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Coulomb effects in three-body reactions with two charged particles

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We present the details of a novel approach to the treatment of Coulomb effects in atomic and nuclear reactions of the three-body type in which two of the particles are charged. Based on three-body integral equations the formalism allows the practical calculation of elastic, inelastic, rearrangement, and breakup processes with full inclusion of the Coulomb repulsion or attraction in a mathematically correct way. No restrictions need to be made concerning the form of the short-range interactions between the three pairs. A particular virtue of our method lies in the fact that it corroborates, and gives precise meaning to, the intuitively anticipated conception of how to describe such reactions.

NUCLEAR REACTIONS Three-body scattering theory. Coulomb effects when two of the particles charged. Screening and renormalization approach. Quasi-particle method. Defined scattering amplitudes for elastic, inelastic, rearrangement and breakup processes, and scattering wave functions. Derived practical integral equations for these quantities.

I. INTRODUCTION

Modern investigations of nuclear reactions involving three fragments are often based on three-body integral equations originating from the work of Watson,[1] Faddeev,[2] Mitra and Bhasin,[3] Amado,[4] and Sitenko and Kharchenko.[5] Although this formalism is quite general, it nevertheless has one important shortcoming: Long-range forces cannot be easily accommodated. For practical purposes this is a serious drawback since in the majority of nuclear reactions at least two charged fragments are taking part. As long as the latter remain bound both in the initial and in the final state, one may treat such processes by the methods of ordinary (short-range) scattering theory. However, in most circumstances it will happen that at least two charged particles separate asymptotically. In fact, these are just the reactions preferred by experimentalists for obvious reasons. Then one must face the question of how to treat the long-range Coulomb potential in the framework of three-body integral equations theory.[6]

In the present paper we restrict ourselves to three-body processes in which *only two* charged particles participate, i.e., the Coulomb force acts in one subsystem only. In order to describe such reactions, essentially two distinct approaches have been developed previously which attempt to formulate three-body integral equations in the presence of the Coulomb force. The one proposed by Noble[8] and by Bencze[9] consists in deriving three-body integral equations similar to those known in ordinary (short-range) theory, with the Coulomb potentials occurring only in the "free" Green's functions and the two-particle transition operators constituting the inhomogeneity and the kernel. In this way well-defined equations are obtained, however, with kernels which are unpleasant three-body operators. Therefore, they do *not* appear to be useful for practical applications. Indeed, in order to make this scheme manageable, drastic and uncontrollable approximations had to be made.[10-12].

Another approach has been put forward by Veselova,[13] who investigated the Faddeev equations for screened Coulomb potentials. After isolating those two-body quantities which generate the Coulomb singularities in the zero screening limit, she explicitly inverted the corresponding part of the Faddeev kernel. By that procedure she arrived at three-body equations having a kernel which can be constructed in terms of two-body transition operators and which is well behaved in that limit. However, the latter property could be shown only for negative energies, i.e., for energies below the breakup threshold.

In the following we present another rigorous approach which combines the advantages of both above mentioned formalisms. It leads to integral equations for elastic, inelastic, rearrangement, and breakup amplitudes the kernels of which are well defined for *all* energies and can be calculated from the *genuine two-body* amplitudes.

Our theory exploits the great similarity between the present problem and that of the scattering of two charged elementary particles. Their close relationship is most clearly exposed by formulating the three-body problem as an effective two-body one. This is accomplished by the quasiparticle formalism of Alt, Grassberger, and Sandhas (AGS)[14] which, therefore, appears to be ideally suited for the following investigations.

Following Ref. 13 we use a screened Coulomb potential. For the scattering of *two* charged particles the screening approach leads for finite values of the screening radius to well-defined scattering amplitudes and wave functions. However, the transition to amplitudes and wave functions for an unscreened Coulomb potential cannot be performed in a straightforward manner. Indeed, it requires the renormalization procedure conjectured by Dalitz[15] and developed by Gorshkov[16] and others.[17-19] In Sec. II this approach to two-charged-particle scattering is presented in some detail since our treatment of the three-body problem closely follows these lines. Section III contains the discussion of the scattering of one neutral and two charged particles, in a model in which we assume that the pairwise short-range forces are rank-one separable potentials, and that the Coulomb force is repulsive. The simplicity of this model enables us to explain our method in a detailed and transparent manner. We demonstrate in particular that the renormalization procedure outlined in Sec. II can be taken over directly in order to define amplitudes for elastic, rearrangement, and breakup processes, as well as scattering wave functions. The above mentioned restrictions are removed in Sec. IV where we allow for arbitrary, in particular local, short-range interactions and also for attractive Coulomb forces.

It should be mentioned that the simple three-body model discussed in Sec. III has already been made the basis of a numerical investigation of elastic proton-deuteron scattering by the present authors. First results for quartet effective range parameters and S-wave phase shifts were reported in Ref. 20, whereas the first results for *pd* cross sections have been published in Ref. 21. A full analysis of *pd* scattering is in preparation.[22]

II. SOME ASPECTS OF THE SCATTERING OF TWO CHARGED PARTICLES

It is well known that many results in (short-range) two-particle-scattering theory do no longer hold for long-range forces of which the Coulomb force is the most important and, therefore, the most thoroughly investigated example. Since the difficulties arising in the three-body problem under consideration turn out to be similar to those encountered for two-charged-particle scattering we start by recapitulating the relevant aspects of the latter. This also serves to fix our notation.

A. Pure Coulomb scattering

Let us first concentrate on the scattering of two particles via a pure Coulomb potential V_C,

$$V_C(r)=g/r\,. \tag{2.1}$$

The corresponding transition operator satisfies the Lippmann-Schwinger (LS) equation

$$T_C(z)=V_C+V_CG_0(z)T_C(z) \tag{2.2}$$

with $G_0(z)=(z-P^2/2\mu)^{-1}$, μ being the reduced mass.

For the following it proves convenient to introduce off-shell scattering states $|\vec{p}_C(z)\rangle$ and off-shell Møller operators $\Omega_C(z)$ via

$$\begin{aligned}|\vec{p}_C(z)\rangle&=\Omega_C(z)|\vec{p}\rangle\\&=[1+G_0(z)T_C(z)]|\vec{p}\rangle\,,\end{aligned} \tag{2.3}$$

where $z=k^2/2\mu\pm i0$ is supposed not to be correlated to the energy $E_p=p^2/2\mu$ associated with the plane wave $|\vec{p}\rangle$.

Although the off-shell matrix elements of T_C, the on-shell amplitudes $T_C(\vec{p}\,',\vec{p})$, and the corresponding on-shell scattering states $|\vec{p}_C^{(\pm)}\rangle$ are explicitly known, they suffer from the well-known diseases.[23] We mention only those which are relevant for the following.

(i) The matrix elements $\langle\vec{p}\,'|T_C(k^2/2\mu\pm i0)|\vec{p}\rangle$ are well defined for $p,p'\neq k$, excluding the forward direction, but do not have a limit when either p or p' or both tend towards their on-shell value k. This is due to the occurrence of characteristic diverging factors. Similar divergences show up also for the scattering states $|\vec{p}_C(k^2/2\mu\pm i0)\rangle$ in the limit p going to k.

(ii) The partial wave series

$$\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\,P_l(\cos\theta)\,T_{Cl}(k) \tag{2.4}$$

does not converge[19,24] (as a function) for any value of the scattering angle θ due to the slow decrease of the "Coulomb phase shifts" with increasing l. This indicates that problems might arise in three- and more-particle scattering if a partial wave decomposition of the Coulomb amplitude or potential is used.

The origin of these and related difficulties lies in the fact that the kernel $V_C G_0(z)$ of the LS equation (2.2), and $G_0(z)V_C$ of the LS equation for $|\vec{p}_C(z)\rangle$, is highly singular for energies $z=\bar{k}^2/2\mu \pm i0$. This is apparent in momentum space where

$$\langle\vec{p}'|V_C G_0(\bar{k}^2/2\mu+i0)|\vec{p}\rangle=\frac{\mu g/\pi^2}{(\vec{p}'-\vec{p})^2(\bar{k}^2+i0-p^2)}\ . \qquad (2.5)$$

Inspection shows that for on-shell values of the momenta $p=p'=\bar{k}$, the singularities of the free Green's function and of the Coulomb potential coincide.[25]

This problem can, however, be avoided by screening the Coulomb potential, i.e., by replacing $V_C(r)$ by a potential[26]

$$V_R(r)=g\,\frac{\exp(-r/R)}{r}\ . \qquad (2.6)$$

The corresponding transition operator $T_R(z)$ fulfills the LS equation (2.2) with V_C replaced by V_R. Similarly, the off-shell Møller operator $\Omega_R(z)$ and scattering states $|\vec{p}_R(z)\rangle$ are defined in analogy to Eq. (2.3) as

$$\begin{aligned}|\vec{p}_R(z)\rangle&=\Omega_R(z)|\vec{p}\rangle\\&=[1+G_0(z)T_R(z)]|\vec{p}\rangle\ .\end{aligned} \qquad (2.7)$$

Since V_R is of short range for finite R all results of conventional scattering theory are valid. In particular, $\Omega_R(\bar{k}^2/2\mu\pm i0)$ becomes the conventional Møller operator mapping the plane wave states $|\vec{p}\rangle$ with $p=\bar{k}$ onto the scattering states $|\vec{p}_R^{(\pm)}\rangle$,

$$\begin{aligned}|\vec{p}_R^{(\pm)}\rangle&=\Omega_R(\bar{k}^2/2\mu\pm i0)|\vec{p}\rangle_{p=\bar{k}}\\&=|\vec{p}_R(\bar{k}^2/2\mu\pm i0)\rangle_{p=\bar{k}}\ .\end{aligned} \qquad (2.8)$$

In order to recover the case of scattering via genuine Coulomb potentials the screening radius R has to become infinite. The performance of this limit is trivial except for the *on-shell* scattering amplitude and the physical wave functions due to the occurrence of violent oscillations.[16,17] However, it has been shown in Refs. 16–19 that these oscillations can be isolated for $R\to\infty$ in the form of a diverging phase factor Z_R,

$$\begin{aligned}&T_R(\vec{p}',\vec{p};\bar{k}^2/2\mu+i0)|_{p=p'=\bar{k}}\\&\quad\overset{R\to\infty}{\sim} Z_R(\bar{k},\mu)T_C(\vec{p}',\vec{p})|_{p=p'=\bar{k}}\end{aligned} \qquad (2.9)$$

and

$$\langle\vec{r}|\vec{p}_R^{(\pm)}\rangle_{p=\bar{k}}\overset{R\to\infty}{\sim} Z_R^{\pm 1/2}(\bar{k},\mu)\langle\vec{r}|\vec{p}_C^{(\pm)}\rangle_{p=\bar{k}}\ . \qquad (2.10)$$

The quantities $T_C(\vec{p}',\vec{p})$ and $\langle\vec{r}|\vec{p}_C^{(\pm)}\rangle$ appearing on the right-hand side of Eqs. (2.9) and (2.10) are the genuine on-shell Coulomb amplitude and scattering wave functions which we are looking for. Writing Z_R as

$$Z_R(\bar{k},\mu)=\exp[2i\phi_R(\bar{k},\mu)]\ , \qquad (2.11)$$

the phase ϕ_R can be determined for arbitrary screening[19] in the limit of large R. In the special case of exponential screening considered here it takes asymptotically the familiar form

$$\phi_R(\bar{k},\mu)=-\eta\,(\ln 2\bar{k}R-C)\ . \qquad (2.12)$$

As is customary we have introduced the "Coulomb parameter" η with $\eta^{-1}=\pm\bar{k}a_B$ (+ for repulsion, − for attraction), $a_B=1/\mu|g|$ being the Bohr radius of the system. $C=0.5772\cdots$ is the Euler number.

B. Coulomb scattering in the presence of short-range potentials

What we have discussed so far remains essentially unchanged if in addition to the Coulomb interaction short-range forces[27] are present, too. We are, thus, again forced to screen the Coulomb potential. Denoting the additional short-range potential by V_s, the total interaction is

$$V^{(R)}=V_R+V_s\ , \qquad (2.13)$$

and depends on the screening radius R. The same holds true also for the corresponding amplitude $T^{(R)}$ which can appropriately be split into a sum of two terms

$$T^{(R)}=T_R+T_{sC}^{(R)}\ . \qquad (2.14)$$

The first of them, T_R, is chosen as the solution of the LS equation (2.2), with V_C replaced by V_R, and describes, therefore, the scattering by the screened Coulomb potential alone. The quantity $T_{sC}^{(R)}$, conventionally called Coulomb-modified strong amplitude, is given according to the two-potential formula by

$$\begin{aligned}T_{sC}^{(R)}(z)&=[1+T_R(z)G_0(z)]V_s[1+G_0(z)T^{(R)}(z)]\\&=\Omega_R^\dagger(z^*)V_s[1+G_0(z)T^{(R)}(z)]\ ,\end{aligned} \qquad (2.15)$$

with $\Omega_R(z)$ defined in Eq. (2.7). This form suggests introducing a quantity $t_{sC}^{(R)}(z)$ via

$$T_{sC}^{(R)}(z)=\Omega_R^\dagger(z^*)t_{sC}^{(R)}(z)\Omega_R(z)\ . \qquad (2.16)$$

Then straightforward algebra leads to the equation

$$t_{sC}^{(R)}(z)=V_s+V_sG_R(z)t_{sC}^{(R)}(z) \qquad (2.17)$$

which demonstrates that $t_{sC}^{(R)}$ is essentially determined by the *short-range* part of the interaction. Here, $G_R(z)$ is the full Green's function for a screened Coulomb potential

$$G_R(z) = (z - H_0 - V_R)^{-1}. \tag{2.18}$$

The representation (2.16) allows an intuitive physical interpretation. Namely, for on-shell values of the momenta $p = p' = \bar{k}$, it becomes

$$\begin{aligned}\langle \vec{p}'|T_{sC}^{(R)}(\bar{k}^2/2\mu + i0)|\vec{p}\rangle &= \langle \vec{p}'|\Omega_R^\dagger(\bar{k}^2/2\mu - i0)t_{sC}^{(R)}(\bar{k}^2/2\mu + i0) \\ &\quad \times \Omega_R(\bar{k}^2/2\mu + i0)|\vec{p}\rangle \\ &= \langle \vec{p}_R'^{(-)}|t_{sC}^{(R)}(\bar{k}^2/2\mu + i0)|\vec{p}_R^{(+)}\rangle, \end{aligned} \tag{2.19}$$

expressing the Coulomb-modified strong amplitude by means of the matrix elements of the operator $t_{sC}^{(R)}$ in the (screened) "Coulomb representation."

In order to recover the scattering by an unscreened Coulomb potential again the limit $R \to \infty$ has to be investigated for $T_{sC}^{(R)}$, and also $t_{sC}^{(R)}$, and the full transition operator $T^{(R)}$. In this limit the kernel of Eq. (2.17) does obviously not develop a singularity of the type (2.5). Consequently all matrix elements $\langle \vec{p}'|t_{sC}^{(R)}(\bar{k}^2/2\mu + i0)|\vec{p}\rangle$ tend smoothly towards their unscreened counterparts. Thus taking into account the large-R behavior (2.10) of the scattering states, we obtain from (2.19) for $p = p' = \bar{k}$

$$\begin{aligned}\langle \vec{p}'|T_{sC}^{(R)}(\bar{k}^2/2\mu + i0)|\vec{p}\rangle &\overset{R\to\infty}{\sim} Z_R(\bar{k}, \mu)T_{sC}(\vec{p}', \vec{p}; \bar{k}^2/2\mu + i0) \\ &= Z_R(\bar{k}, \mu)\langle \vec{p}_C'^{(-)}|\, t_{sC}^{(\infty)}\left(\frac{k^2}{2\mu} + i0\right)|\vec{p}_C^{(+)}\rangle. \end{aligned} \tag{2.20}$$

Here $T_{sC}(\vec{p}', \vec{p}; \bar{k}^2/2\mu + i0)$ is the Coulomb-modified strong amplitude for an *unscreened* Coulomb potential. This result together with Eq. (2.9) implies that also the full amplitude (2.14) behaves for on-shell values of the momenta as

$$\begin{aligned}T^{(R)}(\vec{p}', \vec{p}; \bar{k}^2/2\mu + i0) &\overset{R\to\infty}{\sim} Z_R(\bar{k}, \mu)[T_C(\vec{p}', \vec{p}) + T_{sC}(\vec{p}', \vec{p}; \bar{k}^2/2\mu + i0)] \\ &= Z_R(\bar{k}, \mu)T(\vec{p}', \vec{p}). \end{aligned} \tag{2.21}$$

In other words, the amplitude describing the scattering off a short-range plus an unscreened Coulomb potential also follows from $T^{(R)}$ by asymptotically factoring out the phase factor Z_R.

The results (2.20) and (2.21) are of considerable practical importance, in that they provide us with a recipe for numerically determining the full unscreened amplitude defined above. It consists in calculating first at some value of R the screened Coulomb-modified amplitude $T_{sC}^{(R)}$ and to "renormalize" its on-shell element by $Z_R^{-1}(\bar{k}, \mu)$. This procedure has to be repeated for increasing R until the asymptotic value, which is the unscreened Coulomb-modified amplitude T_{sC}, is reached. To it we then add the analytically known unscreened pure Coulomb amplitude T_C. This yields according to Eq. (2.21) the desired amplitude T. In practical tests it turned out that for proton-proton scattering stable results could be obtained in this way already for $R \sim 30$ fm.

III. SIMPLE THREE-BODY MODEL FOR SCATTERING OF ONE NEUTRAL AND TWO CHARGED PARTICLES

In the present section we are going to describe how the scattering of two charged particles reviewed in Sec. II has to be modified when a third neutral particle is present in addition. In order to keep the presentation as clear as possible we will assume for the moment that the short-range forces acting between the three particles are described by separable potentials of rank one. Complications arising from higher rank separable potentials or from local ones (including finite size corrections) will be dealt with in the next section.

A. Two-body input and kinematics

Let the particles 1 and 2 be charged (with charges Z_1e and Z_2e) and particle 3 be neutral. According to our assumption the potentials for the three subsystems are

$$V_\alpha = |\chi_\alpha\rangle\lambda_\alpha\langle\chi_\alpha|, \quad \text{for } \alpha = 1, 2, \tag{3.1a}$$

$$V_3 = |\chi_3\rangle\lambda_3\langle\chi_3| + V_R, \tag{3.1b}$$

where we take the (point-) Coulomb potential[28] to be screened as in Eq. (2.6), with $g = Z_1Z_2e^2$. The corresponding subsystem amplitudes are[29] [recall Eqs. (2.14) and (2.16)]

$$\hat{T}_\alpha(z_\alpha) = |\chi_\alpha\rangle\hat{\Delta}_\alpha(z_\alpha)\langle\chi_\alpha|, \quad \text{for } \alpha = 1, 2, \tag{3.2a}$$

$$\hat{T}_3^{(R)}(z_3) = \hat{T}_R(z_3) + \hat{\Omega}_R^\dagger(z_3^*)|\chi_3\rangle\hat{\Delta}_3(z_3)\langle\chi_3|\hat{\Omega}_R(z_3), \tag{3.2b}$$

with z_α denoting the energy of the subsystem of the two particles β and γ. In order to distinguish energy-dependent two-body operators to be read in the two-particle space from those in the three-particle space, we characterize from now on the former by a caret. Furthermore, for notational simplicity we do not indicate any energy dependence of the form factors $|\chi_\alpha\rangle$. Then the Coulomb-distorted form factor occurring in Eq. (3.2b) can be expressed in momentum space as overlap between $|\chi_3\rangle$ and the off-shell scattering state (2.3),

$$\langle\vec{p}|\hat{\Omega}_R^\dagger(z_3^*)|\chi_3\rangle = \langle\vec{p}_R(z_3^*)|\chi_3\rangle . \tag{3.3}$$

Finally, we have for $\alpha=1,2$

$$\hat{\Delta}_\alpha^{-1}(z_\alpha) = \lambda_\alpha^{-1} - \langle\chi_\alpha|\hat{G}_0(z_\alpha)|\chi_\alpha\rangle \tag{3.4a}$$

and,[30] as follows most easily from Eq. (2.17),

$$\hat{\Delta}_3^{-1}(z_3) = \lambda_3^{-1} - \langle\chi_3|\hat{G}_R(z_3)|\chi_3\rangle \tag{3.4b}$$

with $\hat{G}_R(z)$ given by Eq. (2.18).

In this section we consider a situation where in each channel precisely one stable bound state exists. This requires the particles 1 and 2 to be equally charged. The cases of oppositely charged particles and of channels with no bound states will be treated in the next section (the latter aspect is relevant, e.g., for proton-deuteron scattering). Then we can write Eqs. (3.4) as

$$\hat{\Delta}_\alpha(z_\alpha) = \frac{\hat{S}_\alpha(z_\alpha)}{z_\alpha - \hat{E}_\alpha}, \quad \alpha = 1,2,3 . \tag{3.5}$$

Hereby, $\hat{E}_\alpha$ is the two-particle binding energy of pair $(\beta+\gamma)$, and $\hat{S}_\alpha(z)$ is defined as[30]

$$\hat{S}_\alpha(z_\alpha) = [\langle\chi_\alpha|\hat{G}_0(\hat{E}_\alpha)\hat{G}_0(z_\alpha)|\chi_\alpha\rangle]^{-1}, \quad \text{for } \alpha=1,2,$$
$$\hat{S}_3(z_3) = [\langle\chi_3|\hat{G}_R(\hat{E}_3)\hat{G}_R(z_3)|\chi_3\rangle]^{-1} . \tag{3.6}$$

Note that for all three subsystems $\hat{S}_\alpha(\hat{E}_\alpha)$ equals the normalization integral of the bound state wave function in subsystem α and is conveniently set equal to 1,

$$\hat{S}_\alpha(\hat{E}_\alpha) = 1, \quad \text{for } \alpha = 1,2,3 . \tag{3.7}$$

In order to investigate the system of the three particles we make use of the quasiparticle formalism which reduces the three-body problem to an effective two-body one. First we will discuss elastic and rearrangement scattering, then the breakup processes.

To begin with let us introduce some notation. We consider three particles with masses m_1, m_2, and m_3 and momenta $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ in the total center-of-mass system. The reduced mass of particles β and γ will be denoted by μ_α:

$$\mu_\alpha^{-1} = m_\beta^{-1} + m_\gamma^{-1}, \tag{3.8}$$

and their relative momentum by

$$\vec{p}_\alpha = \frac{m_\gamma\vec{k}_\beta - m_\beta\vec{k}_\gamma}{m_\beta + m_\gamma} . \tag{3.9}$$

Furthermore, we denote the reduced mass of particle α and the system of particles $(\beta+\gamma)$ by M_α,

$$M_\alpha^{-1} = m_\alpha^{-1} + (m_\beta + m_\gamma)^{-1}, \tag{3.10}$$

and the corresponding relative momentum by

$$\vec{q}_\alpha = \frac{(m_\beta + m_\gamma)\vec{k}_\alpha - m_\alpha(\vec{k}_\beta + \vec{k}_\gamma)}{m_\alpha + m_\beta + m_\gamma} . \tag{3.11}$$

In what follows we will often encounter another type of genuine two-particle system. It consists of a particle of mass m_α and momentum $\vec{k}_\alpha$, and another "particle" of mass $(m_\beta + m_\gamma)$ and momentum $(\vec{k}_\beta + \vec{k}_\gamma)$ with *no internal structure or motion* of the constituents. The free Hamiltonian of this two-body system is $Q_\alpha^2/2M_\alpha$, where $\vec{Q}_\alpha$ is the relative momentum operator acting only in the subspace spanned by the eigenstates $|\vec{q}_\alpha\rangle$ with eigenvalues given by (3.11), $\vec{Q}_\alpha|\vec{q}_\alpha\rangle = \vec{q}_\alpha|\vec{q}_\alpha\rangle$.

In order to distinguish operators acting in this two-particle system from those discussed above and in Sec. II the former will be characterized by an index[31] Q. For instance, the free Green's function is

$$\hat{G}_0^Q(z_\alpha) = (z_\alpha - Q_\alpha^2/2M_\alpha)^{-1}, \tag{3.12}$$

whereas

$$\hat{G}_R^Q(z_\alpha) = (z_\alpha - Q_\alpha^2/2M_\alpha - V_R^Q)^{-1} \tag{3.13}$$

is the total Green's function for the system of particle α and "particle" $(\beta+\gamma)$ interacting via a screened Coulomb potential V_R^Q. The corresponding transition operator, denoted by $\hat{T}_R^Q(z_\alpha)$, satisfies the LS equation (2.2) with V_C replaced by V_R^Q, and $\hat{G}_0$ by $\hat{G}_0^Q$,

$$\hat{T}_R^Q(z_\alpha) = V_R^Q + V_R^Q\hat{G}_0^Q(z_\alpha)\hat{T}_R^Q(z_\alpha), \tag{3.14}$$

and yields in momentum space the (screened) Coulomb scattering amplitude for the considered two-particle system, $\langle\vec{q}_\alpha'|\hat{T}_R^Q(z_\alpha)|\vec{q}_\alpha\rangle = \hat{T}_R^Q(\vec{q}_\alpha', \vec{q}_\alpha; z_\alpha)$. In analogy to Eq. (2.7) the off-shell Møller operator

$$\hat{\Omega}_R^Q(z_\alpha) = 1 + \hat{G}_0^Q(z_\alpha)\hat{T}_R^Q(z_\alpha) \tag{3.15}$$

is introduced which, when applied to the free states $|\vec{q}_\alpha\rangle$, yields the corresponding (off-shell) scattering states $|\vec{q}_{\alpha,R}(z_\alpha)\rangle$. The latter are for $z_\alpha = \bar{q}_\alpha^2/2M_\alpha \pm i0$ and $q_\alpha = \bar{q}_\alpha$ the on-shell states [compare Eq. (2.8)]

$$|\vec{q}_{\alpha,R}^{(\pm)}\rangle = |\vec{q}_{\alpha,R}(\bar{q}_\alpha^2/2M_\alpha \pm i0)\rangle = \hat{\Omega}_R^Q(\bar{q}_\alpha^2/2M_\alpha \pm i0)|\vec{q}_\alpha\rangle \tag{3.16}$$

describing the scattering of particle α off the "center of mass of particles β and γ."

Later on we will need the on-shell amplitudes $\hat{T}_R^Q(\vec{q}_\alpha', \vec{q}_\alpha; \bar{q}_\alpha^2/2M_\alpha + i0)$ and wave functions $|\vec{q}_{\alpha,R}^{(\pm)}\rangle$ in the limit $R\to\infty$. Recalling that both of them are genuine two-body quantities we can immediately take over the results (2.9) and (2.10). Denoting the corresponding counterparts for an unscreened Coulomb potential by an index C, we have for

$$q_\alpha = q'_\alpha = \bar{q}_\alpha \;,$$

$$\hat{T}^{\mathcal{Q}}_R(\vec{q}'_\alpha, \vec{q}_\alpha; \bar{q}_\alpha^{\,2}/2M_\alpha + i0) \overset{R\to\infty}{\sim} Z_R(\bar{q}_\alpha, M_\alpha)\hat{T}^{\mathcal{Q}}_C(\vec{q}'_\alpha, \vec{q}_\alpha) \tag{3.17}$$

and

$$|\vec{q}^{\,(\pm)}_{\alpha,R}\rangle_{q_\alpha = \bar{q}_\alpha} \overset{R\to\infty}{\sim} Z_R^{\,\pm 1/2}(\bar{q}_\alpha, M_\alpha)|\vec{q}^{\,(\pm)}_{\alpha,C}\rangle_{q_\alpha = \bar{q}_\alpha} \;. \tag{3.18}$$

B. Elastic and rearrangement scattering

Even under the simplifying assumptions about the short-range interactions, detailed in Eqs. (3.1), in the present problem the strictly local Coulomb potential occurs in subsystem 3. Hence a formalism for describing three-body processes is needed which allows us to fully take into account local parts of the interactions. This is the case in the AGS quasiparticle approach[14] which, therefore, represents the adequate tool for the following investigation. As was shown there the quasiparticle equations for elastic and rearrangement scattering are

$$\mathcal{T}^{(R)}_{\beta\alpha} = \mathcal{V}^{(R)}_{\beta\alpha} + \sum_\gamma \mathcal{V}^{(R)}_{\beta\gamma}\, \mathcal{G}_{0;\gamma}\mathcal{T}^{(R)}_{\gamma\alpha} \;, \tag{3.19}$$

where we have explicitly indicated the dependence on the screening radius R. All quantities occurring here are still operators with respect to the plane wave states describing the free motion of one particle and the bound system of the other two in the initial and the final state. In fact,

$$\mathcal{T}^{(R)}_{\beta\alpha}(\vec{q}'_\beta, \vec{q}_\alpha) = \langle \vec{q}'_\beta | \mathcal{T}^{(R)}_{\beta\alpha}(E + i0)|\vec{q}_\alpha\rangle \tag{3.20}$$

are just the physical bound-state scattering amplitudes if the momenta q_α and q'_β are equal to their on-shell values $\bar{q}_\alpha$ and $\bar{q}'_\beta$, respectively, defined by means of

$$E = E'_\beta = \bar{q}'^{\,2}_\beta/2M_\beta + \hat{E}_\beta = E_\alpha = \bar{q}_\alpha^{\,2}/2M_\alpha + \hat{E}_\alpha \,. \tag{3.21}$$

The effective free Green's function $\mathcal{G}_{0;\alpha}$ is essentially[32] determined by Δ_α of Eq. (3.5),

$$\begin{aligned}\langle \vec{q}'_\alpha | \mathcal{G}_{0;\alpha}(z)|\vec{q}_\alpha\rangle &= \delta(\vec{q}'_\alpha - \vec{q}_\alpha)(z - q_\alpha^{\,2}/2M_\alpha - \hat{E}_\alpha)^{-1} \\ &= \langle \vec{q}'_\alpha | \hat{G}^{\mathcal{Q}}_0(z - \hat{E}_\alpha)|\vec{q}_\alpha\rangle \,.\end{aligned} \tag{3.22}$$

For the second equality use had been made of Eq. (3.12).

Due to the assumed separability of the nuclear potentials and the fact that the (nonseparable) Coulomb potential[28] V_R acts in one subsystem only, the effective potential $\mathcal{V}^{(R)}$ has the following *exact* and *closed* representation[33]

$$\begin{aligned}\langle \vec{q}'_\beta | \mathcal{V}^{(R)}_{\beta\alpha}(z)|\vec{q}_\alpha\rangle = [\hat{S}_\beta(z - q'^{\,2}_\beta/2M_\beta)]^{1/2}\langle \vec{q}'_\beta | \langle \chi_\beta | \{\bar{\delta}_{\beta\alpha}[G_0(z) + (\delta_{\beta 3} + \delta_{\alpha 3} + \bar{\delta}_{\beta 3}\bar{\delta}_{\alpha 3})G_0(z)T_R(z)G_0(z)] \\ + \delta_{\beta\alpha}\bar{\delta}_{\alpha 3}G_0(z)T_R(z)G_0(z)\}|\chi_\alpha\rangle|\vec{q}_\alpha\rangle[\hat{S}_\alpha(z - q_\alpha^{\,2}/2M_\alpha)]^{1/2}\end{aligned} \tag{3.23}$$

[as usual, $\bar{\delta}_{\beta\alpha} = (1 - \delta_{\beta\alpha})$, and $G_0(z) = (z - H_0)^{-1}$ is the three-particle free Green's function]. The potential $\mathcal{V}^{(R)}$ depends on the screening radius R via the screened Coulomb amplitude T_R (and in an unessential manner via $\hat{S}_3$, as discussed above[30]). The individual terms on the right-hand side of Eq. (3.23) are represented in diagrammatical form in Fig. 1.

We will now proceed as follows. First we demonstrate that the kernel of Eq. (3.19) becomes, in the limit $R\to\infty$, as singular as for the genuine two-particle case [cf. Eq. (2.5)]. Then the most singular part of the kernel will be isolated in such a form that the renormalization procedure discussed in Sec. II can be taken over directly.

As indicated, the effective free Green's function (3.22) has precisely the form of a genuine two-particle free Green's function taken at a two-particle energy $z_\alpha = z - \hat{E}_\alpha$. This becomes even more apparent for physical three-body energies $z = \bar{q}_\alpha^{\,2}/2M_\alpha + \hat{E}_\alpha + i0$ in which case $z_\alpha = \bar{q}_\alpha^{\,2}/2M_\alpha + i0$. Then Eq. (3.22) reduces to $2M_\alpha \langle \vec{q}'_\alpha | (\bar{q}_\alpha^{\,2} + i0 - Q_\alpha^{\,2})^{-1}|\vec{q}_\alpha\rangle$.

What remains to be shown is that $\mathcal{V}^{(R)}_{\beta\alpha}(\vec{q}'_\beta, \vec{q}_\alpha; z)$ reveals in the limit $R\to\infty$ the same singular behavior as the two-body Coulomb potential. As

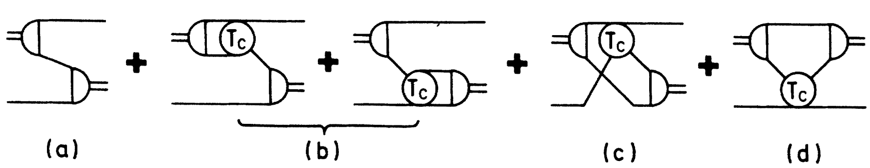

FIG. 1. Diagrammatical representation of the exact effective potential (3.23). Semicircles indicate the form factors.

suggested by Fig. 1 this can originate only from diagram *d*. In fact, its most singular part in the limit $R\to\infty$ has analytically the form

$$\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}V_R(\vec{q}\,'_\alpha-\vec{q}_\alpha)R_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)\,, \tag{3.24}$$

with V_R being the *genuine two-particle* screened Coulomb potential. The term (3.24) is graphically represented by Fig. 2. The decisive point to note is that due to its locality the screened Coulomb potential V_R, as it occurs in Eq. (3.24), depends only on the momentum transfer *from the (charged) particle* α *to the center of mass of particles* β *and* γ. Consequently, $V_R(\vec{q}\,'_\alpha-\vec{q}_\alpha)$ can be *identified* with the matrix elements of the operator V_R^Q introduced in Eq. (3.13) which directly describes the Coulomb interaction between particle α and the center of mass of β and γ, $V_R(\vec{q}\,'_\alpha-\vec{q}_\alpha)\equiv\langle\vec{q}\,'_\alpha|V_R^Q|\vec{q}_\alpha\rangle$. The factor R_α in which the whole

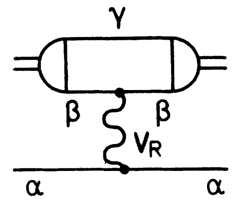

FIG. 2. The part (3.24) of the effective potential (3.23) which arises from diagram (d) of Fig. 1 (with $\alpha\neq 3$) and gives rise to the longest-range contribution of $\mathcal{V}^{(R)}$ in the limit $R\to\infty$.

nonlocality resides is defined by

$$R_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)=[\hat{S}_\alpha(z-q_\alpha'^2/2M_\alpha)]^{1/2}F_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)\times[\hat{S}_\alpha(z-q_\alpha^2/2M_\alpha)]^{1/2} \tag{3.25}$$

with

$$F_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)=\int\frac{d^3k\,\chi_\alpha^*(\vec{k})\chi_\alpha(\vec{k}+\vec{D}_\alpha)}{[k^2/2\mu_\alpha-(z-q_\alpha'^2/2M_\alpha)][(\vec{k}+\vec{D}_\alpha)^2/2\mu_\alpha-(z-q_\alpha^2/2M_\alpha)]}\,. \tag{3.26}$$

In the last equation we have introduced

$$\vec{D}_\alpha=m_\gamma/(m_\beta+m_\gamma)(\vec{q}\,'_\alpha-\vec{q}_\alpha) \tag{3.27}$$

which is, apart from a mass ratio, the momentum transfer from the charged particle α to the other charged particle β.

The quantity $F_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)$ has a simple physical interpretation. Namely when the absolute values of the momenta $\vec{q}\,'_\alpha$ and $\vec{q}_\alpha$ coincide with the on-shell momentum $\bar{q}_\alpha$, then $F_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;\bar{q}_\alpha^2/2M_\alpha+\hat{E}_\alpha)$ is just the body form factor of the bound system α, normalized to unity for $\vec{D}_\alpha=0$. This fact, together with Eq. (3.7), immediately shows that for $\vec{q}\,'_\alpha=\vec{q}_\alpha$ and $q_\alpha=\bar{q}_\alpha$, when the singularities of $V_R(\vec{q}\,'_\alpha-\vec{q}_\alpha)$ and of the effective Green's function (3.22) coincide,

$$R_\alpha(\vec{q}_\alpha,\vec{q}_\alpha;\bar{q}_\alpha^2/2M_\alpha+\hat{E}_\alpha)=1 \tag{3.28}$$

holds. Thus, indeed, the kernel of Eq. (3.19) does exhibit in the limit of zero screening the *same* singular behavior[34] as the two-particle kernel (2.5) but only in the relative momentum variable $\vec{q}_\alpha$ between particle α and the center of mass of particles β and γ. In the Appendix it is proven that all other contributions to $\mathcal{V}_{\beta\alpha}^{(R)}$ are less singular than the term (3.24), implying that in coordinate space they decrease faster than the Coulomb potential for large distances. These observations will allow us to take over the screening procedure for two-charged-particle scattering sketched in Sec. II.

In order to investigate the behavior of $[R_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)-1]$ we introduce for $\alpha=1,2$ an auxiliary function B_α depending only on the magnitudes of the momenta $\vec{q}_\alpha$ and $\vec{q}\,'_\alpha$

$$B_\alpha(q'_\alpha,q_\alpha;z)=\int\frac{d^3k\,|\chi_\alpha(\vec{k})|^2[\hat{S}_\alpha(z-q_\alpha'^2/2M_\alpha)\hat{S}_\alpha(z-q_\alpha^2/2M_\alpha)]^{1/2}}{[k^2/2\mu_\alpha-(z-q_\alpha'^2/2M_\alpha)][k^2/2\mu_\alpha-(z-q_\alpha^2/2M_\alpha)]}\,. \tag{3.29}$$

Comparison with the definition (3.6) of $\hat{S}_\alpha$ shows that half on shell

$$B_\alpha(q'_\alpha,\bar{q}_\alpha;E_\alpha+i0)=[\hat{S}_\alpha(E_\alpha+i0-q_\alpha'^2/2M_\alpha)]^{-1/2}, \tag{3.30}$$

and for arbitrary $q'_\alpha=q_\alpha$

$$B_\alpha(q_\alpha,q_\alpha;z)=R_\alpha(\vec{q}_\alpha,\vec{q}_\alpha;z)\,. \tag{3.31}$$

With the help of B_α we decompose $R_\alpha(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)$, taking into account Eq. (3.28), as follows[37]:

$$R_\alpha(\vec{q}'_\alpha, \vec{q}_\alpha; z) = 1 + [B_\alpha(q'_\alpha, q_\alpha; z) - 1] + [R_\alpha(\vec{q}'_\alpha, \vec{q}_\alpha; z) - B_\alpha(q'_\alpha, q_\alpha; z)] . \tag{3.32}$$

The foregoing discussion makes clear that the first bracket of (3.32) vanishes if $q'_\alpha = q_\alpha = \bar{q}_\alpha$, and the second bracket if $\vec{q}'_\alpha = \vec{q}_\alpha$.

We are now in the position to detail the singularity structure of that part of the kernel of Eq. (3.19) which originates from (3.24). Inserting there the splitting (3.32) we recognize that the first term of it gives the most singular contribution. The second and the third terms are less singular on account of the fact that either the pole of $\mathcal{G}_{0;\alpha}$ or that of the Coulomb potential V_R (in the limit $R \to \infty$) are killed off. We are, therefore, led to the following decomposition of the effective potential $\mathcal{V}^{(R)}$,

$$\mathcal{V}^{(R)}_{\beta\alpha} = \tilde{\mathcal{V}}^{(R)}_{\beta\alpha} + \mathcal{V}'^{(R)}_{\beta\alpha} , \tag{3.33}$$

with

$$\tilde{\mathcal{V}}^{(R)}_{\beta\alpha}(\vec{q}'_\beta, \vec{q}_\alpha; z) = \delta_{\beta\alpha}\bar{\delta}_{\alpha 3} V_R(\vec{q}'_\alpha - \vec{q}_\alpha) \equiv \delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\langle \vec{q}'_\alpha | V^Q_R | \vec{q}_\alpha\rangle \tag{3.34}$$

and

$$\begin{aligned}\mathcal{V}'^{(R)}_{\beta\alpha}(\vec{q}'_\beta, \vec{q}_\alpha; z) = {} & [\hat{S}_\beta(z - q'^2_\beta/2M_\beta)\hat{S}_\alpha(z - q_\alpha^{\,2}/2M_\alpha)]^{1/2}\langle \vec{q}'_\beta|\langle\chi_\beta|\{\bar{\delta}_{\beta\alpha}[G_0(z) + G_0(z)T_R(z)G_0(z)] \\ & + \delta_{\beta\alpha}\bar{\delta}_{\alpha 3}G_0(z)[T_R(z) - V_R]G_0(z)\}|\chi_\alpha\rangle|\vec{q}_\alpha\rangle \\ & + \delta_{\beta\alpha}\bar{\delta}_{\alpha 3}V_R(\vec{q}'_\alpha - \vec{q}_\alpha)\{[B_\alpha(q'_\alpha, q_\alpha; z) - 1] + [R_\alpha(\vec{q}'_\alpha, \vec{q}_\alpha; z) - B_\alpha(q'_\alpha, q_\alpha; z)]\} .\end{aligned} \tag{3.35}$$

Equation (3.33) provides the desired separation of the effective potential into a long-range part ($\tilde{\mathcal{V}}^{(R)}$) which describes the pure (screened) Coulomb scattering of the charged particle α from the center of mass of particles β and γ, and a shorter-range contribution ($\mathcal{V}'^{(R)}$). The former is depicted diagrammatically in Fig. 3.

Thus we are in a situation analogous to that encountered in Sec. II for the genuine two-particle scattering and can repeat the development starting with Eq. (2.13). Applying again the two-potential formalism leads to the following decomposition of the effective two-body transition operator $\mathcal{T}^{(R)}_{\beta\alpha}$,

$$\mathcal{T}^{(R)}_{\beta\alpha} = \tilde{\mathcal{T}}^{(R)}_{\beta\alpha} + \mathcal{T}^{(R)}_{sC,\beta\alpha} . \tag{3.36}$$

Here, the operator $\tilde{\mathcal{T}}^{(R)}$ is defined by a LS equation with potential $\tilde{\mathcal{V}}^{(R)}$,

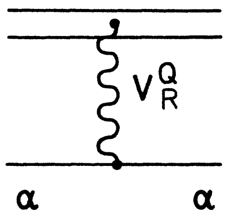

FIG. 3. Genuine two-body (screened) Coulomb potential V^Q_R between charged particle α and the center of mass of particles β and γ identified as the most singular part of the diagram displayed in Fig. 2, and constituting $\tilde{\mathcal{V}}^{(R)}$, Eq. (3.34).

$$\tilde{\mathcal{T}}^{(R)}_{\beta\alpha} = \tilde{\mathcal{V}}^{(R)}_{\beta\alpha} + \sum_\gamma \tilde{\mathcal{V}}^{(R)}_{\beta\gamma}\mathcal{G}_{0;\gamma}\tilde{\mathcal{T}}^{(R)}_{\gamma\alpha} = \tilde{\mathcal{V}}^{(R)}_{\beta\alpha} + \sum_\gamma \tilde{\mathcal{T}}^{(R)}_{\beta\gamma}\mathcal{G}_{0;\gamma}\tilde{\mathcal{V}}^{(R)}_{\gamma\alpha} , \tag{3.37}$$

and the quantity $\mathcal{T}^{(R)}_{sC}$ by

$$\mathcal{T}^{(R)}_{sC,\beta\alpha} = \sum_{\gamma\delta}(1 + \tilde{\mathcal{T}}^{(R)}\mathcal{G}_0)_{\beta\gamma}\mathcal{V}'^{(R)}_{\gamma\delta}(1 + \mathcal{G}_0\mathcal{T}^{(R)})_{\delta\alpha} . \tag{3.38}$$

Both terms (3.37) and (3.38) are easily interpreted. For this purpose we define an amplitude $\tilde{T}_R(z)$ via

$$\tilde{\mathcal{T}}^{(R)}_{\beta\alpha}(z) = \delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\tilde{T}_R(z) \tag{3.39}$$

making use of the specific structure (3.34) of $\tilde{\mathcal{V}}^{(R)}$. Equation (3.37) can then immediately be transformed into an equation for $\tilde{T}_R(z)$, which in momentum space reads as [recall Eq. (3.22)]

$$\tilde{T}_R(\vec{q}'_\alpha, \vec{q}_\alpha; z) = V_R(\vec{q}'_\alpha - \vec{q}_\alpha) + \int d^3q''_\alpha \, \frac{V_R(\vec{q}'_\alpha - \vec{q}''_\alpha)\tilde{T}_R(\vec{q}''_\alpha, \vec{q}_\alpha; z)}{(z - \hat{E}_\alpha) - q''^2_\alpha/2M_\alpha} . \tag{3.40}$$

This, however, coincides with the momentum representation of the LS equation (3.14) at a two-particle energy $z_\alpha = z - \hat{E}_\alpha$. Thus for physical three-particle energies (3.21), we have z_α

$=\bar{q}_\alpha{}^2/2M_\alpha+i0$ so that

$$\tilde{T}_R(\vec{q}\,'_\alpha, \vec{q}_\alpha;\ z=\bar{q}_\alpha{}^2/2M_\alpha+\hat{E}_\alpha+i0) \equiv \langle\vec{q}\,'_\alpha|\hat{T}^Q_R(\bar{q}_\alpha{}^2/2M_\alpha+i0)|\vec{q}_\alpha\rangle \quad (3.41)$$

equals the screened Coulomb amplitude for *two* particles of masses m_α and $(m_\beta+m_\gamma)$.

The second amplitude (3.38) will be called Coulomb-modified strong amplitude in analogy to expression (2.15) in the genuine two-particle case. Let us rewrite it in order to make its structure more transparent. The first step is to recognize that, if we define a quantity $\Omega^{(R)}_\alpha$ as

$$\Omega^{(R)}_\alpha(z)=[1+\mathcal{G}_0(z)\tilde{\mathcal{T}}^{(R)}(z)]_{\alpha\alpha}, \quad (3.42)$$

then it takes, because of the structure of $\tilde{\mathcal{T}}^{(R)}$ discussed above, the explicit form

$$\Omega^{(R)}_\alpha(z)=\begin{cases}\hat{\Omega}^Q_R(z-\hat{E}_\alpha), & \text{for } \alpha\neq 3,\\ 1, & \text{for } \alpha=3.\end{cases} \quad (3.43)$$

Here $\hat{\Omega}^Q_R(z-\hat{E}_\alpha)$ is the "off-shell" Møller operator introduced in Eq. (3.15), at a two-particle energy $z_\alpha=z-\hat{E}_\alpha$. Thus for physical energies $z=\bar{q}_\alpha{}^2/2M_\alpha+\bar{E}_\alpha+i0$, the Møller operator $\hat{\Omega}^Q_R(\bar{q}_\alpha{}^2/2M_\alpha+i0)$ maps the plane waves $|\vec{q}_\alpha\rangle$ with $q_\alpha=\bar{q}_\alpha$ onto the scattering states[38] $|\vec{q}^{\,(+)}_{\alpha,R}\rangle$ [recall Eq. (3.16)].

In analogy to Eq. (2.16) we then define an operator $t^{(R)}_{sC}(z)$ via

$$\mathcal{T}^{(R)}_{sC,\beta\alpha}(z)=\Omega^{(R)\dagger}_\beta(z^*)t^{(R)}_{sC,\beta\alpha}(z)\Omega^{(R)}_\alpha(z) \quad (3.44)$$

which by simple algebra can be shown to fulfill

$$t^{(R)}_{sC,\beta\alpha}(z)=\mathcal{V}'^{(R)}_{\beta\alpha}(z)+\sum_\gamma \mathcal{V}'^{(R)}_{\beta\gamma}(z)\mathcal{G}^{(R)}_\gamma(z)t^{(R)}_{sC,\gamma\alpha}(z). \quad (3.45)$$

Here the similarity to Eq. (2.17) is emphasized by the notation

$$\mathcal{G}^{(R)}_\gamma(z)=\begin{cases}\hat{G}^Q_R(z-\hat{E}_\gamma), & \text{for } \gamma\neq 3,\\ \mathcal{G}_{0;3}(z)=\hat{G}^Q_0(z-\hat{E}_3), & \text{for } \gamma=3,\end{cases} \quad (3.46)$$

where $\hat{G}^Q_R(z-\hat{E}_\gamma)$ is the screened Coulomb Green's function (3.13) at a two-particle energy $z_\gamma=z-\hat{E}_\gamma$. On the energy shell Eq. (3.44), therefore, expresses the Coulomb-modified strong amplitude $\mathcal{T}^{(R)}_{sC,\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha)$ as the matrix element of $t^{(R)}_{sC,\beta\alpha}$ in the "(screened) Coulomb representation"

$$\mathcal{T}^{(R)}_{sC,\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha;\ E+i0) = \langle\vec{q}\,'^{(-)}_{\beta,R}|t^{(R)}_{sC,\beta\alpha}(E+i0)|\vec{q}^{\,(+)}_{\alpha,R}\rangle. \quad (3.47)$$

Now the decomposition (3.36) of the effective two-body amplitude together with (3.39) and (3.44) makes the execution of the limit $R\to\infty$ a simple task. As discussed in Sec. II and at the beginning of Sec. III, after renormalization the two-particle amplitude (3.41) approaches on the energy shell the corresponding pure Coulomb amplitude [cf. Eq. (3.17)]. Consequently we find from Eq. (3.39)

$$\lim_{R\to\infty} Z_{R,\beta}{}^{-1/2}(\bar{q}_\beta', M_\beta)\tilde{\mathcal{T}}^{(R)}_{\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha;\ E+i0) \times Z_{R,\alpha}{}^{-1/2}(\bar{q}_\alpha, M_\alpha)=\mathcal{T}_{C,\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha) = \delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\hat{T}^Q_C(\vec{q}\,'_\alpha, \vec{q}_\alpha), \quad (3.48)$$

where for the sake of compact notation a quantity

$$Z_{R,\alpha}(\bar{q}_\alpha, M_\alpha)=\begin{cases}Z_R(\bar{q}_\alpha, M_\alpha), & \text{for } \alpha\neq 3,\\ 1, & \text{for } \alpha=3,\end{cases} \quad (3.49)$$

has been introduced, Z_R being the renormalization factor defined in Sec. II [cf. Eqs. (2.11) and (2.12)].

Next we investigate the large-R behavior of $\mathcal{T}^{(R)}_{sC}$. For this purpose we recall that the kernel of Eq. (3.45) for $t^{(R)}_{sC}$ does not show in the zero screening limit a singular behavior of the type (2.5), as has been discussed above. The implication is that $t^{(R)}_{sC}$ is well behaved in this limit. Hence in Eq. (3.47) only the scattering states $|\vec{q}^{\,(\pm)}_{\alpha,R}\rangle$ for $\alpha=1$ and 2 develop diverging factors which are, according to Eq. (3.18), known to be the same as in the two-body case. Thus, after having renormalized the on-shell amplitude (3.47) the limit $R\to\infty$ can be performed

$$\lim_{R\to\infty} Z_{R,\beta}{}^{-1/2}(\bar{q}\,'_\beta, M_\beta)\mathcal{T}^{(R)}_{sC,\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha; E+i0) \times Z_{R,\alpha}{}^{-1/2}(\bar{q}_\alpha, M_\alpha) = \mathcal{T}_{sC,\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha;\ E+i0) = \langle\vec{q}\,'^{(-)}_{\beta,C}|t^{(\infty)}_{sC,\beta\alpha}(E+i0)|\vec{q}^{\,(+)}_{\alpha,C}\rangle, \quad (3.50)$$

yielding the Coulomb-modified strong amplitude corresponding to an unscreened Coulomb potential.

Finally, recalling Eq. (3.36) we have proven that on shell the limits

$$\lim_{R\to\infty} Z_{R,\beta}{}^{-1/2}(\bar{q}\,'_\beta, M_\beta)\mathcal{T}^{(R)}_{\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha;\ E+i0) \times Z_{R,\alpha}{}^{-1/2}(\bar{q}_\alpha, M_\alpha) = \{\mathcal{T}_{C,\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha)+\mathcal{T}_{sC,\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha;\ E+i0)\} = \mathcal{T}_{\beta\alpha}(\vec{q}\,'_\beta, \vec{q}_\alpha) \quad (3.51)$$

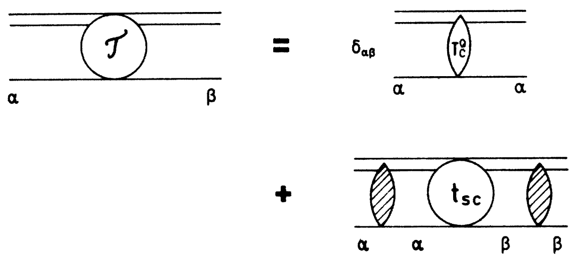

FIG. 4. The final result (3.51) for the elastic and rearrangement amplitudes. The first diagram on the right-hand side represents the two-body Coulomb amplitude (3.48) for the scattering of the charged particle α off the center of mass of the other two particles. The second diagram illustrates the Coulomb-modified strong amplitude (3.50), with the initial and final state Coulomb distortions being characterized by a shaded blob.

exist, and equal the amplitudes for elastic and rearrangement scattering of one neutral and two charged particles, interacting via short-range and unscreened Coulomb potentials. Their representation as a sum of the two contributions (3.48) and (3.50) is depicted in graphical form in Fig. 4.

We would like to emphasize that this approach provides us not only with a correct but also practical formalism for calculating Coulomb corrections in three-body systems, as demonstrated in Refs. 20–22. The decisive point is that, as for the scattering of two charged particles, the Coulomb-modified strong amplitude $\mathcal{T}_{sc}$ can be calculated via a partial wave expansion of, e.g., Eqs. (3.50) and (3.45). To this amplitude we then have to add coherently the analytically known two-particle Coulomb amplitude which in *not* calculable by a partial wave expansion (cf. the discussion at the beginning of Sec. II), in order to obtain the full amplitude via Eq. (3.51).

C. Breakup reactions

Let us envisage the situation of particle α impinging on the bound pair $(\beta+\gamma)$ leading to three free particles. The corresponding amplitude $\mathcal{T}^{(R)}_{0\alpha}(\vec{p}\,',\vec{q}\,';\vec{q}_\alpha)$ can be written as matrix element of an effective transition operator $\mathcal{T}^{(R)}_{0\alpha}$ between plane waves describing the relative motion of the incoming and outgoing particles,

$$\mathcal{T}^{(R)}_{0\alpha}(\vec{p}\,',\vec{q}\,';\vec{q}_\alpha)=\langle\vec{p}\,',\vec{q}\,'|\mathcal{T}^{(R)}_{0\alpha}(E+i0)|\vec{q}_\alpha\rangle . \quad (3.52)$$

This is the physical amplitude if the absolute values of the momenta p', q', and q_α are equal to their corresponding on-shell values defined by

$$E=E_\alpha=\bar{q}_\alpha{}^2/2M_\alpha+\hat{E}_\alpha=E_0'=\bar{p}'^2/2\mu+\bar{q}'^2/2M . \quad (3.53)$$

The relative momenta $(\vec{p}\,',\vec{q}\,')$ can be chosen as any one of the three equivalent sets $(\vec{p}_1{}',\vec{q}_1{}')$, $(\vec{p}_2{}',\vec{q}_2{}')$, or $(\vec{p}_3{}',\vec{q}_3{}')$ defined in Eqs. (3.9) and (3.11).

As is well known[14] the three-body breakup operators $\mathcal{T}^{(R)}_{0\alpha}$ can be calculated either by quadrature from the elastic and rearrangement amplitudes $\mathcal{T}^{(R)}_{\beta\alpha}$

$$\mathcal{T}^{(R)}_{0\alpha}(z)=\mathcal{V}^{(R)}_{0\alpha}(z)+\sum_\beta \mathcal{V}^{(R)}_{0\beta}(z)\mathcal{G}_{0;\beta}(z)\mathcal{T}^{(R)}_{\beta\alpha}(z) , \quad (3.54)$$

or with the help of the integral equation

$$\mathcal{T}^{(R)}_{0\alpha}(z)=\mathcal{V}^{(R)}_{0\alpha}(z)+\sum_\beta \mathcal{T}^{(R)}_{0\beta}(z)\mathcal{G}_{0;\beta}(z)\mathcal{V}^{(R)}_{\beta\alpha}(z) . \quad (3.55)$$

The elastic and rearrangement potentials $\mathcal{V}^{(R)}_{\beta\alpha}$ and amplitudes $\mathcal{T}^{(R)}_{\beta\alpha}$ are those discussed in the preceding subsection, and $\mathcal{G}_{0;\alpha}$ is the effective free Green's function (3.22). For separable nuclear potentials considered in this section [cf. Eq. (3.1)] the effective break-up potential $\mathcal{V}^{(R)}_{0\alpha}$ can again be given in an *exact* and *closed* form.[33] For this purpose it is most convenient to choose as the final-state variables those characterizing channel 3, since the relative momentum between the two charged particles is then simply $\vec{p}_3{}'$. In this case we have[39,28]

$$\mathcal{V}^{(R)}_{0\alpha}(\vec{p}_3{}',\vec{q}_3{}';\vec{q}_\alpha;z)=\langle\vec{q}_3{}'|\langle\vec{p}_3{}'|[1+T_R(z)G_0(z)]|\chi_\alpha\rangle|\vec{q}_\alpha\rangle \times[\hat{S}_\alpha(z-q_\alpha{}^2/2M_\alpha)]^{1/2} , \quad (3.56)$$

or when we make use of the off-shell Møller operators and scattering states, defined in Eq. (2.7),

$$\begin{aligned}\mathcal{V}^{(R)}_{0\alpha}(\vec{p}_3{}',\vec{q}_3{}';\vec{q}_\alpha;z)&=\langle\vec{q}_3{}'|\langle\vec{p}_3{}'|\Omega_R^\dagger(z^*)|\chi_\alpha,\vec{q}_\alpha\rangle\,[\hat{S}_\alpha(z-q_\alpha{}^2/2M_\alpha)]^{1/2}\\&=\langle\vec{p}_3{}'|\hat{\Omega}_R^\dagger(z^*-q_3'^2/2M_3)\langle\vec{q}_3{}'|\chi_\alpha,\vec{q}_\alpha\rangle\,[\hat{S}_\alpha(z-q_\alpha{}^2/2M_\alpha)]^{1/2}\\&=\langle\vec{p}_{3,R}{}'(z^*-q_3'^2/2M_3)|\langle\vec{q}_3{}'|\chi_\alpha,\vec{q}_\alpha\rangle\,[\hat{S}_\alpha(z-q_\alpha{}^2/2M_\alpha)]^{1/2} .\end{aligned} \quad (3.57)$$

The various terms occurring in the definition (3.56) are graphically displayed in Fig. 5. Note that on the energy shell the breakup potential (3.57) becomes simply [recall Eq. (3.7)]

$$\mathcal{V}^{(R)}_{0\alpha}(\vec{p}_3{}',\vec{q}_3{}';\vec{q}_\alpha;E+i0)=\langle\vec{q}_3{}'|\langle\vec{p}_{3,R}'^{(-)}|\chi_\alpha,\vec{q}_\alpha\rangle \quad (3.58)$$

and vanishes for $\alpha=3$ due to energy conservation ($\mathcal{V}^{(R)}_{03}$ corresponds to the first two diagrams of

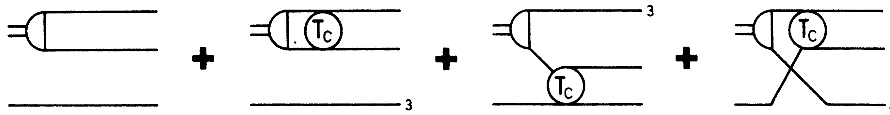

FIG. 5. The exact effective breakup potential (3.56) represented in diagrammatical form.

Fig. 5).

Our discussion of breakup processes will be based on Eq. (3.54), although all results could be derived also from Eq. (3.55). When the splitting (3.36) of $\mathcal{T}^{(R)}_{\beta\alpha}$ is inserted in Eq. (3.54) it induces a similar decomposition of $\mathcal{T}^{(R)}_{0\alpha}$,

$$\mathcal{T}^{(R)}_{0\alpha}=\tilde{\mathcal{T}}^{(R)}_{0\alpha}+\mathcal{T}^{(R)}_{sC,0\alpha}\,, \tag{3.59}$$

with

$$\tilde{\mathcal{T}}^{(R)}_{0\alpha}=\mathcal{V}^{(R)}_{0\alpha}+\sum_{\beta}\mathcal{V}^{(R)}_{0\beta}\mathcal{G}_{0;\beta}\tilde{\mathcal{T}}^{(R)}_{\beta\alpha} \tag{3.60}$$

and

$$\mathcal{T}^{(R)}_{sC,0\alpha}=\sum_{\beta}\mathcal{V}^{(R)}_{0\beta}\mathcal{G}_{0;\beta}\mathcal{T}^{(R)}_{sC,\beta\alpha}\,. \tag{3.61}$$

In order to interpret the transition operator $\tilde{\mathcal{T}}^{(R)}_{0\alpha}$ we make use of the form (3.39) for $\tilde{\mathcal{T}}^{(R)}_{\beta\alpha}$ and of the definition (3.42) of $\Omega^{(R)}_{\alpha}$, to write

$$\tilde{\mathcal{T}}^{(R)}_{0\alpha}(z)=\mathcal{V}^{(R)}_{0\alpha}(z)\Omega^{(R)}_{\alpha}(z)\,. \tag{3.62}$$

In momentum space this expression becomes, for on-shell values of the momenta, $q_3'=\bar{q}_3'$, $p_3'=\bar{p}_3'$, $q_\alpha=\bar{q}_\alpha$, very simple and transparent [recall Eq. (3.58) and the remark following it],

$$\begin{aligned}\tilde{\mathcal{T}}^{(R)}_{0\alpha}(\vec{p}_3{}',\vec{q}_3{}';\vec{q}_\alpha;E+i0)&=\bar{\delta}_{\alpha3}\int d^3q_\alpha'\,\langle\vec{p}^{\,\prime(-)}_{3,R}|\vec{k}\rangle\chi_\alpha(\vec{k}')\\&\quad\times[\hat{S}_\alpha(E_\alpha+i0-q_\alpha'^2/2M_\alpha)]^{1/2}\langle\vec{q}_\alpha{}'|\vec{q}^{\,(+)}_{\alpha,R}\rangle\,,\end{aligned} \tag{3.63}$$

with $\vec{k}$ and $\vec{k}'$ being the well-known linear combinations of $\vec{q}_3{}'$ and $\vec{q}_\alpha{}'$. It involves only a quadrature of genuine two-body screened Coulomb scattering wave functions and represents the pure screened Coulomb breakup.

The Coulomb-modified strong breakup amplitude $\mathcal{T}^{(R)}_{sC,0\alpha}$ can be handled in analogy to the nonbreakup case. Inserting expression (3.44) for $\mathcal{T}^{(R)}_{sC,\beta\alpha}$ into Eq. (3.61) suggests the introduction of the breakup analog of (3.45) via

$$\mathcal{T}^{(R)}_{sC,0\alpha}(z)=\Omega^{\dagger}_{R}(z^*)t^{(R)}_{sC,0\alpha}(z)\Omega^{(R)}_{\alpha}(z)\,. \tag{3.64}$$

It is related to $t^{(R)}_{sC,\beta\alpha}$ via

$$t^{(R)}_{sC,0\alpha}(z)=\sum_{\beta}|\chi_\beta\rangle[S_\beta(z)]^{1/2}\mathcal{G}^{(R)}_{\beta}(z)t^{(R)}_{sC,\beta\alpha}(z) \tag{3.65}$$

with $\mathcal{G}^{(R)}_{\beta}$ defined by Eq. (3.46). We remark that by making use of the integral equation (3.45) for $t^{(R)}_{sC,\beta\alpha}$ we obtain immediately an integral equation[40] for $t^{(R)}_{sC,0\alpha}$,

$$\begin{aligned}t^{(R)}_{sC,0\alpha}&=\sum_{\beta}|\chi_\beta\rangle[S_\beta(z)]^{1/2}\mathcal{G}^{(R)}_{\beta}\mathcal{V}'^{(R)}_{\beta\alpha}\\&\quad+\sum_{\beta}t^{(R)}_{sC,0\beta}\mathcal{G}^{(R)}_{\beta}\mathcal{V}'^{(R)}_{\beta\alpha}\end{aligned} \tag{3.66}$$

which would, of course, also follow if the integral equation (3.55) for the breakup amplitude $\mathcal{T}^{(R)}_{0\alpha}$ were used as a starting point. When writing Eq. (3.64) in momentum space its structure becomes again transparent. Indeed, for on-shell values of the momenta the Coulomb-modified strong breakup amplitude is given as

$$\begin{aligned}&\mathcal{T}^{(R)}_{sC,0\alpha}(\vec{p}_3{}',\vec{q}_3{}';\vec{q}_\alpha;E+i0)\\&\quad=\langle\vec{q}_3{}'|\langle\vec{p}^{\,\prime(-)}_{3,R}|t^{(R)}_{sC,0\alpha}(E+i0)|\vec{q}^{\,(+)}_{\alpha,R}\rangle\end{aligned} \tag{3.67}$$

clearly displaying the (screened) Coulomb distortion of the plane waves describing the relative motion of the different two charged particles in the initial[38] and final states.

The transition to unscreened Coulomb potentials is now as obvious as for the nonbreakup case. Since the kernel of Eq. (3.66) contains only the potential part $\mathcal{V}'^{(R)}_{\beta\alpha}$ which has no infinite range contributions, the limit to zero screening in $t^{(R)}_{sC,0\alpha}$ can be performed as it was the case for $t^{(R)}_{sC,\beta\alpha}$. Thus the only quantities where divergences for $R\to\infty$ occur in Eqs. (3.63) and (3.67) are the genuine two-body scattering states $|\vec{p}^{\,\prime(-)}_{3,R}\rangle$ and $|\vec{q}^{\,(+)}_{\alpha,R}\rangle$. We can, therefore, again apply [using Eqs. (2.10) and (3.18)] the renormalization procedure described above to find that on the energy shell[41]

$$\begin{aligned}&\lim_{R\to\infty}Z_R^{-1/2}(\bar{p}_3',\mu_3)\tilde{\mathcal{T}}^{(R)}_{0\alpha}(\vec{p}_3{}',\vec{q}_3{}';\vec{q}_\alpha;E+i0)\\&\quad\times Z_{R,\alpha}^{-1/2}(\bar{q}_\alpha,M_\alpha)\\&=\mathcal{T}_{C,0\alpha}(\vec{p}_3{}',\vec{q}_3{}';\vec{q}_\alpha)\\&=\bar{\delta}_{\alpha3}\langle\vec{q}_3{}'|\langle\vec{p}^{\,\prime(-)}_{3,C}|\chi_\alpha\rangle[S_\alpha(E_\alpha+i0)]^{1/2}|\vec{q}^{\,(+)}_{\alpha,C}\rangle\end{aligned} \tag{3.68}$$

and

$$\lim_{R\to\infty} Z_R^{-1/2}(\bar{p}_3', \mu_3)\mathcal{T}^{(R)}_{sC,0\alpha}(\vec{p}_3', \vec{q}_3'; \vec{q}_\alpha; E+i0) \times Z_{R,\alpha}^{-1/2}(\bar{q}_\alpha, M_\alpha)$$
$$= \mathcal{T}_{sC,0\alpha}(\vec{p}_3', \vec{q}_3'; \vec{q}_\alpha; E+i0)$$
$$= \langle \vec{q}_3' | \langle \vec{p}^{\,1(-)}_{3,C} | t^{(\infty)}_{sC,0\alpha}(E+i0) | \vec{q}^{\,(+)}_{\alpha,C} \rangle . \quad (3.69)$$

Here $\mathcal{T}_{C,0\alpha}$ and $\mathcal{T}_{sC,0\alpha}$ are the pure Coulomb and the Coulomb-modified strong breakup amplitudes for unscreened Coulomb potentials. The diverging phase factors Z_R and $Z_{R,\alpha}$ are defined in Eqs. (2.11) and (3.49), respectively. Putting these results together, we have proven that also the on-shell element of the full breakup amplitude (3.59) tends after renormalization towards the physical breakup amplitude for an unscreened Coulomb potential

$$\lim_{R\to\infty} Z_R^{-1/2}(\bar{p}_3', \mu_3)\mathcal{T}^{(R)}_{0\alpha}(\vec{p}_3', \vec{q}_3'; \vec{q}_\alpha; E+i0) \times Z_{R,\alpha}^{-1/2}(\bar{q}_\alpha, M_\alpha)$$
$$= [\mathcal{T}_{C,0\alpha}(\vec{p}_3', \vec{q}_3'; \vec{q}_\alpha) + \mathcal{T}_{sC,0\alpha}(\vec{p}_3', \vec{q}_3'; \vec{q}_\alpha; E+i0)]$$
$$= \mathcal{T}_{0\alpha}(\vec{p}_3', \vec{q}_3'; \vec{q}_\alpha) . \quad (3.70)$$

Equation (3.70) is represented in diagrammatical form in Fig. 6.

That this approach to breakup reactions is also practical will be demonstrated in a later publication. The decisive point is here as in the nonbreakup case that the Coulomb-modified strong amplitude $\mathcal{T}_{sC,0\alpha}$ can be calculated, e.g., by means of a partial wave expansion of (3.69) together with Eq. (3.65) or (3.66). And the pure Coulomb breakup described by (3.68), which would not be obtainable by a partial wave expansion, can, however, be evaluated, e.g., by quadrature of analytically known two-body Coulomb scattering wave functions.

D. Scattering states

Sometimes it is desirable to work with scattering wave functions instead of with scattering amplitudes. We, therefore, discuss how the formalism developed above can be implemented in a wave function approach to three-body processes.[42] Let $|\Psi^{(+)}_{\alpha,\vec{q}_\alpha,R}\rangle$ denote a scattering state characterized by a channel state $|\psi_\alpha\rangle|\vec{q}_\alpha\rangle$, with $|\psi_\alpha\rangle$ being the bound-state wave function of pair $(\beta+\gamma)$ and $\vec{q}_\alpha$ being the channel relative momentum. In the quasiparticle approach it can be obtained as a sum of three terms,[14]

$$|\Psi^{(+)}_{\alpha,\vec{q}_\alpha,R}\rangle = \sum_\beta G_0(E_\alpha+i0)\mathcal{V}^{(R)}_{0\beta}(E_\alpha+i0)|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta . \quad (3.71)$$

Here we have introduced the effective two-body states $|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta$ which are defined as

$$|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta = \lambda_\beta\langle\chi_\beta|\Psi^{(+)}_{\alpha,\vec{q}_\alpha,R}\rangle$$
$$= [\delta_{\beta\alpha} + \mathcal{G}_{0;\beta}(E_\alpha+i0)\mathcal{T}^{(R)}_{\beta\alpha}(E_\alpha+i0)]|\vec{q}_\alpha\rangle . \quad (3.72)$$

They represent that part of the scattering wave function in which particles α and γ, with $\alpha,\gamma\neq\beta$, emerge as a correlated pair (spectator wave functions), and are obviously only vectors in the relative momentum space of the two colliding frag-

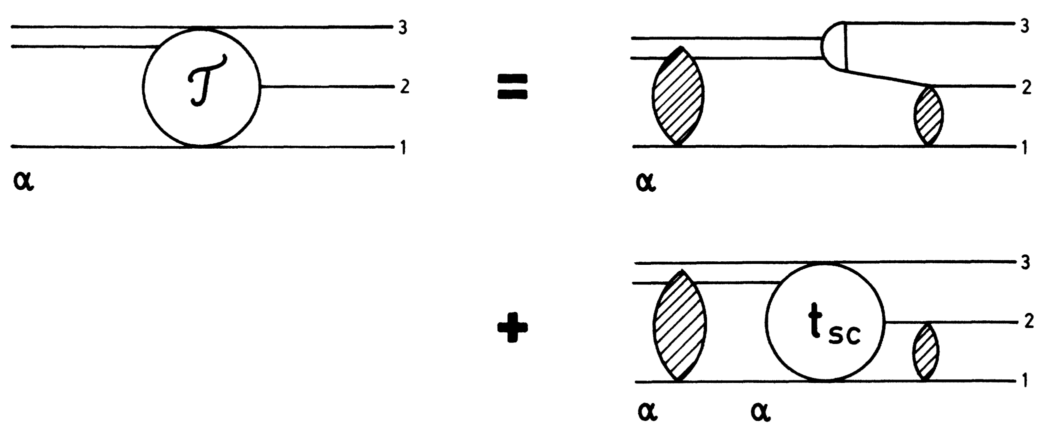

FIG. 6. The final result (3.70) for the breakup amplitudes described as a sum of the pure Coulomb and the Coulomb-modified strong breakup amplitudes (3.68) and (3.69), respectively. The large shaded blob has the same interpretation as in Fig. 4, whereas the small one indicates the Coulomb distortion between the charged particles 1 and 2.

ments. Note that Eq. (3.72) can be interpreted as a mapping of the plane waves $|\vec{q}_\alpha\rangle$ onto the effective two-body scattering states by means of effective two-body Møller operators, as has been emphasized in Ref. 43. With the help of the integral equation (3.19) for $\mathcal{T}^{(R)}$ one easily derives[14,43,44] an equation for these effective states,

$$|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta = \delta_{\beta\alpha}|\vec{q}_\alpha\rangle + \mathcal{G}_{0;\beta}(E_\alpha + i0) \times \sum_\gamma \mathcal{V}^{(R)}_{\beta\gamma}(E_\alpha + i0)|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\gamma . \quad (3.73)$$

Recall that in all equations (3.71)–(3.73) the magnitude of the momentum $\vec{q}_\alpha$ is fixed by the on-shell condition (3.21) to the value $|\vec{q}_\alpha| = \bar{q}_\alpha$. Inserting in Eq. (3.72) the splitting (3.36) for the amplitude $\mathcal{T}^{(R)}_{\beta\alpha}$ leads to the following expression[38]

$$|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta = \delta_{\beta\alpha}|\vec{q}^{\,(+)}_{\alpha,R}\rangle + \mathcal{G}^{(R)}_\beta(E_\alpha + i0)t^{(R)}_{sC,\beta\alpha}(E_\alpha + i0)|\vec{q}^{\,(+)}_{\alpha,R}\rangle \quad (3.74)$$

in which the long-range Coulomb distortion of the two incoming fragments is explicitly separated out by means of the (screened) Coulomb scattering states $|\vec{q}^{\,(+)}_{\alpha,R}\rangle$ introduced in Eq. (3.16). The effective two-body operators $\mathcal{G}^{(R)}_\beta$ and $t^{(R)}_{sC;\beta\alpha}$ are defined by Eqs. (3.46) and (3.45), respectively. When making use of the latter equation one can easily rewrite Eq. (3.74) into an integral equation for $|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta$,

$$|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta = \delta_{\beta\alpha}|\vec{q}^{\,(+)}_{\alpha,R}\rangle + \mathcal{G}^{(R)}_\beta(E_\alpha + i0) \times \sum_\gamma \mathcal{V}'^{(R)}_{\beta\gamma}(E_\alpha + i0)|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\gamma \quad (3.75)$$

which we could have derived also from the integral equation (3.73). Note that this result is nothing else than a distorted wave representation of the effective two-body states, with only the shorter-range part $\mathcal{V}'^{(R)}$ of the effective potential $\mathcal{V}^{(R)}$ occurring in the kernel.

The limit $R\to\infty$ is now most easily investigated by means of Eq. (3.74). We simply have to recall that, as discussed after Eq. (3.49), $t^{(R)}_{sC,\beta\alpha}$ is well behaved in that limit. Furthermore, the screened Coulomb Green's function $\hat{G}^Q_R$ occurring in $\mathcal{G}^{(R)}_\beta$ for $\beta = 1, 2$ [recall the definition (3.46)] goes over into the unscreened one. Thus there remains only the singular behavior arising from the screened Coulomb scattering states $|\vec{q}^{\,(+)}_{\alpha,R}\rangle$. Taking into account Eq. (3.18) we can conclude that after renormalization we obtain in the limit $R\to\infty$ the effective two-body wave function for an unscreened Coulomb potential,[38]

$$\lim_{R\to\infty} Z_{R,\alpha}^{-1/2}(\bar{q}_\alpha, M_\alpha)|\Psi^{(R)}_{\alpha,\vec{q}_\alpha}\rangle_\beta = |\Psi_{\alpha,\vec{q}_\alpha}\rangle_\beta = \delta_{\beta\alpha}|\vec{q}^{\,(+)}_{\alpha,C}\rangle + \mathcal{G}^{(\infty)}_\beta(E_\alpha + i0)t^{(\infty)}_{sC;\beta\alpha}(E_\alpha + i0)|\vec{q}^{\,(+)}_{\alpha,C}\rangle . \quad (3.76)$$

Going back to the definition (3.71) for the screened three-body scattering state $|\Psi^{(+)}_{\alpha,\vec{q}_\alpha,R}\rangle$ we find

$$\lim_{R\to\infty} Z_{R,\alpha}^{-1/2}(\bar{q}_\alpha, M_\alpha)|\Psi^{(+)}_{\alpha,\vec{q}_\alpha,R}\rangle = \sum_\beta G_0(E_\alpha + i0)\mathcal{V}^{(\infty)}_{0\beta}(E_\alpha + i0)|\Psi_{\alpha,\vec{q}_\alpha}\rangle_\beta = |\Psi^{(+)}_{\alpha,\vec{q}_\alpha}\rangle \quad (3.77)$$

which provides us with the desired three-body scattering wave function for an unscreened Coulomb potential. In the last step explicit use has been made of the form (3.56) for $\mathcal{V}^{(R)}_{0\beta}$ to show that

$$G_0(z)\mathcal{V}^{(R)}_{0\beta}(z) = G_R(z)|\chi_\beta\rangle[S_\beta(z)]^{1/2} . \quad (3.78)$$

And in the limit $R\to\infty$ the genuine two-body Green's functions G_R goes over into G_C.

It is worthwhile pointing to an interesting feature of the Coulomb scattering wave function constructed in our approach; namely, inspection of Eq. (3.76) reveals that the spectator wave functions $\langle\vec{\rho}'_\beta|\Psi_{\alpha,\vec{q}_\alpha}\rangle_\beta$ show the typical Coulomb-type behavior in the two-fragment relative variable $\vec{\rho}'_\beta$, starting at the threshold of channel β (for $\beta = 1, 2$). The Coulomb-type behavior which occurs in the variable $\vec{r}'_3$ between the charged particles 1 and 2 for energies above the breakup threshold is only introduced when going over to the full scattering wave function $\langle\vec{\rho}'_3, \vec{r}'_3|\Psi^{(+)}_{\alpha,\vec{q}_\alpha}\rangle$. The explicit separation of these effects might be advantageous for practical applications.

Similar equations can be obtained for the state $|\Psi^{(-)}_{\alpha,\vec{q}_\alpha}\rangle$ which is asymptotically characterized by a free wave $|\psi_\alpha\rangle|\vec{q}_\alpha\rangle$ and incoming spherical waves, and for the states $|\Psi^{(\pm)}_0\rangle$ characterized by the plane waves $|\vec{p}, \vec{q}\rangle$. Three-body bound-state wave functions follow, of course, from Eq. (3.71) in the limit $R\to\infty$, with the effective two-body states calculated from the homogeneous version of Eq. (3.72).

IV. THREE-PARTICLE SCATTERING WITH GENERAL SHORT-RANGE TWO-BODY POTENTIALS

In the preceding section we have developed a formalism for describing the scattering of three particles two of which are charged, but with the restrictions that the short-range forces are represented by rank-one separable potentials and that the Coulomb force is repulsive. These limitations are now shown to have been only of technical nature. We, indeed, demonstrate that the formulas derived before are valid, with unessential modifications, also for general short-range interactions and for attractive Coulomb potentials.

Two important differences to the simple case discussed in Sec. III emerge when general two-particle interactions are admitted. First, excitation of the bound systems in the initial and/or final state becomes possible. Secondly, there may also occur "channels" which do not correspond to asymptotically allowed configurations. This happens either if the energy is not high enough (energetically closed channels), or if the situations described by them are no physical channels at all (the latter will be termed unphysical channels; a well-known example is the "channel" $p+d^*$ in proton-deuteron scattering where d^* denotes the neutron-proton pair in the $^1S_0$ state). As is shown below the unphysical and the closed physical channels pose no problem whatsoever. That is, the need for generalizing the formalism developed in Sec. III is caused solely by the occurrence of the physical, energetically allowed channels.

For the following investigation we employ again the quasiparticle formalism of Ref. 14.

A. Repulsive Coulomb force

We first treat the situation of two equally charged particles which is of paramount interest for nuclear reactions.

1. Two-body input

Starting from arbitrary short-range two-body potentials V_γ^s we decompose them into a separable part and a remainder V_γ'. Thus for $\alpha=1$ and 2 we have

$$V_\alpha = \sum_{r=1}^{N_\alpha} |\chi_{\alpha r}\rangle \lambda_{\alpha r} \langle \chi_{\alpha r}| + V_\alpha' . \tag{4.1a}$$

In subsystem 3 consisting of the two charged particles 1 and 2, the (screened) Coulomb force[28] V_R acts in addition to V_3^s. Since we require V_R to be repulsive only the short-range force V_3^s is capable of producing bound states. Thus a suitable decomposition of the subsystem potential V_3 is[45]

$$V_3 = \sum_{r=1}^{N_3} |\chi_{3r}\rangle \lambda_{3r} \langle \chi_{3r}| + V_3' + V_R . \tag{4.1b}$$

The subsystem amplitudes corresponding to these potentials are[29]

$$\hat{T}_\alpha(z_\alpha) = \sum_{r,s=1}^{N_\alpha} |\hat{\Phi}_{\alpha r}(z_\alpha)\rangle \hat{\Delta}_{\alpha, rs}(z_\alpha) \langle \hat{\Phi}_{\alpha s}(z_\alpha^*)| + \hat{T}_\alpha'(z_\alpha) , \tag{4.2}$$

with $\hat{T}_\alpha'$ fulfilling the LS equations

$$\hat{T}_\alpha'(z_\alpha) = V_\alpha' + V_\alpha' \hat{G}_0(z_\alpha) \hat{T}_\alpha'(z_\alpha) , \quad \text{for } \alpha = 1, 2 , \tag{4.3a}$$

$$\hat{T}_3'(z_3) = (V_3' + V_R) + (V_3' + V_R) \hat{G}_0(z_3) \hat{T}_3'(z_3) . \tag{4.3b}$$

For the following it proves advantageous to rewrite Eq. (4.3b) in analogy to Eq. (2.14) in order to explicitly display the pure Coulomb contribution. This is again accomplished with the help of the two-potential formula yielding

$$\hat{T}_3'(z_3) = \hat{T}_R(z_3) + \hat{\Omega}_R^\dagger(z_3^*) \hat{t}_{sC,3}'(z_3) \hat{\Omega}_R(z_3) . \tag{4.4}$$

Here $\hat{T}_R(z)$ and $\hat{\Omega}_R(z)$ are the transition operator and the off-shell Møller operator for a screened Coulomb potential, introduced in Eq. (2.7), and $\hat{t}_{sC,3}'$ fulfills the LS equation (2.17) with the short-range potential V_3'. The form factors $|\hat{\Phi}_{\alpha r}(z_\alpha)\rangle$ occurring in Eq. (4.2) are defined for $\alpha = 1$ and 2 by

$$|\hat{\Phi}_{\alpha r}(z_\alpha)\rangle = [1 + \hat{T}_\alpha'(z_\alpha) \hat{G}_0(z_\alpha)] |\chi_{\alpha r}\rangle , \tag{4.5a}$$

and for $\alpha = 3$ by

$$\begin{aligned} |\hat{\Phi}_{3r}(z_3)\rangle &= [1 + \hat{T}_3'(z_3) \hat{G}_0(z_3)] |\chi_{3r}\rangle \\ &= \hat{\Omega}_R^\dagger(z_3^*) [1 + \hat{t}_{sC,3}'(z_3) \hat{G}_R(z_3)] |\chi_{3r}\rangle \end{aligned} \tag{4.5b}$$

[G_R is the screened Coulomb Green's function (2.18)]. Finally, the elements of the matrix $\hat{\Delta}_\alpha$ are determined by

$$[\hat{\Delta}_\alpha^{-1}(z_\alpha)]_{rs} = \lambda_{\alpha r}^{-1} \delta_{rs} - \langle \chi_{\alpha r} | \hat{G}_0(z_\alpha) | \hat{\Phi}_{\alpha s}(z_\alpha) \rangle \tag{4.6}$$

for $\alpha = 1, 2, 3$.

Let us make a few remarks. The first one concerns the number of separable terms in Eq. (4.1). Assume the existence of n_α stable bound states in subsystem α. Then for the quasiparticle approach to be applicable the number N_α of separable terms in the decomposition (4.1), which determines the size of the matrix equation (4.14) below, must be at least as large as n_α so that the remainder V_α' is too weak for producing any bound states. However, in practical applications it might be advisable to choose $N_\alpha > n_\alpha$ (this aspect is discussed in detail in Ref. 46). Then these additional separable terms give rise to $(N_\alpha - n_\alpha)$ "unphysical" channels. The second remark concerns the special case that the short-range potential in subsystem α is a separable potential of rank M_α. The simplest possibility[47] is to choose $N_\alpha = M_\alpha$ which implies $V_\alpha' = 0$. From Eqs. (4.3) it then follows that $\hat{T}_\alpha' = 0$ for $\alpha = 1$ and 2, and $\hat{T}_3' = \hat{T}_R$. This situation is clearly included in the above formalism. Finally, we draw the attention to a simplifying change in notation as compared to Sec. III. In contrast to the convention observed there, any implicit dependence of various subsystem quantities on the screening radius is no longer indicated by a superscript R. That is, only a subscript R is used, as before, to label the pure screened Coulomb quantities.

In order to proceed further it is necessary to briefly recapitulate[48] the properties of $\hat{\Delta}_\alpha(z)$. Let us denote the binding energies and wave functions of the n_α possible bound states in subsystem α by $\hat{E}_{\alpha r}$ and $|\psi_{\alpha r}\rangle$, $r=1,\ldots,n_\alpha$ $(n_\alpha \leq N_\alpha)$, respectively. We assume that the first n_α separable terms in (4.1) are adequately chosen so that all bound state poles of $\hat{T}_\alpha$ show up in $\hat{\Delta}_\alpha$, i.e., $\hat{T}'_\alpha$ is nonpolar. For the following it proves advantageous to require for the form factors the conditions[49]

$$\lambda_{\alpha r}\langle\chi_{\alpha r}|\psi_{\alpha s}\rangle=\delta_{rs}, \quad \text{for } r=1,\ldots,N_\alpha , \tag{4.7}$$

which imply the subsequent relations at the energy $\hat{E}_{\alpha r}$

$$\begin{aligned}|\psi_{\alpha r}\rangle&=\hat{G}_0(\hat{E}_{\alpha r})[1+\hat{T}'_\alpha(\hat{E}_{\alpha r})\hat{G}_0(\hat{E}_{\alpha r})]|\chi_{\alpha r}\rangle\\ &=\hat{G}_0(\hat{E}_{\alpha r})|\hat{\Phi}_{\alpha r}(\hat{E}_{\alpha r})\rangle .\end{aligned} \tag{4.8}$$

As a consequence of (4.7), $\hat{\Delta}_{\alpha,rs}(z_\alpha)$ behaves for z_α in the neighborhood of a binding energy $\hat{E}_{\alpha m}$ as

$$\hat{\Delta}_{\alpha,rs}(z_\alpha)\underset{\sim}{{}^{z_\alpha\to\hat{E}_{\alpha m}}}\frac{\delta_{rm}\delta_{ms}}{z_\alpha-\hat{E}_{\alpha m}}, \quad \text{for } m=1,\ldots,n_\alpha . \tag{4.9}$$

That is, the matrix $\hat{\Delta}_\alpha$ contains the bound state poles *only* in its first n_α diagonal elements; all other matrix elements $\hat{\Delta}_{\alpha,rs}$ being, therefore, nonpolar. In order to clearly expose this important fact we rewrite these terms as [cf. Eq. (3.5)]

$$\hat{\Delta}_{\alpha,rr}(z_\alpha)\equiv\frac{\hat{S}_{\alpha,r}(z_\alpha)}{z_\alpha-\hat{E}_{\alpha r}} \quad \text{for } r=1,\ldots,n_\alpha \tag{4.10}$$

with

$$\hat{S}_{\alpha,r}(\hat{E}_{\alpha r})=1 , \tag{4.11}$$

leaving all other elements of $\hat{\Delta}_\alpha$ unchanged.

2. *Elastic, inelastic, and rearrangement scattering*

Also for arbitrary short-range two-body interactions the elastic, inelastic and rearrangement amplitudes $\mathcal{T}^{(R)}_{\beta n,\alpha m}(\vec{q}'_\beta,\vec{q}_\alpha)$ can be expressed as matrix elements of effective two-particle transition operators between plane wave states

$$\mathcal{T}^{(R)}_{\beta n,\alpha m}(\vec{q}'_\beta,\vec{q}_\alpha)=\langle\vec{q}'_\beta|\mathcal{T}^{(R)}_{\beta n,\alpha m}(E+i0)|\vec{q}_\alpha\rangle . \tag{4.12}$$

Here the relative momenta of the two fragments in the initial and the final states are restricted to their on-shell values $q'_\beta=\bar{q}'_{\beta n}$ and $q_\alpha=\bar{q}_{\alpha m}$ defined by

$$E=E'_{\beta n}=\frac{\bar{q}'^2_{\beta n}}{2M_\beta}+\hat{E}_{\beta n}=E_{\alpha m}=\frac{\bar{q}_{\alpha m}{}^2}{2M_\alpha}+\hat{E}_{\alpha m} . \tag{4.13}$$

The indices n and m denote collectively all quantum numbers which are necessary for a complete characterization of the subsystem bound states, and are numbered by $1,\ldots,n_\alpha$ for m and by $1,\ldots,n_\beta$ for n.

The transition operators $\mathcal{T}^{(R)}_{\beta n,\alpha m}$ fulfill the multichannel, effective two-body LS equations

$$\begin{aligned}\mathcal{T}^{(R)}_{\beta n,\alpha m}(z)&=\mathcal{V}^{(R)}_{\beta n,\alpha m}(z)\\ &\quad+\sum_\gamma\sum_{k,l=1}^{N_\gamma}\mathcal{V}^{(R)}_{\beta n,\gamma k}(z)\mathcal{G}_{0;\gamma,kl}(z)\mathcal{T}^{(R)}_{\gamma l,\alpha m}(z)\\ &=\mathcal{V}^{(R)}_{\beta n,\alpha m}(z)\\ &\quad+\sum_\gamma\sum_{k,l=1}^{N_\gamma}\mathcal{T}^{(R)}_{\beta n,\gamma k}(z)\mathcal{G}_{0;\gamma,kl}(z)\mathcal{V}^{(R)}_{\gamma l,\alpha m}(z) .\end{aligned} \tag{4.14}$$

Note that the dimension of these matrix relations is determined by the numbers N_β and N_α of separable terms in Eq. (4.1). According to our convention the $n_\beta\times n_\alpha$ physical amplitudes occupy the upper n_β rows and n_α columns of the matrix $\mathcal{T}^{(R)}_{\beta n,\alpha m}$. The effective free Green's function $\mathcal{G}_0$ is given by[50]

$$\langle\vec{q}'_\alpha|\mathcal{G}_{0;\alpha,kl}(z)|\vec{q}_\alpha\rangle=\delta(\vec{q}'_\alpha-\vec{q}_\alpha)\hat{\Delta}_{\alpha,kl}(z-q_\alpha{}^2/2M_\alpha) , \tag{4.15}$$

generalizing Eq. (3.22). From the above discussion of the singularity structure of $\hat{\Delta}_\alpha$ follows that the diagonal elements $\langle\vec{q}'_\alpha|\mathcal{G}_{0;\alpha,kk}(z)|\vec{q}_\alpha\rangle$ are proportional to $(z-q_\alpha{}^2/2M_\alpha-\hat{E}_{\alpha k})^{-1}$ for $k=1,\ldots,n_\alpha$, whereas no other element can become polar.

The effective potential $\mathcal{V}^{(R)}_{\beta n,\alpha m}$ is now much more complicated than in Sec. III. In fact, it is itself determined by the solution of an integral equation. Let the three-body operators $U'_{\beta\alpha}$ fulfill Faddeev-type equations with the nonpolar subsystem amplitudes T'_γ introduced in Eq. (4.2):

$$U'_{\beta\alpha}=\bar{\delta}_{\beta\alpha}G_0{}^{-1}+\sum_\gamma\bar{\delta}_{\beta\gamma}T'_\gamma G_0U'_{\gamma\alpha} . \tag{4.16}$$

Then the matrix elements of $U'_{\beta\alpha}$ between the modified form factors (4.5) yield $\mathcal{V}^{(R)}_{\beta n,\alpha m}$ via

$$\langle\vec{q}'_\beta|\mathcal{V}^{(R)}_{\beta n,\alpha m}(z)|\vec{q}_\alpha\rangle=\langle\vec{q}'_\beta|\langle\hat{\Phi}_{\beta n}(z^*-q'^2_\beta/2M_\beta)|G_0(z)U'_{\beta\alpha}(z)G_0(z)|\hat{\Phi}_{\alpha m}(z-q_\alpha{}^2/2M_\alpha)\rangle|\vec{q}_\alpha\rangle . \tag{4.17}$$

We mention, however, that it is not necessary to solve the integral Eq. (4.16) exactly. Instead, expanding $U'_{\beta\alpha}$ in powers of T'_γ and inserting it in the definition (4.17) yields a practical calculational scheme, the so-called quasi-Born expansion of the effective potential[51] (the question of the convergence of the quasi-Born series for short-range potentials is discussed in Ref. 46). In the present investigation we are, however, interested only in the problems arising from the additional action of the Coulomb force between the two charged particles.

Let us briefly sketch the further procedure which follows closely the one employed in Sec. III. We first demonstrate that for $R\to\infty$ the kernel of the LS Eq. (4.14) develops a "Coulomb-type singularity." When investigating its most singular part we will find that, despite the formal complications due to the complexity of $\mathcal{V}^{(R)}$ and $\mathcal{G}_0$, it can be isolated in a form which very much resembles Eq. (2.5). Thus again the renormalization approach presented in Sec. II can be applied to define the physical amplitudes for the various scattering processes with an unscreened Coulomb potential.

From the preceding discussion of the pole structure of $\mathcal{G}_0$ it is evident that a singular behavior of the kernel

$$\mathcal{K}^{(R)}_{\beta n,\alpha m}(z)=\sum_r \mathcal{V}^{(R)}_{\beta n,\alpha r}(z)\mathcal{G}_{0;\alpha,rm}(z) \qquad (4.18)$$

in the limit $R\to\infty$ can occur only for values of the index m specifying the physical channels[52]; i.e., it suffices to consider m in the range from 1 to n_α.

In the Appendix we demonstrate in detail by iterating Eq. (4.16) for $U'_{\beta\alpha}$ that only one term of the resulting quasi-Born expansion of $\mathcal{V}^{(R)}$ can become as singular as the Coulomb potential in the zero screening limit. It is, in fact, the direct generalization of Eq. (3.24), graphically represented by diagram d of Fig. 1 and analytically given by the expression

$$\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}V_R(\vec{q}\,'_\alpha-\vec{q}_\alpha)F_{\alpha,nm}(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)\,. \qquad (4.19)$$

Denoting the momentum space representation of the modified form factors by

$$\langle\vec{k}\,|\hat{\Phi}_{\alpha m}(z-q_\alpha{}^2/2M_\alpha)\rangle=\hat{\Phi}_{\alpha m}(\vec{k};z-q_\alpha{}^2/2M_\alpha)$$

and by

$$\langle\hat{\Phi}_{\alpha n}(z^*-q'^2_\alpha/2M_\alpha)\,|\vec{k}\rangle=\hat{\Phi}^*_{\alpha n}(\vec{k};z^*-q'^2_\alpha/2M_\alpha)\,,$$

the quantity $F_{\alpha,nm}(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)$ is defined by [cf. Eq. (3.26)]

$$F_{\alpha,nm}(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)=\int\frac{d^3k\,\hat{\Phi}^*_{\alpha n}(\vec{k};z^*-q'^2_\alpha/2M_\alpha)\hat{\Phi}_{\alpha m}(\vec{k}+\vec{D}_\alpha;z-q_\alpha{}^2/2M_\alpha)}{[k^2/2\mu_\alpha-(z-q'^2_\alpha/2M_\alpha)][(\vec{k}+\vec{D}_\alpha)^2/2\mu_\alpha-(z-q_\alpha{}^2/2M_\alpha)]}\,, \qquad (4.20)$$

with $\vec{D}_\alpha$ introduced in Eq. (3.27). By combining the result (4.19) with the form (4.15) for $\mathcal{G}_0$ [recall Eq. (4.10)], it becomes apparent that this part of the kernel (4.18) exhibits for $R\to\infty$ a singular behavior which is characteristically similar to the kernel (2.5) for two-charged-particle scattering.

We point out that for on-shell values of the momenta, $|\vec{q}\,'_\alpha|=\bar{q}'_{\alpha n}$ and $|\vec{q}_\alpha|=\bar{q}_{\alpha m}$ $(n,m=1,\ldots,n_\alpha)$, which make the channel energies $E'_{\alpha n}$ and $E_{\alpha m}$ equal [cf. Eq. (4.13)], the quantity $F_{\alpha,nm}(\vec{q}\,'_\alpha,\vec{q}_\alpha;E_{\alpha m}=E'_{\alpha n})$ is the transition form factor leading from the bound state m to the bound state n of the pair $(\beta+\gamma)$ [to derive this result use must be made of the relation (4.8)]. The elastic form factors $F_{\alpha,mm}(\vec{q}\,'_\alpha,\vec{q}_\alpha;E_{\alpha m})$ are, as usual, normalized to one for zero momentum transfer. This fact enables us to further simplify the most singular part of the kernel (4.18).

To do so it is necessary to investigate in more detail that part of $\mathcal{K}^{(R)}_{\beta n,\alpha m}$ which arises from the potential term (4.19). In a compact notation it can be written as

$$\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}V_R(\vec{q}\,'_\alpha-\vec{q}_\alpha)(z-q_\alpha{}^2/2M_\alpha-\hat{E}_{\alpha m})^{-1}\times R_{\alpha,nm}(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)\,, \qquad (4.21)$$

where we have introduced the smooth nonpolar function

$$R_{\alpha,nm}(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)=\sum_r F_{\alpha,nr}(\vec{q}\,'_\alpha,\vec{q}_\alpha;z)\times\hat{\Delta}_{\alpha,rm}(z-q_\alpha{}^2/2M_\alpha)\times(z-q_\alpha{}^2/2M_\alpha-\hat{E}_{\alpha m})\,. \qquad (4.22)$$

We first of all note that no Coulomb-type singularities can occur in the nondiagonal terms of (4.21). In the case of nondegenerate energy levels, if $\hat{E}_{\alpha m}\neq\hat{E}_{\alpha n}$ for $n\neq m$, the on-shell momenta $\bar{q}_{\alpha m}$ and $\bar{q}'_{\alpha n}$ are different. Thus, the singularities of $V_R(\vec{q}\,'_\alpha-\vec{q}_\alpha)$ for $R\to\infty$ and of $[z-q_\alpha{}^2/2M_\alpha-\hat{E}_{\alpha m}]^{-1}$ do *not* coincide on the energy shell. For degenerate levels where $\bar{q}_{\alpha m}$ equals $\bar{q}'_{\alpha n}$ for $n\neq m$, the nondiagonal elements of $R_{\alpha,nm}$ even vanish on shell. For, in the sum in Eq. (4.22) only the contribution with $r=m$ survives due to the pole behavior (4.10) of $\hat{\Delta}_{\alpha,rm}$, but then $F_{\alpha,nm}$ vanishes due to the orthogonality of the bound-state wave functions. Accordingly only the diagonal terms of (4.21) remain to be investigated. Here we know from the above discussion [recall Eq. (4.11)] that for $\vec{q}_\alpha'=\vec{q}_\alpha$ and $|\vec{q}_\alpha|=\bar{q}_{\alpha m}$

$$R_{\alpha,mm}(\vec{q}_\alpha,\vec{q}_\alpha;E_{\alpha m})=F_{\alpha,mm}(\vec{q}_\alpha,\vec{q}_\alpha;E_{\alpha m})=1\ . \tag{4.23}$$

This suggests the following decomposition[53]

$$R_{\alpha,nm}(\vec{q}_\alpha',\vec{q}_\alpha;z)=\delta_{nm}+[R_{\alpha,nm}(\vec{q}_\alpha,\vec{q}_\alpha;z)-\delta_{nm}] +[R_{\alpha,nm}(\vec{q}_\alpha',\vec{q}_\alpha;z) - R_{\alpha,nm}(\vec{q}_\alpha,\vec{q}_\alpha;z)]\ . \tag{4.24}$$

The first bracket on the right-hand side vanishes for $n=m$ and $|\vec{q}_\alpha|=\bar{q}_{\alpha m}$ for physical energies $z=E_{\alpha m}+i0$, whereas the second one vanishes for $\vec{q}_\alpha'=\vec{q}_\alpha$. Thus, inserting the decomposition (4.24) in (4.21) we find that its first term gives rise to the most singular part of $\mathcal{K}^{(R)}$, which we denote $\tilde{\mathcal{K}}^{(R)}$

$$\tilde{\mathcal{K}}^{(R)}_{\beta n,\alpha m}(\vec{q}_\beta',\vec{q}_\alpha;z)=\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\delta_{nm}V_R(\vec{q}'_\alpha-\vec{q}_\alpha)\times[z-q_\alpha^2/2M_\alpha-\hat{E}_{\alpha m}]^{-1} =\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\delta_{nm}\langle\vec{q}_\alpha'|V_R^Q\hat{G}_0^Q(z-\hat{E}_{\alpha m})|\vec{q}_\alpha\rangle \tag{4.25}$$

[for the occurrence of V_R^Q see the discussion following Eq. (3.24)]. We recall that here the range of m is restricted to the values $1,\ldots,n_\alpha$. This result entails, in complete analogy to Sec. III, a splitting of the kernel $\mathcal{K}^{(R)}$ into two terms,

$$\mathcal{K}^{(R)}_{\beta n,\alpha m}(z)=\tilde{\mathcal{K}}^{(R)}_{\beta n,\alpha m}(z)+\mathcal{K}'^{(R)}_{\beta n,\alpha m}(z)\ , \tag{4.26}$$

the first of which becomes as singular in the zero screening limit as the genuine two-body kernel (2.5) while the second one is well behaved.

We, therefore, can apply the machinery developed in Sec. III also to the present general case. This is most easily done for

$$\mathcal{F}^{(R)}_{\beta n,\alpha m}(z)=\sum_{r=1}^{N_\alpha}\mathcal{T}^{(R)}_{\beta n,\alpha r}(z)\mathcal{G}_{0;\,\alpha,rm}(z) \tag{4.27}$$

for which the LS equation (4.14) takes (in matrix notation) the simple form

$$\mathcal{F}^{(R)}=\mathcal{K}^{(R)}+\mathcal{K}^{(R)}\mathcal{F}^{(R)}=\mathcal{K}^{(R)}+\mathcal{F}^{(R)}\mathcal{K}^{(R)}\ . \tag{4.28}$$

Use of the decomposition (4.26) then allows $\mathcal{F}^{(R)}$ to be written as a sum of two terms

$$\mathcal{F}^{(R)}=\tilde{\mathcal{F}}^{(R)}+\mathcal{F}^{(R)}_{sC}\ . \tag{4.29}$$

Here $\tilde{\mathcal{F}}^{(R)}$ is defined by the LS equation

$$\tilde{\mathcal{F}}^{(R)}=\tilde{\mathcal{K}}^{(R)}+\tilde{\mathcal{K}}^{(R)}\tilde{\mathcal{F}}^{(R)}=\tilde{\mathcal{K}}^{(R)}+\tilde{\mathcal{F}}^{(R)}\tilde{\mathcal{K}}^{(R)}\ , \tag{4.30}$$

which is quite simple on account of the structure of its kernel (4.25). Hence its solution is easily found to be [cf. Sec. III, Eqs. (3.39)–(3.41)]

$$\tilde{\mathcal{F}}^{(R)}_{\beta n,\alpha m}(z)=\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\delta_{nm}\hat{T}_R^Q(z-\hat{E}_{\alpha m})\hat{G}_0^Q(z-\hat{E}_{\alpha m})\ ,\quad m=1,\ldots,n_\alpha \tag{4.31}$$

with the quantities $\hat{T}_R^Q$ and $\hat{G}_0^Q$ introduced in Eqs. (3.14) and (3.12), respectively.

The second term $\mathcal{F}^{(R)}_{sC}$ then admits a representation analogous to Eq. (3.38). Here, however, we aim immediately at the LS equation fulfilled by $\mathcal{F}^{(R)}_{sC}$. For this purpose it proves convenient to generalize the distortion operator (3.42) as

$$\Omega^{(R)}_{\alpha m}(z)=[1+\tilde{\mathcal{F}}^{(R)\dagger}(z^*)]_{\alpha m,\alpha m}\ , \tag{4.32}$$

or explicitly [recall Eq. (3.15)]

$$\Omega^{(R)}_{\alpha m}(z)=\begin{cases}\hat{\Omega}_R^Q(z-\hat{E}_{\alpha m}) & \text{for } \alpha=1,2, \text{ and } m=1,\ldots,n_\alpha\ ,\\ 1 & \text{for } \alpha=1,2 \text{ and } m=n_\alpha+1,\ldots,N_\alpha\ ,\\ 1 & \text{for } \alpha=3\ .\end{cases} \tag{4.33}$$

It represents an off-shell (screened) Coulomb Møller operator for the (open) physical channels describing the scattering of one of the charged particles relative to the center of mass of the other two. With its help we find the integral equation

$$\mathcal{F}^{(R)}_{sC;\,\beta n,\alpha m}(z)=\Omega^{(R)\dagger}_{\beta n}(z^*)\mathcal{K}'^{(R)}_{\beta n,\alpha m}(z)\Omega^{(R)\dagger}_{\alpha m}(z^*) +\Omega^{(R)\dagger}_{\beta n}(z^*)\sum_{\gamma r}\mathcal{K}'^{(R)}_{\beta n,\gamma r}(z)\mathcal{F}^{(R)}_{sC;\,\gamma r,\alpha m}(z)\ . \tag{4.34}$$

The above results can be converted into relations for the transition amplitudes by multiplying them from the right with $\mathcal{G}_0^{-1}$. In the case of Eq. (4.29) this leads to the decomposition

$$\mathcal{T}^{(R)}(z)=\tilde{\mathcal{T}}^{(R)}(z)+\mathcal{T}^{(R)}_{sC}(z)\ . \tag{4.35}$$

From Eq. (4.31) then follows that on the energy shell (4.13)

$$\langle\vec{q}_\beta'|\tilde{\mathcal{T}}^{(R)}_{\beta n,\alpha m}(E+i0)|\vec{q}_\alpha\rangle =\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\delta_{nm}\hat{T}_R^Q(\vec{q}_\alpha',\vec{q}_\alpha;\bar{q}_{\alpha m}^{\,2}/2M_\alpha+i0) \tag{4.36}$$

for the (open) physical channels ($m=1,\ldots,n_\alpha$). And the Coulomb-modified strong amplitude $\mathcal{T}^{(R)}_{sC}$ can again be written as the matrix element of an operator $t^{(R)}_{sC}$ in the (screened) Coulomb representation

$$\langle \vec{q}'_\beta | \mathcal{T}^{(R)}_{sC;\beta n,\alpha m}(E+i0) | \vec{q}_\alpha \rangle = \langle \vec{q}'^{(-)}_{\beta,R} | t^{(R)}_{sC;\beta n,\alpha m}(E+i0) | \vec{q}^{(+)}_{\alpha,R} \rangle \ . \tag{4.37}$$

The scattering states[38] $|\vec{q}^{(\pm)}_{\alpha,R}\rangle$ with $q_\alpha = \bar{q}_{\alpha m}$ have been defined in Eq. (3.16). The quantity $t^{(R)}_{sC;\beta n,\alpha m}$ fulfills for $m=1,\ldots,n_\alpha$ a LS equation of the type (3.45) with an effective potential

$$\mathcal{V}'^{(R)}_{\beta n,\alpha m}(z) = \mathcal{K}'^{(R)}_{\beta n,\alpha m}(z)[\hat{G}^Q_0(z-\hat{E}_{\alpha m})]^{-1} \ , \tag{4.38}$$

containing only the shorter-range Coulomb contributions, and a kernel

$$\sum_{\gamma r} \mathcal{K}'^{(R)}_{\beta n,\gamma r}(\delta_{\gamma\alpha}\delta_{rm} + \bar{\mathcal{F}}^{(R)}_{\gamma r,\alpha m}) = \begin{cases} \mathcal{V}'^{(R)}_{\beta n,\alpha m}\hat{G}^Q_R(z-\hat{E}_{\alpha m}) & \text{for } \alpha=1,2 \text{ and } m=1,\ldots,n_\alpha \\ \mathcal{K}'^{(R)}_{\beta n,\alpha m} & \text{for } \alpha=1,2, \text{ and } m=n_\alpha+1,\ldots,N_\alpha \ , \\ \mathcal{K}'^{(R)}_{\beta n,\alpha m} & \text{for } \alpha=3 \ . \end{cases} \tag{4.39}$$

With the help of Eqs. (4.36) and (4.37) the performance of the limit screening radius R going to infinity is an easy task. Renormalizing the on-shell amplitudes for the (open) physical channels $\tilde{\mathcal{T}}^{(R)}_{\beta n,\alpha m}(\vec{q}'_\beta,\vec{q}_\alpha;E+i0)$ and $\mathcal{T}^{(R)}_{sC;\beta n,\alpha m}(\vec{q}'_\beta,\vec{q}_\alpha;E+i0)$ with the phase factors $Z_{R,\beta}{}^{-1/2}(\bar{q}'_{\beta n},M_\beta)$ and $Z_{R,\alpha}{}^{-1/2}(\bar{q}_{\alpha m},M_\alpha)$, introduced in Eq. (3.49), we find that they tend for $R\to\infty$ towards the pure two-body elastic Coulomb amplitudes and the Coulomb modified strong amplitudes, respectively, for unscreened Coulomb potentials. The same holds true, of course, also for the full amplitude (4.35). We, thus, have succeeded in generalizing the formalism of Sec. III B to arbitrary short-range potentials leading to a well-defined theory of elastic, and rearrangement processes including Coulomb effects.

3. Breakup reactions

Breakup processes can be treated in analogy to Sec. III. Again we start from an equation which expresses the breakup amplitudes by means of the nonbreakup operators [cf. Eq. (3.54)]

$$\mathcal{T}^{(R)}_{0,\alpha m}(z) = \mathcal{V}^{(R)}_{0,\alpha m}(z) + \sum_\beta \sum_{r,s=1}^{N_\beta} \mathcal{V}^{(R)}_{0,\beta r}(z)\mathcal{G}_{0;\beta,rs}(z)\mathcal{T}^{(R)}_{\beta s,\alpha m}(z) \ . \tag{4.40}$$

The amplitudes for the transition from the two-fragment configuration (α,m) to the final three-free particle channel is then given by the matrix elements of $\mathcal{T}^{(R)}_{0,\alpha m}$ between the corresponding plane wave states

$$\mathcal{T}^{(R)}_{0,\alpha m}(\vec{p}',\vec{q}';\vec{q}_\alpha) = \langle \vec{p}',\vec{q}' | \mathcal{T}^{(R)}_{0,\alpha m}(E+i0) | \vec{q}_\alpha \rangle \ , \tag{4.41}$$

provided the momenta fulfill the on-shell condition

$$E = E_{\alpha m} = \bar{q}_{\alpha m}{}^2/2M_\alpha + \hat{E}_{\alpha m} = E'_0 = \frac{\bar{p}'^2}{2\mu} + \frac{\bar{q}'^2}{2M} \ . \tag{4.42}$$

Here $\{\vec{p}',\vec{q}'\}$ can be any one of the equivalent sets of momenta $\{\vec{p}'_\alpha,\vec{q}'_\alpha\}$ defined in Eqs. (3.9) and (3.11). However, since particle three is neutral, the choice $\{\vec{p}'_3,\vec{q}'_3\}$ is the most appropriate one. The nonbreakup amplitudes $\mathcal{T}^{(R)}_{\beta n,\alpha m}$ and the effective free Green's function $\mathcal{G}_{0;\beta,nm}$ are those discussed before.

Similar to the rearrangement potential (4.17) also the breakup potential $\mathcal{V}^{(R)}_{0,\alpha m}$ is defined[50] by the solution $U'_{0\alpha}$ of the Faddeev-type equation (4.16),

$$\langle \vec{p}'_3,\vec{q}'_3 | \mathcal{V}^{(R)}_{0,\alpha m}(z) | \vec{q}_\alpha \rangle = \langle \vec{p}'_3,\vec{q}'_3 | U'_{0\alpha}(z)G_0(z) | \Phi_{\alpha m}(z - q_\alpha{}^2/2M_\alpha)\rangle | \vec{q}_\alpha \rangle \ . \tag{4.43}$$

It is important to realize that the expression (4.43) contains the long-range distortion of the motion of the outgoing two charged particles 1 and 2. The latter can be explicitly extracted by making use of the following relation[54]

$$U'_{0\alpha} = \delta_{\beta\alpha}G_0{}^{-1} + (1+T'_\beta G_0)U'_{\beta\alpha} \tag{4.44}$$

which holds for arbitrary β. Choosing $\beta=3$ and taking into account the form (4.4) for the remainder amplitude T'_3, the quantity $U'_{0\alpha}(z)G_0(z)|\Phi_{\alpha m}(z)\rangle$ can be written as [recall Eq. (4.5)]

$$U'_{0\alpha}(z)G_0(z)|\Phi_{\alpha m}(z)\rangle = \Omega_R^\dagger(z^*)[1+W'_{3\alpha}(z)]\,|\chi_{\alpha m}\rangle\,, \tag{4.45}$$

with

$$W'_{3\alpha} = t'_{sC,3}G_R[\delta_{\alpha 3} + U'_{3\alpha}G_0(1+T'_\alpha G_0)] + \bar\delta_{\alpha 3}T'_\alpha G_0 + \sum_\beta \bar\delta_{\beta\alpha}U'_{3\beta}G_0T'_\beta G_0(1+T'_\alpha G_0)\,. \tag{4.46}$$

Here Ω_R is the off-shell Møller operator introduced in Eq. (2.7). The virtue of this decomposition is that $W'_{3\alpha}\neq 0$ only if the remainders V'_α of the *short-range* forces, defined in Eq. (4.1), do not vanish.[55] We thus end up with the following representation of the breakup potential (4.43),

$$\langle \vec p_3{}', \vec q_3{}'|\mathcal{V}_{0,\alpha m}^{(R)}(z)|\vec q_\alpha\rangle = \langle \vec p\,'_{3,R}(z^* - q_3'^2/2M_3)|\langle \vec q_3{}'|[1+W'_{3\alpha}(z)]\,|\chi_{\alpha m}\rangle|\vec q_\alpha\rangle \tag{4.47}$$

which explicitly displays the (screened) Coulomb distorted wave for the outgoing charged particles 1 and 2.

Let us remark that for $W'_{3\alpha}=0$ the expression (4.47) is the generalization of the simple pure Coulomb breakup potential used in Sec. III, to the case of many subsystem bound states. As discussed there, the contribution with $\alpha=3$ vanishes on the energy shell due to energy conservation. The term proportional to $W'_{3\alpha}$ depends on the remainder subsystem amplitudes T'_α and can be made sufficiently small. Therefore, a practical method for calculating these corrections is obtained by expanding Eq. (4.16) for $U'_{3\alpha}$, occurring in Eq. (4.46), into powers of T'_α. This procedure leads to the quasi-Born expansion of the effective potential.

We now proceed as in Sec. III. Introducing the decomposition (4.35) of the rearrangement amplitudes $\mathcal{T}^{(R)}_{\beta n,\alpha m}$ in the right-hand side of Eq. (4.40) we are led to a similar splitting of $\mathcal{T}^{(R)}_{0,\alpha m}$,

$$\mathcal{T}^{(R)}_{0,\alpha m} = \tilde{\mathcal{T}}^{(R)}_{0,\alpha m} + \mathcal{T}^{(R)}_{sC;\,0,\alpha m}\,. \tag{4.48}$$

Here the first term is defined as

$$\tilde{\mathcal{T}}^{(R)}_{0,\alpha m}(z) = \Omega_R^\dagger(z^*)\left(|\chi_{\alpha m}\rangle + \sum_\beta \sum_{n,r=1}^{N_\beta} |\chi_{\beta r}\rangle \mathcal{G}_{0;\,\beta,rn}(z)\tilde{\mathcal{T}}^{(R)}_{\beta n,\alpha m}(z)\right). \tag{4.49}$$

Taking into account the result (4.36) for $\tilde{\mathcal{T}}^{(R)}_{\beta n,\alpha m}$, we obtain on the energy shell[38]

$$\tilde{\mathcal{T}}^{(R)}_{0,\alpha m}(\vec p_3{}', \vec q_3{}'; \vec q_\alpha; E+i0) = \bar\delta_{\alpha 3}\langle \vec q_3{}'|\langle \vec p\,'^{(-)}_{3,R}|A_{0,\alpha m}(E+i0)|\vec q^{(+)}_{\alpha,R}\rangle\,. \tag{4.50}$$

For a concise notation we have introduced the abbreviation (for $m=1,\ldots,n_\alpha$)

$$A_{0,\alpha m}(z) = \sum_{n=1}^{N_\alpha} |\chi_{\alpha n}\rangle \mathcal{G}_{0;\,\alpha,nm}(z)[\hat G_0^Q(z-\hat E_{\alpha m})]^{-1} \tag{4.51}$$

which is nothing else but a peculiar off-shell continuation of $|\chi_{\alpha m}\rangle$. Thus the amplitude (4.50) represents the pure (screened) Coulomb breakup amplitude, generalizing the expression (3.63).

The Coulomb-modified strong breakup amplitude $\mathcal{T}^{(R)}_{sC;\,0,\alpha m}$ contains the whole effect of the short-range interactions. For on-shell values of the momenta, $q_3'=\bar q_3'$, $p_3'=\bar p_3'$, $q_\alpha=\bar q_{\alpha m}$, it can be brought into a form similar to Eq. (3.67)

$$\mathcal{T}^{(R)}_{sC;\,0,\alpha m}(\vec p_3{}', \vec q_3{}'; \vec q_\alpha; E+i0) = \langle \vec q_3{}'|\langle \vec p\,'^{(-)}_{3,R}|t^{(R)}_{sC;\,0,\alpha m}(E+i0)|\vec q^{(+)}_{\alpha,R}\rangle\,. \tag{4.52}$$

The operator $t^{(R)}_{sC;\,0,\alpha m}(z)$ can be determined from the corresponding nonbreakup operator $t^{(R)}_{sC;\,\beta n,\alpha m}(z)$ introduced in Eq. (4.37), by means of

$$t^{(R)}_{sC;\,0,\alpha m}(z) = \sum_n W'_{3\alpha}(z)|\chi_{\alpha n}\rangle \mathcal{G}_{0;\,\alpha,nm}(z)[\hat G_0^Q(z-\hat E_{\alpha m})]^{-1} + \sum_\beta \sum_{n,r=1}^{N_\beta}[1+W'_{3\beta}(z)]\,|\chi_{\beta n}\rangle\,\mathcal{G}_{0;\,\beta,nr}(z)[1+\bar{\mathfrak{F}}^{(R)}(z)]_{\beta r,\beta r}\,t^{(R)}_{sC;\,\beta r,\alpha m}(z) \tag{4.53}$$

with $m=1,\ldots,n_\alpha$. We only mention that an integral equation can also be derived directly for $t^{(R)}_{sC;0,\alpha m}$, generalizing Eq. (3.66).

The transition to an unscreened Coulomb potential for the breakup amplitudes (4.50) and (4.52) proceeds now along the familiar lines. First we recall that $t^{(R)}_{sC;\,\beta n,\alpha m}$ is well behaved in the zero screening limit. The same holds true also for $t^{(R)}_{sC;\,0,\alpha m}$ since, as inspection of Eq. (4.53) reveals, for on-shell values of the momenta the poles from the effective free Green's function $\mathcal{G}_0$ can never coincide with the singularity of the potential V_R in the limit $R\to\infty$. The implication is that in this limit all matrix elements $\langle \vec p_3{}', \vec q_3{}'|t^{(R)}_{sC;\,0,\alpha m}|\vec q_\alpha\rangle$ tend towards the corresponding expressions calculated with an unscreened Coulomb potential. In other words, the singular behavior for large R in the on-shell amplitudes (4.50) and (4.52) arises exclusively from the scattering states $\langle \vec{p}_{3,R}'^{(-)}|$ and $|\vec{q}_{\alpha,R}^{(+)}\rangle$ (for $\alpha=1$ and 2). But in this case we know how to proceed. Namely, the amplitudes have to be renormalized by $Z_R^{-1/2}(\bar{p}_3', \mu_3)$ and $Z_{R,\alpha}^{-1/2}(\bar{q}_{\alpha m}, M_\alpha)$, defined in Eqs. (2.11) and (3.49). Taking afterwards the limit $R\to\infty$ we obtain the pure Coulomb breakup and the Coulomb-modified strong breakup amplitude, respectively, for an unscreened Coulomb potential. The same holds true then also for the full breakup amplitude $\mathcal{T}_{0,\alpha m}^{(R)}(\vec{p}_3', \vec{q}_3'; \vec{q}_\alpha)$ on account of Eq. (4.48). These results generalize Eqs. (3.68)–(3.70). Accordingly, we have at our disposal a well defined theory for calculating breakup processes for arbitrary two-body short-range potentials.

4. Scattering states

We, finally, generalize the discussion of the scattering states in Sec. III D to arbitrary short-range interactions. Let $|\Psi_{\alpha m,\vec{q}_\alpha,R}^{(+)}\rangle$ be a scattering state characterized by an incoming two-fragment channel state $|\psi_{\alpha m}\rangle|\vec{q}_\alpha\rangle$. It can be expressed in analogy to Eq. (3.71) by the set of effective two-body states $|\Psi_{\alpha m,\vec{q}_\alpha}^{(R)}\rangle_{\beta n}$ (spectator states) according to

$$|\Psi_{\alpha m,\vec{q}_\alpha,R}^{(+)}\rangle=\sum_{\beta n} G_0(E_{\alpha m}+i0)\mathcal{V}_{0,\beta n}^{(R)}(E_{\alpha m}+i0)|\Psi_{\alpha m,\vec{q}_\alpha}^{(R)}\rangle_{\beta n} \tag{4.54}$$

with $m=1,\dots,n_\alpha$ and $n=1,\dots,N_\beta$. The latter are defined by the analog of Eq. (3.72),

$$|\Psi_{\alpha m,\vec{q}_\alpha}^{(R)}\rangle_{\beta n}=\left(\delta_{\beta\alpha}\delta_{nm}+\sum_{r=1}^{N_\beta}\mathcal{G}_{0;\beta,nr}(E_{\alpha m}+i0)\ \mathcal{T}_{\beta r,\alpha m}^{(R)}(E_{\alpha m}+i0)\right)|\vec{q}_\alpha\rangle\ , \tag{4.55}$$

and fulfill the integral equations

$$|\Psi_{\alpha m,\vec{q}_\alpha}^{(R)}\rangle_{\beta n}=\delta_{\beta\alpha}\delta_{nm}|\vec{q}_\alpha\rangle+\sum_{r=1}^{N_\beta}\mathcal{G}_{0;\beta,nr}(E_{\alpha m}+i0)\sum_\gamma\sum_{s=1}^{N_\gamma}\mathcal{V}_{\beta r,\gamma s}^{(R)}(E_{\alpha m}+i0)|\Psi_{\alpha m,\vec{q}_\alpha}^{(R)}\rangle_{\gamma s}\ . \tag{4.56}$$

For physical states the absolute value of the momentum $\vec{q}_\alpha$ has to be equal to its on-shell value, $q_\alpha=\bar{q}_{\alpha m}$, which is related to the channel energy $E_{\alpha m}$ via Eq. (4.13).

We could now proceed as in Sec. III D. However, in order not to duplicate the previous derivation we start from another point of view; namely, by inserting Eq. (4.55) in Eq. (4.54) it follows that the full scattering states are simply given as

$$|\Psi_{\alpha m,\vec{q}_\alpha,R}^{(+)}\rangle=G_0(E_{\alpha m}+i0)\mathcal{T}_{0,\alpha m}^{(R)}(E_{\alpha m}+i0)|\vec{q}_\alpha\rangle \quad (\alpha\neq 0)\ , \tag{4.57}$$

where $\mathcal{T}_{0,\alpha m}^{(R)}$ is the breakup operator defined by Eq. (4.40), and $|\vec{q}_\alpha|=\bar{q}_{\alpha m}$. We can now take over directly the results of the preceding subsection. Insertion of the decomposition (4.48) of $\mathcal{T}_{0,\alpha m}^{(R)}$ in the right-hand side of Eq. (4.57) leads to two terms the first of which is according to (4.49) and (4.51) equal to

$$G_R(E_{\alpha m}+i0)A_{0,\alpha m}(E_{\alpha m}+i0)|\vec{q}_{\alpha,R}^{(+)}\rangle \quad \text{with } q_\alpha=\bar{q}_{\alpha m}\ . \tag{4.58}$$

The second contribution depends on the breakup operator $t_{sC;0,\alpha m}^{(R)}$ via

$$G_R(E_{\alpha m}+i0)t_{sC;0,\alpha m}^{(R)}(E_{\alpha m}+i0)|\vec{q}_{\alpha,R}^{(+)}\rangle \quad \text{with } q_\alpha=\bar{q}_{\alpha m}\ . \tag{4.59}$$

Thus for the full scattering state we obtain

$$|\Psi_{\alpha m,\vec{q}_\alpha,R}^{(+)}\rangle=G_R(E_{\alpha m}+i0)\times[A_{0,\alpha m}(E_{\alpha m}+i0)+t_{sC;0,\alpha m}^{(R)}(E_{\alpha m}+i0)]\times|\vec{q}_{\alpha,R}^{(+)}\rangle\ . \tag{4.60}$$

This representation makes the performance of the zero-screening limit a simple task. We only have to take into account that $t_{sC;0,\alpha m}^{(R)}$ is well behaved in that limit approaching the breakup operator $t_{sC;0,\alpha m}^{(\infty)}$ calculated for an unscreened Coulomb potential. Also G_R goes over to the unscreened Coulomb Green's function G_C. By making use of Eq. (3.18) we thus conclude that[38]

$$\begin{aligned}\lim_{R\to\infty} Z_{R,\alpha}^{-1/2}(\bar{q}_{\alpha m},M_\alpha)|\Psi_{\alpha m,\vec{q}_\alpha,R}^{(+)}\rangle &=G_C(E_{\alpha m}+i0)[A_{0,\alpha m}(E_{\alpha m}+i0)\\ &\quad+t_{sC;0,\alpha m}^{(\infty)}(E_{\alpha m}+i0)]\,|\vec{q}_{\alpha,C}^{(+)}\rangle\\ &=|\Psi_{\alpha m,\vec{q}_\alpha}^{(+)}\rangle\ ,\end{aligned} \tag{4.61}$$

i.e., knowledge of the breakup operator $t^{(\infty)}_{sC;\,0,\alpha m}$ allows the determination of the full scattering state. But we emphasize once more that also the generalizations of Eqs. (3.74)–(3.77) to the case of arbitrary short-range two-body interactions are easily derived. We finally remark that since the structure of Eq. (4.61) is identical to the one of Eq. (3.77) together with (3.76), the discussion following Eq. (3.78) concerning the Coulomb-type behavior of the scattering wave function can be taken over without any change.

B. Attractive Coulomb force

If the two charged particles 1 and 2 are oppositely charged the treatment of the subsystem 3 has to be slightly modified. For, in addition to the bound states arising from the short-range part V_3^s if there are any, we have the infinity of Coulomb-type bound states. Thus the subsystem interaction $V_3 = V_3^s + V_R$ must be decomposed as

$$V_3 = \sum_{r=1}^{N_3} |\chi_{3r}\rangle \lambda_{3r} \langle \chi_{3r}| + V_3' , \qquad (4.62)$$

where, in contrast to Eq. (4.1b), the separable terms account for the first N_3 bound states of the full potential V_3. The amplitude $\hat{T}_3'$ corresponding to the rest potential V_3' can, however, again be rewritten in the form (4.4), the operator $\hat{t}'_{sC,3}$ fulfilling the LS equation (2.17) with the short-range potential $V_3^s - \sum_{r=1}^{N_3} |\chi_{3r}\rangle \lambda_{3r} \langle \chi_{3r}|$. And, *per constructionem*, $\hat{T}_3'$ contains only the bound-state poles at energies $\hat{E}_{3n}$ for $n > N_3$, the other ones ($n \le N_3$) residing in $\hat{\Delta}_3$. With this proviso the whole development of the previous subsection can be repeated, leading again to a correct definition of the various three-body scattering amplitudes and wave functions for energies below the (N_3+1)-th threshold (in the breakup region N_3 is infinite, of course).

V. DISCUSSION

In the preceding sections we have presented a formalism for describing the scattering of one neutral and two charged particles which interact via arbitrary short-range potentials plus, in addition, the long-range Coulomb force. This approach which is based on the quasiparticle formulation of the three-body problem, could be proven both to be mathematically correct and to actually provide us with the desired physical scattering amplitudes. The starting point was the observation that the Coulomb contributions to the occurring effective two-body potentials, which describe the scattering of one charged particle off the correlated pair of the other two, can be decomposed very naturally into an infinite-range and a shorter-range part. The former can be interpreted as the Coulomb interaction between the charged particle and the *center of mass of the other two* (*uncorrelated*) *particles* (a type of interaction which has to be introduced artificially in conventional approaches) and has exactly the same form as in the genuine two-charged particle scattering. Consequently, the methods developed there for handling the long-range distortion could be taken over directly to the three-body case.

The present formalism is valid for all types of processes which can occur in a general three-body system, *at all energies*. It, moreover, substantiates the physical picture conjectured intuitively for these reactions. In order to clarify this remark let us envisage once more the situation where one of the charged particles impinges on the bound system of the other charged and the neutral particle. The infinite-range part of the effective potential between these two fragments gives rise to a distortion of the incoming plane wave converting the latter into a Coulomb scattering state. At this stage the neutral particle plays, as expected, no role (except by defining a characteristic reduced mass for the two fragments). For finite distances the three particles undergo complicated interactions due to shorter-range Coulombic effects and due to the genuine short-range forces. In case the final state consists again of only two charged fragments a final state Coulomb distortion, similar to the one in the initial state, takes place. If, however, three free particles are produced, then only the movement of the two charged ones will be influenced by the long-range Coulomb force resulting in a Coulomb scattering state for these two particles. Of course, in addition we always have the pure Coulomb ("Rutherford") contributions.

This separation of the amplitudes for all possible processes into a pure Coulomb and a Coulomb-modified strong part in our approach is also desirable from a practical point of view. For, the latter can be calculated as in ordinary short-range theory by means of integral equations which are one-dimensional after partial wave projection. Note, however, that nowhere a partial wave expansion of the two-body Coulomb T matrices which occur in the effective potentials needs to be made. The first (pure Coulomb) parts of the amplitudes are *not* obtainable from angular momentum projected expressions, the reason being the nonconvergence of the partial wave expansion. However, for the nonbreakup amplitudes the pure Coulomb contribution is just the genuine two-particle Coulomb amplitude which is analytically known. And in the breakup case it can be obtained, e.g., by quadrature of explicitly known two-body Coulomb scattering wave functions. We mention that a similarly simple structure results also for the effective two-body (spectator) scattering wave functions from which the full scattering wave functions can be constructed.

An important point is the wide range of application of this method. The most obvious candidate is proton-deuteron scattering for which first numerical results are already available.[20,21] Further results will be presented in a subsequent publication.[22] Other interesting candidates are electromagnetic and weak processes in the (p,p,n) system. But our treatment of the Coulomb force can be implemented straightforwardly in any N-body theory provided that only two charged fragments occur in the initial and the final state. This opens up a host of other possibilities. Some of them which could be handled with present-day numerical experience are the four-nucleon system ($d+d$, $p+{}^3$H, and $n+{}^3$He going over to all possible final states including four free particles) and three-body models of complex nuclear reactions with two of the fragments involved being charged [e.g., (d,p), (d,pn), (p,pn), ..., reactions[56]]. Other applications lie in the field of atomic and molecular processes (n+H, ion-ion-neutral atom scattering, ...). Methods for actually performing such calculations will be described in Ref. 22.

It is worthwhile pointing to the interesting possibility that reactions of the type mentioned above might allow us to learn something about the (short-range) two-particle forces. This hope is based on the fact that here we have the rare opportunity to investigate the interference between the known Coulomb and an unknown (short-range) interaction *in the same subsystem*. For instance, for the *ppn* system the strong off-shell sensitivity which impedes testing charge symmetry in nucleon-nucleon scattering[57] might carry over to an off-shell sensitivity of various proton-deuteron observables.

We, moreover, touch upon the question of extending this approach to the scattering of three charged particles. It is quite obvious that the formalism presented in this paper can *without any additional complication* be extended to three-charged particle scattering for *nonbreakup* reactions (at any energy). The breakup into three free charged particles[58] which is not so obvious is presently under investigation.

A final remark concerns the simplicity and the physically convincing structure of the final representations (4.35)–(4.37), (4.48), (4.50), (4.52), and (4.61), which indicates that here a very general feature of the charged-particle collision problem is displayed. In other words, the final results are independent of the quasiparticle method used for their derivation. This aspect is clarified in a Møller operator approach based on time-dependent scattering theory which is the content of a publication under preparation.

APPENDIX

We have to prove the assertion that the most singular part in the limit $R\to\infty$ of the kernel of the LS equation (3.19) arises from the contribution (3.24) to the effective potential (3.23), and in the general case of the LS equation (4.14) from the part (4.25). For this purpose the singularity structure of the various terms occurring in the definitions (3.35) and (4.26) of $\mathcal{V}'$ and $\mathcal{K}'$, respectively, has to be examined. It is obvious that in the present context we need *not* care about the singularities which exist even for short-range potentials.[59] Rather, only those are of interest here which arise in the zero screening limit for on-shell values of the momenta where $\mathcal{G}_0$ has its poles. The result of this investigation will be that none of the terms of $\mathcal{V}'$ shows a singular behavior of the "dangerous" type (3.24) but only a more harmless one (the latter is termed "not dangerous"), and similarly for $\mathcal{K}'$. In order to simplify the notation the unessential factors $[\hat{S}_\beta(z-q_\beta^2/2M_\beta)\hat{S}_\alpha(z-q_\alpha^2/2M_\alpha)]^{1/2}$ are omitted. Throughout this Appendix we consider R to be infinite. This is indicated by a superscript ∞ on the effective potential.

We proceed in two steps. In the first one we deal with the simple model discussed in Sec. III (separable short-range potentials of rank one, repulsive Coulomb force), and investigate term by term the various contributions to $\mathcal{V}'$. Thereupon, it is demonstrated that in the general case treated in Sec. IV, taking into account nonseparable rest terms T'_γ and attractive Coulomb forces does not alter the conclusions.

The effective potential $\mathcal{V}'$ of Eq. (3.35) is graphically represented in Fig. 1 except for diagram (d) in which T_C has to be replaced by $T_C - V_C$ (the contribution proportional to V_C is just the singular part (3.24)). Let us start with the first of the diagrams (b) [diagram (a) does not contain Coulomb effects at all]. The corresponding expression is

$$\mathcal{V}^{(b,\infty)}_{\beta\alpha}(\vec{q}'_\beta,\vec{q}_\alpha;z)=\bar{\delta}_{\beta\alpha}\delta_{\alpha 3}\frac{\chi^*_\beta(\vec{p}'_\beta)\phi_\alpha(\vec{p}_\alpha;z)}{z-p_\alpha{}^2/2\mu_\alpha-q_\alpha{}^2/2M_\alpha}\,, \tag{A1}$$

where the relative momenta $\vec{p}'_\beta, \vec{p}_\alpha$ are, according to Eq. (3.9), linear combinations of the external momenta, $\vec{p}'_\beta=-\vec{q}_\alpha-m_\alpha/(m_\gamma+m_\alpha)\vec{q}'_\beta$ and $\vec{p}_\alpha=m_\beta/(m_\beta+m_\gamma)\vec{q}_\alpha+\vec{q}'_\beta$. The Coulomb-modified form factor is defined by

$$\phi_\alpha(\vec{p}_\alpha;z)=\int \frac{d^3k\hat{T}_C(\vec{p}_\alpha,\vec{k};z-q_\alpha{}^2/2M_\alpha)\chi_\alpha(\vec{k})}{(z-k^2/2\mu_\alpha-q_\alpha{}^2/2M_\alpha)} \tag{A2}$$

and reduces at the on-shell point, $z=E+i0$ with E given by Eq. (3.21) and $|\vec{q}_\alpha|=\bar{q}_\alpha$, to

$$\phi_\alpha(\vec{p}_\alpha;E)=\int \frac{d^3k\hat{T}_C(\vec{p}_\alpha,\vec{k};\hat{E}_\alpha)\chi_\alpha(\vec{k})}{\hat{E}_\alpha-k^2/2\mu_\alpha} \ . \tag{A3}$$

Since the two-body binding energy $\hat{E}_\alpha$ is negative the Coulomb amplitude appears here for negative energies where it is nonsingular (except for the singularity in forward direction which, however, is smoothed out by the integration). Consequently $\phi_\alpha(\vec{p}_\alpha;z)$ is not dangerous. The same holds true for the second diagram of (b).

Next we consider graph (c) of Fig. 1. Its analytical representation is

$$\mathcal{V}^{(c,\infty)}_{\beta\alpha}(\vec{q}'_\beta,\vec{q}_\alpha;z)=\bar{\delta}_{\beta\alpha}\bar{\delta}_{\beta 3}\bar{\delta}_{\alpha 3}\int \frac{d^3k\chi^*_\beta(\vec{p}'_\beta)\chi_\alpha(\vec{p}_\alpha)\ \hat{T}_C(\vec{p}'_\gamma,\vec{p}_\gamma;z-k^2/2M_\gamma)}{(z-p'^2_\beta/2\mu_\beta-q'^2_\beta/2M_\beta)\,(z-p_\alpha{}^2/2\mu_\alpha-q_\alpha{}^2/2M_\alpha)} \tag{A4}$$

with

$$\vec{p}'_\beta=\vec{k}+m_\gamma/(m_\gamma+m_\alpha)\vec{q}'_\beta,\quad \vec{p}_\alpha=-\vec{k}-m_\gamma/(m_\beta+m_\gamma)\vec{q}_\alpha,\quad \vec{p}'_\gamma=-\vec{q}'_\beta-m_\beta/(m_\alpha+m_\beta)\vec{k},$$

and

$$\vec{p}_\gamma=\vec{q}_\alpha+m_\alpha/(m_\alpha+m_\beta)\vec{k}\,.$$

On the energy shell the denominator in (A4) becomes simply $(\hat{E}_\beta-p'^2_\beta/2\mu_\beta)(\hat{E}_\alpha-p_\alpha{}^2/2\mu_\alpha)$, i.e., it can never vanish. In other words, $\hat{T}_C(\vec{p}'_\gamma,\vec{p}_\gamma;E+i0-k^2/2M_\gamma)$ is integrated over with the bound-state wave functions $\psi^*_\beta(\vec{p}'_\beta)=\langle\chi_\beta|\hat{G}_0(\hat{E}_\beta)|\vec{p}'_\beta\rangle$ and $\psi_\alpha(\vec{p}_\alpha)=\langle\vec{p}_\alpha|\hat{G}_0(\hat{E}_\alpha)|\chi_\alpha\rangle$. Furthermore, when $\vec{q}'_\beta$ and $\vec{q}_\alpha$ are fixed by the on-shell condition, the momenta $\vec{p}'_\gamma$ and $\vec{p}_\gamma$ can never reach their respective on-shell values where $\hat{T}_C(\vec{p}'_\gamma,\vec{p}_\gamma;\ E+i0-k^2/2M_\gamma)$ would pick up a singular phase factor. Due to these reasons $\mathcal{V}^{(c,\infty)}_{\beta\alpha}(\vec{q}'_\beta,\vec{q}_\alpha;z)$ is not dangerous.

There remains finally graph (d) to be examined, with T_C replaced by T_C-V_C. Denoting this contribution by $\mathcal{V}'^{(d,\infty)}_{\beta\alpha}$ we obtain [with $\vec{p}'_\alpha=-\vec{k}-m_\gamma/(m_\beta+m_\gamma)\vec{q}'_\alpha$, $\vec{p}_\alpha=-\vec{k}-m_\gamma/(m_\beta+m_\gamma)\vec{q}_\alpha$, $\vec{p}'_\gamma=\vec{q}'_\alpha+m_\alpha/(m_\alpha+m_\beta)\vec{k}$, and $\vec{p}_\gamma=\vec{q}_\alpha+m_\alpha/(m_\alpha+m_\beta)\vec{k}$]

$$\mathcal{V}'^{(d,\infty)}_{\beta\alpha}(\vec{q}'_\beta,\vec{q}_\alpha;z)=\delta_{\beta\alpha}\bar{\delta}_{\alpha 3}\int \frac{d^3k\chi^*_\alpha(\vec{p}'_\alpha)\chi_\alpha(\vec{p}_\alpha)(\hat{T}_C-V_C)(\vec{p}'_\gamma,\vec{p}_\gamma;z-k^2/2M_\gamma)}{(z-p'^2_\alpha/2\mu_\alpha-q'^2_\alpha/2M_\alpha)(z-p_\alpha{}^2/2\mu_\alpha-q_\alpha{}^2/2M_\alpha)} \ . \tag{A5}$$

By means of arguments similar to those employed for $\mathcal{V}^{(c,\infty)}$ it can be shown that also $\mathcal{V}'^{(d,\infty)}$ is not dangerous. However, it is rewarding to investigate in more detail the behavior of (A5) on the energy shell, i.e., for values of z, $\vec{q}'_\beta$, and $\vec{q}_\alpha$ fixed by the relation (3.21). In fact, from the hypergeometric function representation[23] of the Coulomb amplitude one can derive

$$(\hat{T}_C-V_C)(\vec{p}'_\gamma,\vec{p}_\gamma;E+i0-k^2/2M_\gamma) \underset{|\vec{q}'_\alpha-\vec{q}_\alpha|\to 0}{\sim} |\vec{q}'_\alpha-\vec{q}_\alpha|^{-1}. \tag{A6}$$

Consequently $\mathcal{V}'^{(d,\infty)}_{\beta\alpha}(\vec{q}'_\beta,\vec{q}_\alpha;E+i0)$ behaves for small momentum transfer as $|\vec{q}'_\alpha-\vec{q}_\alpha|^{-1}$ times a nonlocal and nonsingular function of $\vec{q}'_\alpha$ and $\vec{q}_\alpha$ being, therefore, not dangerous.[60]

This proves our assertion. It is a simple but useful exercise to check the arguments of the Appendix in a perturbation expansion of the Coulomb amplitude $\hat{T}_C$ for simple, e.g., Yamaguchi, types of form factors for which the integrals can be done analytically.[61]

The demonstration that for general (e.g., local) two-body short-range potentials no other term except (4.19) of the effective potential (4.17) displays for $R\to\infty$ a Coulomb-type behavior can be performed in close analogy to the above discussion. For this purpose we iterate the integral equation (4.16) for $U'_{\beta\alpha}$,

$$U'_{\beta\alpha}=\bar{\delta}_{\beta\alpha}G_0{}^{-1}+\sum_\gamma \bar{\delta}_{\beta\gamma}\bar{\delta}_{\gamma\alpha}T'_\gamma + \sum_{\gamma\epsilon}\bar{\delta}_{\beta\gamma}\bar{\delta}_{\gamma\epsilon}\bar{\delta}_{\epsilon\alpha}T'_\gamma G_0 T'_\epsilon+\cdots, \tag{A7}$$

insert this series in the defining Eq. (4.17) of the effective potential, and collect terms of equal power in T'_γ (quasi-Born expansion). The zeroth order contribution does not contain the Coulomb interaction at all. In first order the generalizations of $\mathcal{V}^{(b)}$ to $\mathcal{V}^{(d)}$ are produced, the *only* dangerous one of these being just the term isolated in Eq. (4.19). The fact that many bound states may occur (either due to the short-range or due to an attractive Coulomb force) requires only trivial modifications of the arguments used above. When second order terms in T'_γ are finally introduced it is apparent from the multiple scattering structure that no two consecutive scatterings via the Coulomb potential V_C can occur; That is, when, say, $\gamma = 3$ implying Coulomb scattering of particles 1 and 2, then ϵ must be different from 3, i.e., T'_ϵ represents a short-range scattering amplitude. In addition T'_3 is integrated over and thereby smoothed out. The same holds true for any higher order term of the series expansion (A7).

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[25]This cannot happen for negative energies which are encountered in the calculation of bound states. Therefore no problems arise in that case.

[26]We restrict ourselves to exponential screening which has advantages in practical calculations. All results to be discussed below can be proven (in a generalized sense) for wide classes of screening functions.

[27]They might include also short-range electromagnetic forces arising, e.g., in the scattering of particles with extended charge distributions.

[28]Since only two of the particles are charged, writing V_R without any subsytem index is unique.

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[30]The dependence on the screening radius R is not made explicit here, since the matrix elements of G_R between normalizable states tend smoothly, for $R \to \infty$, towards the corresponding matrix elements of the Green's function for an unscreened Coulomb potential.

[31]An unambiguous notation would require the superscript Q_α, indicating the two-body system in which these operators act. However, in order to simplify our notation we label only the energy argument z_α by a subscript α.

[32]It proves convenient to depart from the definitions of Ref. 14 so that $\mathcal{G}_{0;\alpha}$ is given by Eq. (3.22) instead of by Δ_α itself. This is simply achieved by multiplying the effective amplitudes $\langle \vec{q}\,'_\beta | \mathcal{T}^{(R)}_{\beta\alpha}(z) | \vec{q}_\alpha \rangle$ of Ref. 14 from the left with $[\hat{S}_\beta(z - q'^2_\beta/2M_\beta)]^{1/2}$ and from the right with $[\hat{S}_\alpha(z - q_\alpha^2/2M_\alpha)]^{1/2}$, and similarly for the effective potentials. The resulting amplitudes and potentials are those used in the present section. However, due to the normalization condition (3.7) these new amplitudes coincide at the on-shell point (3.21) with those of Ref. 14.

[33]See Ref. 14, Sec. 5.

[34]Since for energies below the lowest threshold the Green's function (3.22) is nonpolar, calculations of three-body bound states can be performed without any difficulty, as discussed before for two-particle scattering (Ref. 25). This has been tried for ^{3}He some time ago by Alessandrini *et al.* (Ref. 35) and by Zankel *et al.* (Ref. 36). However, both papers contain errors which invalidate their conclusions. A more complete investigation is being performed by the present authors.

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[37]In order to obtain the splitting (3.32) completely

symmetric in $\vec{q}'_\alpha$ and $\vec{q}_\alpha$ we were led to introduce the function B_α. Retaining this symmetry appears, however, to be of practical importance since it reflects the symmetry of the original effective potential $\mathfrak{v}^{(R)}_{\beta\alpha}$, and, therefore, helps to reduce waste of computer time and memory.

[38]For the sake of compact notation we introduce also the symbol $|\vec{q}\,^{(\pm)}_{3,R}\rangle$ to denote the plane waves $|\vec{q}_3\rangle$ which describe the relative motion of the neutral particle 3 and the charged particle (1+2).

[39]Due to our preference of the form (3.22) for $\mathcal{G}_{0;\alpha}$ also the breakup amplitude $\mathcal{T}^{(R)}_{0\alpha}(\vec{p}',\vec{q}';\vec{q}_\alpha;z)$ and the effective potential $\mathfrak{v}^{(R)}_{0\alpha}(\vec{p}',\vec{q}';\vec{q}_\alpha;z)$ differ from those given in Ref. 14 by a factor $[\hat{S}_\alpha(z-q_\alpha{}^2/2M_\alpha)]^{1/2}$. Since this factor becomes equal to 1 on the energy shell [cf. Eq. (3.7)] it does not affect the on-shell amplitudes (compare footnote 32).

[40]For $V_R=0$ it reduces to the familiar equation used for breakup calculations with at least two neutral particles (e.g., neutron-deuteron breakup).

[41]It is a simple matter to devise a splitting (3.33) with an expression for $\tilde{\mathfrak{v}}$ which is off-the-energy-shell different from and less symmetric than (3.34), but which leads to a Coulomb breakup amplitude (3.68) with the factor $[S_\alpha(E_\alpha+i0)]^{1/2}$ missing. In this case $\mathcal{T}_{C,0\alpha}$ assumes the simple form

$$\mathcal{T}_{C,0\alpha}(\vec{p}\,'_3,\vec{q}\,'_3;\vec{q}_\alpha) = \frac{1}{(2\pi)^{3/2}}\int d^3r d^3r' \exp[i\vec{q}\,'_3(b_\alpha\vec{r}+d_\alpha\vec{r}\,')] \times\langle\vec{p}\,'^{(-)}_{3,C}|\vec{r}\rangle\chi_\alpha(\vec{r}\,')\langle a_\alpha\vec{r}+c_\alpha\vec{r}\,'|\vec{q}\,^{(+)}_{\alpha,C}\rangle\,,$$

where the constants a_α, b_α, c_α, and d_α are certain mass ratios. Integrals of this type may be tackled by methods used in electron-atom scattering calculations.

[42]A related wave function approach to proton-deuteron scattering with separable potentials has been proposed by V. F. Kharchenko and S. A. Storoshenko (unpublished); see also A. M. Veselova (unpublished).

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[47]For an alternative procedure, see Ref. 46.

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[49]If the form factors $|\chi_{\alpha r}\rangle$ depend on the subsystem energy z_α, the conditions (4.7) are imposed only for $z_\alpha=\hat{E}_{\alpha r}$. We mention that the latter are satisfied automatically by form factors chosen, e.g., as in the unitary pole or the Hilbert-Schmidt expansion approach.

[50]In contrast to Sec. III, redefining the effective amplitudes and potentials (see footnote 32) does not enhance the transparency of the following derivation. We, thus, adopt the customary conventions of Ref. 14.

[51]From the general expression (4.17) we easily recover the form of $\mathfrak{v}^{(R)}_{\beta\alpha}$ used in the previous section. Indeed, comparing the simple subsystem amplitudes (3.2) with Eqs. (4.2)–(4.4) we find $T'_\gamma=\delta_{\gamma3}T_R$. Then Eq. (4.16) can be solved explicitly to yield $U'_{\beta\alpha}=\bar{\delta}_{\beta\alpha}G_0{}^{-1}+\bar{\delta}_{\beta3}\bar{\delta}_{\alpha3}T_R$. Inserting this result in Eq. (4.17) and taking into account Eqs. (4.5) leads to the effective potential (3.23).

[52]To be precise only the *open* physical channels do contribute. For unphysical and closed channels the elements of $\mathcal{G}_{0;\alpha}$ are nonpolar so that the corresponding elements of $\mathfrak{X}^{(R)}$ are quite harmless. In order to avoid the notational distinction between closed and open physical channels we assume the three-body energy z to be high enough so that all physical channels are open.

[53]We could quite simply obtain a splitting completely symmetric in $\vec{q}\,'_\alpha$ and $\vec{q}_\alpha$ by a procedure analogous to that employed in Sec. III [cf. Eq. (3.32) and footnote 37]. However, we renounce this in order to save some formulas.

[54]E. O. Alt, P. Grassberger, and W. Sandhas (unpublished).

[55]In Sec. III the short-range potentials V^s_α were chosen purely separable so that $V'_\alpha\equiv0$ for all α. Consequently, only the first term in the decomposition (4.45) had survived there.

[56]Recent reviews of the conventional approach to deuteron stripping reactions are, e.g., N. Austern, *Direct Nuclear Reaction Theories* (Wiley, New York, 1970); G. Baur and D. Trautmann, Phys. Rep. 25C, 293, 1976.

[57]H. Kumpf, Yad. Fiz. 17, 1156 (1973) [Sov. J. Nucl. Phys. 17, 602, 1973]; P. U. Sauer, Phys. Rev. Lett. 32, 626, 1974.

[58]For a discussion of the asymptotic behavior of the wave function for three free charged particles, see Merkuriev, Ref. 7.

[59]An exposition of the singularity structure of the effective potential for short-range forces of the Yukawa type can be found in H. Ziegelmann, *Das Dreikörperproblem der Quantenmechanik oberhalb der Aufbruchsschwelle* (Burg, Basel, 1976).

[60]Note that a local potential of the form $V(\vec{q}\,'-\vec{q})\sim|\vec{q}\,'-\vec{q}|^{-1}$ behaves in co-ordinate space like $V(r)\sim r^{-2}$. However, the nonlocal factor appearing in $\mathfrak{v}'^{(d,\infty)}_{\beta\alpha}$ produces an additional decrease for large distances so that the whole expression is, in fact, of shorter range.

[61]For diagrams (b) this is easily checked also to all orders in V_C for a special class of form factors by means of the explicit analytical expressions for the Coulomb-distorted form factors given in Ref. 62.

[62]W. W. Zachary, J. Math. Phys. 12, 1379 (1971); 14, 2018 (1973); Ž. Bajzer, Nuovo Cimento A22, 300 (1974); van Haeringen and van Wageningen, Ref. 23.