Black sphere model for the linewidths of P-state protonium

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The annihilation widths of nP $\bar{p}p$ bound states (protonium) are calculated in the Wentzel-Kramers-Brillouin approximation for a totally absorbing sphere, including the coupling to $\bar{n}n$ states. It is shown that the annihilation occurs preferentially in one of the two possible isospin eigenstates, and that on the average it is 100 times stronger than the radiative $2P$-$1S$ transition. A simple formula for the n dependence of the annihilation widths is also given.

[NUCLEAR REACTIONS. Protonium $\bar{p}p$, calculated p-state Γ.]

Recently we presented a black sphere model for the linewidths of kaonic and antiprotonic atoms,[1] in which the widths are calculated in the WKB approximation from the barrier penetration factor

$$\Gamma = n^{-3} m\alpha^2 (2\pi)^{-1} e^{-2I}, \qquad (1)$$
$$I = \int_R^{r_1} \kappa(r)dr\,, \quad \kappa = (2mV_{\text{eff}} - 2mE)^{1/2}\,.$$

The effective potential V_{eff} included only the Coulomb and centrifugal potentials, and the integration in I started from an "absorption radius" R which was kept as a free parameter, with a given dependence on the nucleon number (the integration end point r_1 is the inner turning point of the Bohr orbit). The main point of our present paper is that for the P states of protonium, R can be calculated from our knowledge of the long-range part of the nucleon-antinucleon potential. With the $\bar{p}p$ and $\bar{n}n$ channels properly coupled and tensor forces included, it turns out that the total effective potential becomes attractive in all P-wave protonium states at radii of about 1 fm. In the WKB approximation, Γ is then given by (1), where $-i\kappa$ is the local wave number in the presence of the full potentials, and R is defined by $\kappa(R) = 0$. The imaginary part of the $\bar{p}p$ optical potential is believed to be negligible at distances larger than 1 fm. It therefore enters the WKB approximation only indirectly in the sense that it eventually absorbs the wave which has penetrated the potential barrier.

The use of the WKB approximation together with an ingoing-wave boundary condition around 0.5 fm has been previously advocated by Ball and Chew.[2] More recent calculations have dropped the WKB approximation but kept the boundary condition[3] which means that the approximation is still needed locally around 0.5 fm, in order to define the meaning of an ingoing wave in the presence of a potential. These authors considered only isospin eigenstates, some of which have repulsive effective potentials down to 0.5 fm. For such states, the absorption rates depend critically on the boundary radius, and the model is inconsistent with a smoothly rising ImV. For our P-wave protonium states, on the other hand, this situation does not occur when the Coulomb potential and the coupling to the closed $\bar{n}n$ channels are properly included. It is then advisable to keep the WKB approximation for the following two reasons:

Firstly, the dependence on the annihilation boundary disappears completely. In fact, $V(r)$ is needed only for $r > R$, where R is 1–2 fm (see Table I). Of course one cannot exclude the possibility that reflection from the region $r < R$ will somewhat weaken the absorption. A measure of this reflection is given by the real part of the complex energy shift. For other antiprotonic atoms, there are indications that this is a minor effect.[4] But in principle, this could be different for protonium.

Secondly, in the WKB approximation the coupled-channel problem is reduced to equivalent one-channel problems by a simple diagonalization of the effective potential. The approximation here is that the diagonalizing matrix $U(r)$ in a given angular momentum state commutes with the kinetic energy operator. For the tensor force, the diagonalization has been given by Christian and Hart,[5] but for $\bar{p}p$ atoms we must also include the coupling to the $\bar{n}n$ states. In the $L = J \pm 1$ states, we thus get four coupled channels. It was just the proper coupling between $\bar{p}p$ and $\bar{n}n$ states that went wrong in a similar paper by Desai.[6] As a result, Desai got

TABLE I. The calculated annihilation widths of the $2P$ protonium states for various values of the poorly known coupling constant combination G_0^2 of Eq. (6). The most likely values of $G_0^2/4\pi$ are 35 or 25; the other values are only given for illustration. Also given is the inner turning point R of the effective potential V_{eff} in the WKB approximation, and the dominant isospin component in the annihilation.

	R (fm)				Γ (eV)			
$G_0^2/4\pi$	$^1P_1$	$^3P_1$	$^3P_0$	$^3P_2$	$^1P_1$	$^3P_1$	$^3P_0$	$^3P_2$
35	1.24	1.34	2.39	1.53	0.029	0.032	0.194	0.045
25	1.13	1.26	2.39	1.48	0.024	0.028	0.193	0.042
15	0.95	1.13	2.38	1.43	0.016	0.022	0.193	0.039
5	0.45	0.90	2.37	1.36	0.004	0.014	0.192	0.035
I_{dom}					0	1	0	0

two different decay rates (one for each isospin) for each $\bar{p}p$ state $^{2S+1}L_J$ of given angular momentum L, spin S, and total angular momentum J, and in the end he simply took the average of these two numbers.

The proper treatment of protonium states must start from the coupled channel Schrödinger equation $H\psi = E\psi$ in the space of $\bar{p}p$ and $\bar{n}n$ states

$$\psi \equiv \begin{Bmatrix} \psi(\bar{p}p) \\ \psi(\bar{n}n) \end{Bmatrix};$$

$$H = \begin{Bmatrix} p^2/2m + V_C + V_{\bar{p}} & V_{\bar{p}\bar{n}} \\ V_{\bar{p}\bar{n}} & p^2/2m + 2\delta m + V_{\bar{n}} \end{Bmatrix}, \quad (2)$$

where $m = \frac{1}{2}m_p$, $\delta m = m_n - m_p$, $V_C = -\alpha/r$, and $V_{\bar{p}} = V_{\bar{n}} \equiv V_0$. V_C is the Coulomb potential and V_0 the diagonal part of the hadronic potential. After partial-wave decomposition, diagonalization of H in the WKB approximation gives rise to simple "eigenpotentials." For $J = L$ or $J = 0$, these are

$$V_\pm = V_0 + \tfrac{1}{2}V_C + \delta m \pm [(\delta m - \tfrac{1}{2}V_C)^2 + V_{\bar{p}\bar{n}}{}^2]^{1/2}. \quad (3)$$

The long-range part of the nondiagonal potential $V_{\bar{p}\bar{n}}$ is due to the exchange of charged pions $V_{\bar{p}\bar{n}} = 2V_\pi$:

$$V_\pi = -\tfrac{1}{3}f^2\left[\vec{\sigma}_1\cdot\vec{\sigma}_2 + \left(1 + \frac{3}{x} + \frac{3}{x^2}\right)S_{12}\right]e^{-x}x^{-1}, \quad (4)$$

$$x \equiv m_\pi r, \quad f^2 = 0.079,$$

where S_{12} is the usual tensor operator. In the case of (3), V_π is small compared with the isospin-breaking part $\delta m - \frac{1}{2}V_C$; we can expand the square root, getting

$$V_-(x \gg 1) = V_0 + V_C - 2V_\pi{}^2(\delta m - \tfrac{1}{2}V_C)^{-1},$$
$$V_+(x \gg 1) = V_0 + 2\delta m + 2V_\pi{}^2(\delta m - \tfrac{1}{2}V_C)^{-1}. \quad (5)$$

We see that V_- reduces to the Coulomb potential for $r \to \infty$. Therefore, if our initial state is a Coulomb bound state, V_- is the appropriate potential. Note also that we always have $V_+ > V_-$.

With $H_-\psi = E\psi$, the isospin content of ψ is a function of r. At distances where the exchange of charged pions is negligible, ψ is $2^{-1/2}(|I=1\rangle + |I=0\rangle)$. As r is decreased into the region of charged pion exchange, ψ rotates towards one of the two isospin eigenstates. When the inner turning point is reached, V_C and δm are already negligible with respect to V_π, and from here on ψ is practically a pure isospin eigenstate. From our remarks following Eq. (5) it follows that ψ has the isospin of the most attractive potential in this region. It turns out to be $I=0$ for the states $^1P_1$, $^3P_0$ and $I=1$ for $^3P_1$. We thus get "dynamical" isospin selection rules for the annihilation channels, which are valid only within the accuracy of the WKB approximation.

For the spin-triplet states with $L = J \pm 1$, one has to diagonalize a 4×4 effective potential matrix. Since pion exchange is responsible both for the tensor force and for the charge exchange, all four degrees of freedom come into play in the same r region, and the diagonalization must be made numerically. The four eigenpotentials never cross each other (see Fig. 1). The eigenpotential which contains the P-wave atomic $\bar{p}p$ bound state is the most attractive one and turns into the $I=0$ potential for small r (this shows that our isospin selection rule is not restricted by the number of coupled channels). In Fig. 2 the wave function for this eigenpotential is decomposed into amplitudes of given orbital angular momentum and isospin.

In the actual calculation, we include ω and σ_0 exchange in V_0, with a common mass m_ω, in addition to π exchange:

$$V_0 = -(4\pi r)^{-1}G_0{}^2e^{-m_\omega r} + V_\pi, \quad G_0{}^2 \equiv G_\omega{}^2 + G_\sigma{}^2. \quad (6)$$

Since Γ is rather insensitive to G_0 within reasonable limits ($G_0^2/4\pi \approx 20$–40), there is little point in improving V_0 at the present stage. There are also some long-range effects such as the atomic fine structure and the pion mass difference which will slightly change Γ.

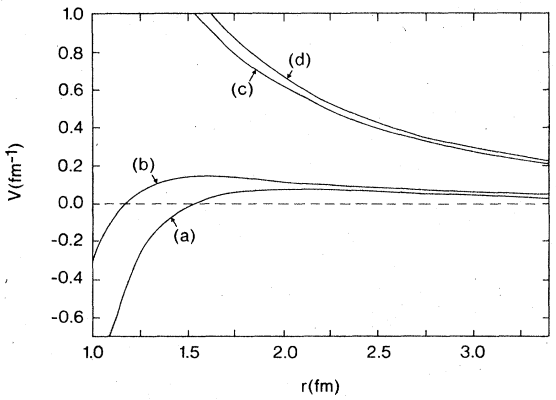

FIG. 1. Eigenvalues of the effective potential matrix V for the $^3P_2$-$^3F_2$ channel as a function of r. The eigenpotentials may be characterized by their behavior at infinity: (a) $\rightarrow \bar{p}p$, p wave; (b) $\rightarrow \bar{n}n$, p wave; (c) $\rightarrow \bar{n}n$, f wave; (d) $\rightarrow \bar{n}n$, f wave. In the neighborhood of the turning points, (a) and (c) correspond to isospin zero states and (b) and (d) to isospin one states.

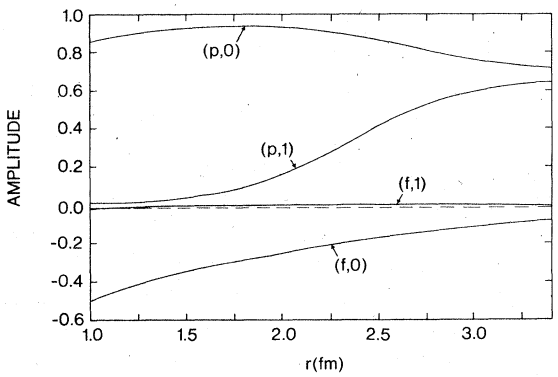

FIG. 2. Decomposition of the eigenstates of the potential $V(a)$ of Fig. 1 into states (L, I) of orbital angular momentum L and isospin I. For small values of r, the state $(p, 0)$ dominates, although a substantial amount of $(f, 0)$ is present. Annihilation from small values of r will preferentially lead to isospin zero final states.

We do not give any widths for the 1S states because the WKB approximation can hardly be trusted in this case. The 2P widths are collected in Table I. For comparison, the radiative 2P-1S width is 3.7×10^{-4} eV. It is likely that all 2P hyperfine states are statistically populated, in which case the average probability of radiative deexcitation of the 2P states is 1.1%. If reflection from the region beyond the inner turning point would decrease the annihilation widths uniformly, by as much as a factor $\frac{1}{2}$, then the radiative deexcitation probability would reach 2.2%. In contrast, the 3D annihilation width is very small compared with the 3D-2P radiative width. Thus, in the absence of Stark mixing, the annihilation is dominantly from the 2P level.

Since Desai's average 2P annihilation widths are of the same order of magnitude as ours, his estimates of Stark mixing and predicted dominance of S-wave capture in liquid hydrogen are hardly affected by our results. One should note that the $^3P_0$ width is almost an order of magnitude larger than the other P-state widths.

For fixed L, the dependence of Γ on the principal quantum number n is obtained as follows: The integral in (1) can be split into two pieces $I = I_1 + I_2$,

$$I_2 = \int_R^{R_c} \kappa(r)dr, \quad I_1 = \int_{R_c}^{r_1} \kappa(r)dr, \tag{7}$$

where R_c is large enough that the hadronic potential can be neglected in I_1 and small enough that the energy E can be neglected in I_2 (this corresponds to putting $n=\infty$ in I_2). The n dependence follows then from the evaluation of I_1 (Ref. 1):

$$I_1 = \lambda \ln(Y + \lambda - y/\lambda) + n \ln(2 - y/n - Y) - Y - \tfrac{1}{2}(n+\lambda)\ln(n^2-\lambda^2) + \lambda \ln n\lambda/y, \tag{8}$$

$$\lambda \equiv l + \tfrac{1}{2}, \quad y \equiv \alpha m R_c, \quad Y \equiv (\lambda^2 - 2y + y^2/n^2)^{1/2}.$$

Using now $y/\lambda \ll 1$, we find

$$\Gamma(\lambda, n) = n^{-3}\left(\frac{n+\lambda}{n-\lambda}\right)^n \left(1 - \frac{\lambda^2}{n^2}\right)^\lambda F(\lambda) \tag{9}$$

which differs but slightly from the n^{-3} dependence assumed by Desai. Estimates based on perturbation theory are drastically different, however, because they assume that the wave function goes as r^l in the annihilation region. Obviously, (9) applies to all kaonic and antiprotonic atoms which have $Z\alpha m R_c \ll l + \frac{1}{2}$, with $R_c \approx R$ in the case of heavier atoms. It cannot be used for pionic atoms, however.

Finally, we wish to criticize the normal optical potential calculations for the linewidths of antiprotonic atoms. Many nuclei of these atoms have zero isospin, in which case pion exchange is completely eliminated if the optical potential is averaged over the nuclear wave function. If nuclear excitations are taken out of the optical potential and treated explicitly, this is no longer the case. Because of the importance of pion exchange for the annihilation widths, nuclear excitation should therefore be essential. In fact, at least for "deutronium" $\bar{p}d$, the closure approximation could give better results than the normal optical potential calculation.

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[1]W. B. Kaufmann and H. Pilkuhn, Phys. Lett. 62B, 165 (1976).
[2]J. S. Ball and G. F. Chew, Phys. Rev. 109, 1385 (1958).
[3]O. D. Dalkarov and F. Myhrer, CERN Report No. TH 228, 1977 (unpublished).
[4]H. Poth *et al.* (unpublished).
[5]R. Christian and E. Hart, Phys. Rev. 77, 441 (1950).
[6]B. R. Desai, Phys. Rev. 119, 1384 (1960).