}^{(+)}(\vec{x})$ by a plane wave, even near threshold, despite the fact that the low-energy on-shell pion-nucleus cross section is miniscule. The reason is that the amplitudes (35) or (36) are so sensitive to the off-shell behavior of the pion-nucleus scattering amplitude that even far from the 3-3 resonance this is the dominant part of the amplitude.[10] However, supposing that one could replace $\chi_{\vec{k}}^{(+)}(\vec{x})$ by $e^{i\vec{k}\cdot\vec{x}}$, we see that, were it possible to neglect the potentials U and V in comparison to the mass, both (35) and (36) would vanish as $k\to 0$, as a consequence of wave-function orthogonality. With realistic potentials, as described in the next section, there can be substantial differences between PS and PV, although probably not as extreme as those found by Miller and Weber[5] in evaluating (32) and (33) directly with $\phi_{\vec{p}}^{+}$ replaced by a Dirac plane wave (orthogonalized to ψ by projection).

Finally, it should be noted that, to the extent the "small" Dirac components can be neglected, a matrix element of the form

$$\int d^3x\,\phi_{\vec{p}}^{(-)\dagger}(\vec{x})\beta\vec{\sigma}\psi(\vec{x})\cdot\vec{W}(\vec{x}), \tag{37}$$

will undergo still more cancellations, and is expected to be of order k/p or $(V_{\text{s.o.}}/V_c)$, exactly as we found for Eq. (21).

D. Estimating the relativistic corrections

In order to estimate the differences between, say, Eq. (26) and Eq. (14), or between Eq. (31) and

Eq. (19), we need a model of the potentials $U(x)$ and $V(x)$. The nonrelativistic reduction of the bound-state Dirac equation, via the elimination of the small components of the wave function, leads to the two-component spinor equation ($E_0 = M - B$)

$$[B + U(\vec{x}) + V(\vec{x})]u(\vec{x}) - \vec{\sigma}\cdot\nabla[2M - B + U(\vec{x}) - V(\vec{x})]^{-1}\vec{\sigma}\cdot\nabla u(\vec{x}) = 0, \qquad (38)$$

and so, clearly, $U(\vec{x}) + V(\vec{x})$ must have something like the shape of a Woods-Saxon potential of depth ~50 MeV, if they are to reproduce the nonrelativistic phenomenology. Letting

$$U(\vec{x}) = U_0 f(r),$$
$$V(\vec{x}) = V_0 f(r), \qquad (39)$$
$$f(r) = \{1 + \exp[(r - R)/a]\}^{-1},$$

we see $(U_0 + V_0)[1 + (U_0 - V_0)/2M] \simeq -53$ MeV. On the other hand, we see that Eq. (38) also contains a spin-orbit interaction of the approximate form

$$\frac{-(U_0 - V_0)}{2M^2[1 + (U_0 - V_0)/4M]}\,\vec{s}\cdot\vec{l}\,\frac{1}{r}\,\frac{1}{dr}f(r);$$

so from the empirical low-energy nucleon-nucleus spin orbit interaction strength,[11] we have

$$\frac{U_0 - V_0}{1 + (U_0 - V_0)/4M} \simeq -600 \text{ MeV}, \qquad (40)$$

so that $U_0 \simeq -295$ MeV and $V_0 \simeq 222$ MeV. If these values are applied to scattering, we find the effective potential in a Schrödinger equation for the Pauli spinor wave function to have a well depth

$$V_{eff}(r = 0) = U_0 + V_0 + \frac{U_0^2 - V_0^2}{2M} + \frac{E_p - M}{M}V_0 = -53 + 0.24(E_p - M) \text{ MeV}. \qquad (41)$$

Equation (41) agrees rather well with the central part of the real potential found in optical model fits to nucleon-nucleus elastic scattering at energies up to 150 MeV[11]; several authors[12] have pointed out that this is not fortuitous, so we probably ought to regard this Dirac equation model for single-nuclear wave functions as not completely foolish.

Let us now inquire how the presence of $V(\vec{x})$ in Eq. (30) introduces corrections to the nonrelativistic form, Eq. (14). We concentrate on the gradient terms, and ignore the small components, integrating by parts to obtain, to lowest order in $V(\vec{x})/M$,

$$B_{\vec{p}}(\vec{k};1) \simeq [(\vec{p}-\vec{k})^2 + \kappa^2]^{-1} i\vec{k}\cdot\int d^3x \left(1 + \frac{V(\vec{x})}{M}\right) e^{i\vec{k}\cdot\vec{x}}[2(\nabla + i\vec{p})\phi_{\vec{p}}^{(-)*}(\vec{x})]\psi(\vec{x})$$
$$- k^2[(\vec{p}-\vec{k})^2 + \kappa^2]^{-1}\int d^3x\, e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{p}}^{(-)*}(\vec{x})\frac{V(\vec{x})}{M}\psi(\vec{x}) + [(\vec{p}-\vec{k})^2 + \kappa^2]^{-1} i\vec{k}\cdot\int d^3x\, e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{p}}^{(-)*}(\vec{x})\frac{\nabla V(\vec{x})}{M}\psi(\vec{x}). \qquad (42)$$

Of the corrections in Eq. (42), the dominant one is evidently that involving $\vec{k}\cdot\nabla V(\vec{x})/M$. Since, for a central potential, $\nabla V(\vec{x})$ mixes partial waves just as $\overleftarrow{\nabla} - \vec{\nabla}$ does, we do no violence to orthogonality effects *per se* by comparing this correction with the "leading term" in plane-wave approximation. Thus, following the local WKB approximation and then setting $\phi_{\vec{p}}^{(-)*}(\vec{r}) = \simeq (e^{-i\vec{p}\cdot\vec{r}}/(2\pi)^{3/2})$, we find

$$B_{\vec{p}}(\vec{k};1) \simeq \frac{\vec{k}\cdot\hat{p}}{p}\tilde{\psi}(\vec{p}-\vec{k})\left(\frac{U_0 + E_p V_0/M}{U_0 + V_0}\right) + \frac{k^2}{2M^2}\tilde{\psi}(\vec{p}-\vec{k})\frac{V_0}{U_0 + V_0} + i(2\pi)^{-3/2}[(\vec{p}-\vec{k})^2 + \kappa^2]^{-1}M^{-1}\int d^3x\,\vec{k}\cdot\hat{x}e^{i(\vec{k}-\vec{p})\cdot\vec{x}}V'(|\vec{x}|)\psi(\vec{x}). \qquad (43)$$

It is easy to see that the first and third terms of Eq. (43) are indeed the dominant ones. At energies typical of photoabsorption, electrodisintegration, or meson absorption, i.e., $k \sim 100$ MeV/c, $p \sim 430$ MeV/c, we find the ratio of the first and third terms to be approximately

$$2i(1 + V_0/M)(U_0/V_0 + E_p/M)\frac{M^2}{p|\vec{p}-\vec{k}|}\frac{R}{[R^2 + (\pi a)^2]^{1/2}} \times \cot(|\vec{p}-\vec{k}|R - \tfrac{1}{2}l\pi)[\cos(|\vec{p}-\vec{k}|R + \cos^{-1}\{R/[R^2 + (\pi a)^2]^{1/2}\})]^{-1}, \qquad (44)$$

where the potential was assumed to have a Woods-Saxon shape with radius R and surface thickness a, as in Eq. (39). The terms tend to be out of phase by a factor i, so their squares must be added to get the differential cross section. Moreover, their angular oscillations tend to be out of phase, as is clear from the trigonometric factors in Eq. (44), so that the smaller will fill in the minima of the larger. Disregarding the (oscillatory) trigonometric factors in (44), we see that the smaller (third) term is about 10–20% (in strength) of the larger (first) term. The third term gradually becomes more important with increasing energy $\omega(k)$, becoming dominant for $\omega \sim 200$ MeV.

Next, we estimate the effect of dropping the small Dirac wave-function components within the framework of the model defined by Eqs. (23) and (39). We see that a small component $\psi_S(\vec{x})$ is related to its corresponding "large" component $\psi_L(\vec{x})$ through the Dirac equation, which gives

$$\psi_S(\vec{x}) = -i[E + M + U(\vec{x}) - V(\vec{x})]^{-1}\vec{\sigma}\cdot\nabla\psi_L(\vec{x}). \quad (45)$$

The first term of Eq. (42) would then contain an additional factor of approximately

$$1 + \frac{p^2 - 2MU_0 - 2E_p V_0}{(E_p + M + U_0 - V_0)(2M - B + U_0 - V_0)}, \quad (46)$$

in addition to terms of order k^2 and of order $\nabla[U(x) - V(x)]$, analogous to the second and third terms in Eq. (42). The additional factor (46) is roughly 1.14, for the case we have been considering [$k \sim 100$ MeV/$c = \omega(k)$, $p \sim 430$ MeV/c].

Although the above analysis for $B_{\vec{p}}(\vec{k};1)$ pertains to inelastic longitudinal electron scattering form factors, it is evident from Eq. (31) that a similar analysis will hold for the convective term in photoabsorption, with the difference that $(\vec{k}\cdot\hat{p})p$ is replaced by $\hat{\epsilon}\cdot\vec{p}$ in the first term of (43), the second term vanishes because of the orthogonality of $\vec{k}$ and $\hat{\epsilon}$, and in the third term $(\vec{k}\cdot\hat{x})/[(\vec{p}-\vec{k})^2+\kappa^2]$ is replaced by $\hat{\epsilon}\cdot\hat{x}$. The ratio of the first and third terms will still be given by (44). The preceding remarks about the relative effect of these terms, and particularly its energy dependence, are especially relevant to the recent analysis of $^{16}O(\gamma,p)$ data in the energy range from 60 to 200 MeV, by Findley and Owens,[13] and Matthews *et al.*[14]

IV. EFFECTS OF INELASTICITY

In order to investigate the effects of inelasticity without becoming lost in complexities, let us consider a single nucleon moving nonrelativistically in a potential which has intrinsic degrees of freedom. The Hamiltonian is

$$H = H_0 + h + v, \quad (47)$$

where H_0 is the Hamiltonian of the intrinsic states

$$H_0|n\rangle = \epsilon_n|n\rangle, \quad (48)$$

h is the single-nucleon Hamiltonian

$$h = -\frac{\nabla^2}{2M} + V(r), \quad (49)$$

and v is the channel-coupling potential. If we express the initial and final nuclear states as linear combinations of intrinsic states,

$$\langle\vec{r}|\Psi_i\rangle = \sum_n \psi_n(\vec{r})|n\rangle, \quad (50)$$

$$\langle\Psi_f|\vec{r}\rangle = \sum_n \langle n|\phi_n^{(-)*}(\vec{r}), \quad (51)$$

then we expect the amplitude to absorb a scalar particle of momentum $\vec{k}$ to be of the form

$$T_{fi}(\vec{k}) = g\Big(\sum_n \int d^3r\,\phi_n^{(-)*}(\vec{r})e^{i\vec{k}\cdot\vec{r}}\psi_n(\vec{r}) + \lambda\sum_{n'}\sum_n \int d^3r\,\phi_{n'}^{(-)*}(\vec{r})\psi_n(\vec{r})F_{n'n}(\vec{k})\Big). \quad (52)$$

The second term of Eq. (52) represents the possibility that the scalar particle can be absorbed by the potential, perhaps accompanied by a transition from state n to state n'. We suppose the probability amplitude for this to happen is given by $\lambda F_{n'n}(\vec{k})$, where the transition form factor $F_{n'n}(\vec{k})$ has the not unnatural property

$$F_{n'n}(\vec{k}=0) = \delta_{n'n}, \quad (53)$$

and is an appropriately continuous function of $\vec{k}$.

The dynamical equations for $\psi_n(\vec{r})$ and $\phi_n^{(-)*}(\vec{r})$ are [we assume $v(r)$ has only off-diagonal matrix elements]

$$\Big(-\frac{\nabla^2}{2M} + V(r)\Big)\psi_n(\vec{r}) + \sum_{m\neq n}\langle n|v(\vec{r})|m\rangle\psi_m(\vec{r}) = (E_i - \epsilon_n)\psi_n(\vec{r}), \quad (54)$$

$$\Big(-\frac{\nabla^2}{2M} + V(r)\Big)\phi_n^{(-)*}(\vec{r}) + \sum_{m\neq n}\phi_m^{(-)*}(\vec{r})\langle m|v(\vec{r})|n\rangle = (E_f - \epsilon_n)\phi_n^{(-)*}(\vec{r}). \quad (55)$$

By virtue of Eqs. (54) and (55), we find

$$(E_f - E_i)\phi_n^{(-)*}(\vec{r})\psi_n(\vec{r}) = -\frac{1}{2M}\nabla\cdot[\psi_n(\vec{r})\nabla\phi_n^{(-)*}(\vec{r}) - \phi_n^{(-)*}(\vec{r})\nabla\psi_n(\vec{r})] + \sum_{m\neq n}[\phi_m^{(-)*}(\vec{r})\psi_n(\vec{r})\langle m|v(\vec{r})|n\rangle - \phi_n^{(-)*}(\vec{r})\psi_m(\vec{r})\langle n|v(\vec{r})|m\rangle]. \quad (56)$$

When we sum over n and integrate, the potential term drops out, leaving the volume integral of a divergence, which vanishes, as before. Thus,

$$\langle\Psi_f|\Psi_i\rangle \equiv \sum_n \int d^3r\,\phi_n^{(-)*}(\vec{r})\psi_n(\vec{r}) = 0\,. \tag{57}$$

A similar relation to Eq. (56), for $\phi_{n'}^{(-)*}(\vec{r})\psi_n(\vec{r})$, is easily found by similar manipulations. By means of (56) and its equivalent with $n' \neq n$, we may show that

$$T_{fi}(\vec{k}) = g\bigg(\frac{i}{2M(E_f-E_i)}\sum_n \int d^3r\,e^{i\vec{k}\cdot\vec{r}}\phi_n^{(-)*}(\vec{r})\vec{k}\cdot(\overleftarrow{\nabla}-\overrightarrow{\nabla})\psi_n(r)$$
$$+\frac{\lambda}{E_f-E_i}\sum_{n'}\sum_n \int d^3r\,\phi_{n'}^{(-)*}(\vec{r})\psi_n(\vec{r})\langle n'|[h+v(\vec{r}),F(\vec{k})]|n\rangle\bigg). \tag{58}$$

We see that Eq. (58) explicitly vanishes at $\vec{k}=0$, precisely as we should have expected from Eq. (52), (53), and (57). Exactly as in Eq. (14), however, the cancellations which are implied by wave-function orthogonality have taken place, so that subsequent approximations will not do violence to them. Although Eq. (58) is in principle exact to all orders in v, we can identify various terms with the leading diagrams in perturbation theory, according to their behavior in the limit of small v: The part of the first term in (58) with $n=0$ corresponds to the lowest-order diagram, Fig. 1(a). The terms in the sum with $n\neq 0$ are a vertex renormalization, Fig. 1(b).

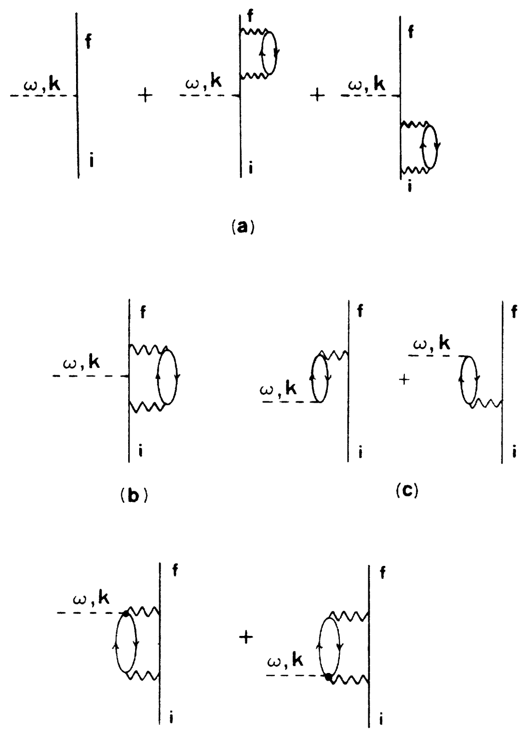

FIG. 1. (a) Lowest order graphs for absorbing a spin-0 particle of energy ω and momentum $\vec{k}$, including the effects of initial-state correlations as well as final state rescattering (with inelasticity). (b) Dispersive correction to the absorption vertex, required for consistency with initial-state correlations and final-state rescattering effects. (c) Correction to the absorption amplitude, which is present if the excitable "core" can interact with the external "probe" independently of the valence nucleon. (d) Possible additional corrections to (c), which could result, e.g., in electrodissociation $(e, e'p)$, if the p-core interaction is charge dependent.

The term involving $[h, F(\vec{k})]$ corresponds, in leading order, to Fig. 1(c), which can be regarded as another sort of vertex correction. Finally, if the commutator $[v(\vec{r}), F(\vec{k})]$ does not vanish, the latter term of (58) gives rise to diagrams like Fig. 1(d). To give a concrete example of a circumstance in which this commutator will not vanish, let us consider longitudinal electroproduction, with the nucleus regarded as a proton outside an excitable "core" of charge Z. The core form factor is given by

$$\lambda F(\vec{k}) = \tfrac{1}{2}Z\int d^3r\,e^{i\vec{k}\cdot\vec{r}}[\rho(\vec{r})+T_3(\vec{r})]\,, \tag{59}$$

where the operators $\rho(\vec{r})$ and $T_3(\vec{r})$ satisfy commutation relations

$$[T_i(\vec{r}),\rho(\vec{r}\,')] = 0\,,\quad i=1,2,3\,,$$
$$[T_i(\vec{r}),T_j(\vec{r}\,')] = \delta(\vec{r}-\vec{r}\,')\epsilon_{ijk}T_k(\vec{r})\,. \tag{60}$$

The interaction $v(\vec{r})$ will in general have a charge-dependent part

$$v(\vec{r}) = v_0[\rho(\vec{r}) - \langle 0|\rho(\vec{r})|0\rangle] + v_1 T(\vec{r})\cdot\tau\,, \tag{61}$$

which will not commute with the isovector part of $\lambda F(\vec{k})$. If we regard the operators $\rho(\vec{r})$ and $T_i(\vec{r})$ as particle-hole operators in the "core" subspace, we see that they excite both collective states as well as continuum states and in this way generate a quite realistic spectrum. The commutator then has the form of a local operator in the valence-particle space, which is at the same time a particle-hole operator in the core space:

$$\lambda[v(\vec{r}), F(\vec{k})] = \tfrac{1}{2} v_1 e^{i\vec{k}\cdot\vec{r}}\{(\tau_1 + i\tau_2)[T_1(\vec{r}) - iT_2(\vec{r})] - (\tau_1 - i\tau_2)[T_1(\vec{r}) + iT_2(\vec{r})]\}, \tag{62}$$

hence the form of Fig. 1(d).

V. SUMMARY AND CONCLUSIONS

By judicious application of the divergence theorem, I have derived a new method for expressing certain reaction amplitudes which are of particular interest in medium energy nuclear physics. The virtue of the new formulation is that it takes into account the orthogonality of initial and final nuclear states *ab initio*, and, therefore, represents in many cases a better starting point for approximations than the schemes presently in use. I have illustrated the use of the technique for nonrelativistic and relativistic one-body problem, as well as in a simple model of a many-body system. Clearly the latter represents a fertile area for further investigation, and one which is absolutely essential to the formulation of a theory which consistently includes absorptive mechanisms in the nuclear dynamics. Certainly, it is not possible to draw completely definitive conclusions about pion or photon absorption without including absorptive effects, although I do not expect the qualitative conclusions drawn herein to be greatly altered thereby.

In the process of illustrating the new method, I have rederived some familiar results, e.g., the equivalence theorem and the suppression of the magnetic term, relative to the convective term, in (γ, p). I have also derived some new and potentially interesting results, particularly with reference to relativistic corrections in the single-body model of (γ, p), which should be investigated further because of their application to the question of the necessity for, e.g., two-step intermediate-Δ mechanisms to explain the higher energy (γ, p) data of Matthews, *et al.*[14]

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