Exactly solvable time-dependent Hartree-Fock equations

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Models of quantum mechanical many-body systems are presented such that the Hartree-Fock equations for them admit analytical solutions, which simulate an isolated atomic nucleus or two nuclei colliding with each other.

[NUCLEAR REACTIONS Scattering theory, time-dependent Hartree-Fock method.]

I. INTRODUCTION

The Hartree-Fock (HF) method combined with simple effective nucleon-nucleon interactions has been very successful in describing gross features of nuclear structure. The random phase approximation, which is a linearized version of the time dependent Hartree-Fock (TDHF) method, has also proved useful for understanding small amplitude collective motions. Hence it is natural to try to apply the TDHF method to more dynamical problems such as nucleus-nucleus scattering.

The TDHF calculation is much more complex than that for HF because one has to solve partial differential equations involving the time variable. Unlike in the HF case, the single-particle states in the TDHF method vary in time, and it is not very easy to visualize how the method works. With a large computer, apparently one can do TDHF calculations without much difficulty, and already a few such calculations have been reported.[1-3] Nevertheless it would be interesting to construct models, even if they are very simple-minded ones, such that the HF and TDHF equations can be solved analytically. Such models would hopefully provide us with insights into how the TDHF method works and how it can possibly be improved. Also, the models can be used for testing computer programs for TDHF calculations.

The purpose of this paper is to present a few models such that the HF and TDHF equations for them can be solved analytically. The TDHF equations for the models are of the form of the so-called nonlinear Schrödinger (NLS) equations for which "soliton solutions" can be constructed easily.[4] Such solutions can simulate an isolated nucleus or nucleus-nucleus scattering. In Sec. II we summarize mathematical apparatus for solving NLS equations. Three models are presented in Secs. III, IV, and V, respectively. In Sec. VI we discuss limitations and possible extensions of the models and method.

II. MATHEMATICAL PRELIMINARIES

Define an $n \times n$ real symmetric matrix $D = (D_{\alpha\beta})$ with

$$D_{\alpha\beta} = \delta_{\alpha\beta} + e_\alpha e_\beta/(\kappa_\alpha + \kappa_\beta)\,, \tag{2.1}$$

where $e_\alpha \equiv A_\alpha{}^{1/2}\exp(\kappa_\alpha x)$ and A_α and κ_α are arbitrarily real positive constants. Introduce n functions $\phi_\alpha(x)$ which are defined by

$$\sum_\beta D_{\alpha\beta}\phi_\beta = e_\alpha\,, \quad \text{or } D\hat{\phi} = \hat{e}\,, \tag{2.2}$$

where $\hat{\phi}$ and $\hat{e}$ are one-column matrices. One can show[4,5] that ϕ_α satisfied the time-independent NLS equation

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \sum_{\beta=1}^{n}\lambda_\beta\phi_\beta{}^2\right)\phi_\alpha = \epsilon_\alpha\phi_\alpha\,, \tag{2.3}$$

where

$$\lambda_\beta = 4\kappa_\beta\,, \qquad \epsilon_\alpha = -(\hbar\kappa_\alpha)^2/2m\,. \tag{2.4}$$

Since Eq. (2.3) is a nonlinear equation, the normalization of ϕ_α is nontrivial. In fact ϕ_α's are all normalized by

$$\int_{-\infty}^{\infty}\phi_\alpha{}^2 dx = 1.$$

To show this we start with[6]

$$\frac{dD}{dx} = \hat{e}\tilde{\hat{e}}\,, \tag{2.5}$$

where $\tilde{\hat{e}}$ is the transposition of $\hat{e}$. Equation (2.5)

follows immediately from Eq. (2.1). Next, differentiating $DD^{-1}=1$ and using Eqs. (2.2) and (2.5), we obtain

$$\begin{aligned}\frac{dD^{-1}}{dx} &= -D^{-1}\frac{dD}{dx}D^{-1}\\ &= -D^{-1}\hat{e}\tilde{\hat{e}}^{-1}\\ &= -\hat{\phi}\tilde{\hat{\phi}}\,,\end{aligned} \tag{2.6}$$

and hence

$$\int_{-\infty}^{\infty}\phi_\alpha(x)\phi_\beta(x)dx = D_{\alpha\beta}^{-1}(-\infty) - D_{\alpha\beta}^{-1}(\infty)\,. \tag{2.7}$$

It is easy to show that $D_{\alpha\beta}^{-1}(-\infty)=\delta_{\alpha\beta}$ and $D_{\alpha\beta}^{-1}(\infty)=0$, and hence,

$$\int_{-\infty}^{\infty}\phi_\alpha(x)\phi_\beta(x)dx = \delta_{\alpha\beta}\,. \tag{2.8}$$

Next we consider time-dependent functions. Define an $n\times n$ Hermitian matrix $\mathfrak{D}$ with

$$\mathfrak{D}_{\alpha\beta}=\delta_{\alpha\beta}+u_\alpha u_\beta^*/(\kappa_\alpha+\kappa_\beta^*)\,, \tag{2.9}$$

where $u_\alpha = A_\alpha^{1/2}\exp(\kappa_\alpha x + i\hbar\kappa_\alpha^2 t/2m)$. A_α is a positive real constant while the κ_α's are arbitrary complex constants with positive real parts. The n functions $\psi_\alpha(x,t)$ which are defined by[5]

$$\mathfrak{D}\hat{\psi}=\hat{u} \tag{2.10}$$

satisfy the n-component NLS equations

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\sum_{\beta=1}^{n}\lambda_\beta|\psi_\beta|^2\right)\psi_\alpha = i\hbar\frac{\partial\psi_\alpha}{\partial t} \tag{2.11}$$

where

$$\lambda_\beta = 4\,\mathrm{Re}\,\kappa_\beta\,. \tag{2.12}$$

The ψ_α's are normalized by

$$\int_{-\infty}^{\infty}\psi_\alpha^*(x,t)\psi_\beta(x,t)dx=\delta_{\alpha\beta} \tag{2.13}$$

which can be proved in the same manner as Eq. (2.8).

A close examination shows that each ψ_α contains one "wave packet" or "soliton." In Ref. 4, we showed how solutions can be constructed such that ψ_α can contain any number of solitons, but we will not use such solutions in this paper.

III. NUCLEUS IN ONE DIMENSION

Consider a system of "nucleons" in one dimension, the interaction between nucleons i and j being

$$V_{ij}=-\frac{\hbar^2}{2m}\frac{g}{3}\delta(x_i-x_j)\,. \tag{3.1}$$

When the spin and isospin of the system are saturated, as we will assume throughout this paper, the TDHF equations are given by Eq. (2.11), with[7]

$$\lambda_\beta = g\,. \tag{3.2}$$

Each orbital accommodates four nucleons, hence the mass number $A=4n$. The single-particle wave function $\psi_\alpha(x,t)$ is normalized by Eq. (2.13).

First, let us consider a stationary state by putting

$$\psi_\alpha(x,t)=\phi_\alpha(x)\exp(-i\epsilon_\alpha t/\hbar)\,. \tag{3.3}$$

Then ϕ_α obeys the HF equation (2.3), with $\lambda_\beta=g$. Since λ_β is independent of β, all single-particle energies ϵ_α are degenerate, which implies that ϕ_α is independent of α. Although n in Eq. (2.3) can formally be any integer, $n=1$ is the only physically meaningful choice which corresponds to a "nucleus" of mass number $A=4$.[8] The Slater determinant for a nucleus of $A=8$ or $12,\ldots$ is identically zero. The explicit form of $\phi_\alpha\equiv\phi$ is immediately found from Eq. (2.2) to be[5]

$$\begin{aligned}\phi(x) &= A^{1/2}e^{\kappa x}/[(A/2\kappa)e^{2\kappa x}+1]\\ &= (\kappa/2)^{1/2}\mathrm{sech}(\kappa x)\,.\end{aligned} \tag{3.4}$$

Here $\kappa=\frac{1}{4}g$, and we took A to be $A=2\kappa$.[9] The single-particle energy is given by $\epsilon=-(\hbar\kappa)^2/2m = -(\hbar g/4)^2/2m$. Equation (3.4) represents a nucleus whose center is at the origin. Note that the choice of the origin is arbitrary.

If we denote the total kinetic and potential energies of the constituent nucleons by T and V, respectively, and the total energy by E, we obtain

$$E=T+V=\tfrac{1}{2}(T+4\epsilon)\,. \tag{3.5}$$

Combining this with the virial theorem for δ-function interactions,[10] i.e., $V=-2T$, we find $T=-4\epsilon/3$, and hence

$$E=\frac{4\epsilon}{3}=-\frac{\hbar^2}{2m}\frac{g^2}{12}\,. \tag{3.6}$$

The TDHF equation (2.11) with $n=1$ has a solution

$$\psi(x,t)=\phi(x-vt)\exp[i(mv/\hbar)x-i\omega t]\,, \tag{3.7}$$

where ϕ is defined by Eq. (3.4), and

$$\hbar\omega=\epsilon+\tfrac{1}{2}mv^2\,. \tag{3.8}$$

This ψ represents an isolated nucleus which is moving with a constant speed v. We note that Eq. (3.7) is simply a consequence of Galilean invariance of the TDHF equation.

McGuire[11] considered the same model as the present one except that the particles are all distinguishable in his case. Recall that eigenstates for a system of distinguishable particles contain those for fermions and for bosons which are of course subjected to the same interactions. McGuire showed that, for any number of particles in his model, there is one and only one bound

state. Moreover, the wave function of the bound state is symmetric with respect to all coordinates involved, and hence it can be regarded as that for a system of bosons. Up to mass number $A=4$, effectively there is no difference between boson and fermion systems because of the spin-isospin degeneracy in the latter. For $A>4$, however, the fermion wave function, if it exists, must be different from that for bosons. McGuire's result, that there exists one and only one bound state for distinguishable particles, implies that there is no bound state containing more than four fermions. This is consistent with our finding that there can be only one orbital in the model under consideration.

Next let us discuss a more interesting case of nucleus-nucleus scattering. Since each nucleus consists of four nucleons, there are altogether eight nucleons, and we need two orbitals to accommodate them. The Slater determinant is

$$\Psi=\frac{1}{\sqrt{2}}\begin{vmatrix}\psi_1(x_1,t) & \psi_1(x_2,t)\\ \psi_2(x_1,t) & \psi_2(x_2,t)\end{vmatrix}. \tag{3.9}$$

Here again the spin and isospin factors in the wave functions have been suppressed. We expect that the two wave functions ψ_1 and ψ_2 are well separated for $t=\pm\infty$, i.e., $\lim_{t\to\pm\infty}\psi_1(x,t)\psi_2(x,t)=0$. The TDHF equation is given by Eq. (2.11), with $\lambda_\beta=g$ and $n=2$. Unlike the case of the stationary state with Eq. (2.3), ψ_1 and ψ_2 will be different functions [orthogonal to each other according to Eq. (2.13)], and Ψ of Eq. (3.9) will not vanish identically.

From Eq. (2.10) we obtain

$$\psi_1=(2\kappa)^{1/2}(\eta_1/\Delta)\exp[i(mv_1/\hbar)x-i\omega_1 t]\times\left(1+\frac{i(v_1-v_2)\eta_2{}^2}{(2\hbar\kappa/m)+i(v_1-v_2)}\right), \tag{3.10}$$

where $\eta_\alpha=\exp[\kappa(x-v_\alpha t)]$, and

$$\Delta=1+\eta_1{}^2+\eta_2{}^2+\frac{[\eta_1\eta_2(v_1-v_2)]^2}{(2\hbar\kappa/m)^2+(v_1-v_2)^2}. \tag{3.11}$$

ψ_2 is obtained from Eq. (3.10) by interchanging subscripts 1 and 2. In deriving ψ_1 and ψ_2, we have put

$$\mathrm{Re}\kappa_1=\mathrm{Re}\kappa_2=\kappa=\tfrac{1}{4}g, \tag{3.12}$$

$$\mathrm{Im}\kappa_\alpha=mv_\alpha/\hbar, \tag{3.13}$$

$$A_1=A_2=2\kappa, \tag{3.14}$$

and

$$\hbar\omega_\alpha=-\frac{\hbar^2}{2m}\mathrm{Re}(\kappa_\alpha{}^2)=-\frac{(\hbar\kappa)^2}{2m}+\frac{mv_\alpha{}^2}{2}. \tag{3.15}$$

If $v_1=v_2$, then ψ_1 and ψ_2 are both reduced to ψ of Eq. (3.7), and the Slater determinant Ψ vanishes. The nucleon density is given by

$$\rho(x,t)=4[|\psi_1(x,t)|^2+|\psi_2(x,t)|^2], \tag{3.16}$$

where the factor 4 is due to the spin-isospin multiplicity.

In order to have a feel for the solution let us examine its asymptotic behavior for $t\to\pm\infty$. Let us put $v_1=0$ and $v_2>0$. Then we find for $t\to\infty$,

$$\psi_1(t\to\infty)=(2\kappa)^{1/2}\exp(\kappa x-i\omega_1 t)/(1+e^{2\kappa x})=\phi(x)e^{-i\omega_1 t}, \tag{3.17}$$

$$\psi_2(t\to\infty)=(2\kappa)^{1/2}\exp[\kappa(x-v_2t)+i(mv_2/\hbar)x-i\omega_2 t]\times\xi/\{1+|\xi|^2\exp[2\kappa(x-v_2t)]\}=\phi(x-c-v_2t)\exp\{i[(mv_2/\hbar)x-\omega_2 t+\chi]\}, \tag{3.18}$$

where $\phi(x)$ is defined by Eq. (3.4), while ξ, c, and χ are defined by

$$\xi=\frac{iv_2}{(2\hbar\kappa/m)+iv_2}=\exp(-\kappa c+i\chi). \tag{3.19}$$

For $t\to\infty$ we obtain

$$\psi_1(t\to-\infty)=\phi(x-c)\exp[-i(\omega_1 t+\chi)], \tag{3.20}$$

$$\psi_2(t\to\infty)=\phi(x-v_2t)\exp\{i[(mv_2/\hbar)x-\omega_2 t]\}. \tag{3.21}$$

More generally, for $v_2>v_1$ we find

$$\psi_1(t\to\infty)=\phi(x-v_1t)e^{-i\omega_1 t},\qquad \psi_1(t\to-\infty)=\phi(x-c-v_1t)\times\exp\{i[(mv_1/\hbar)x-\omega_1 t-\chi]\}, \tag{3.22}$$

while $\psi_2(t\to\pm\infty)$ remain the same as those of Eqs. (3.18) and (3.21). The parameter ξ in this general case is defined by Eq. (3.19), with v_2 replaced by v_2-v_1.

There is another interesting situation in which we can visualize ψ_1 and ψ_2 relatively easily. For $t=0$, we find that $\eta_1=\eta_2$, and hence,

$$\psi_\alpha(x,0)=f_\alpha(x)\exp(imv_\alpha x/\hbar). \tag{3.23}$$

Here $f_\alpha(x)$ is a function which can be easily found out from Eq. (3.11) by putting $\eta_1=\eta_2$; $f_1(x)$ and $f_2(x)$ are complex and related by $f_1(x)=f_2^*(x)$. For the densities in each orbital, we obtain

$$|\psi_1(x,0)|^2=|\psi_2(x,0)|^2=\tfrac{1}{4}\rho(x,0). \tag{3.24}$$

Therefore all eight nucleons are distributed in the same manner. Nevertheless, the Slater determinant does not vanish because of the phase factor of Eq. (3.10). One may have wondered that, because of the Pauli principle, the two orbitals try to avoid each other such that their density overlap is minimized. Our model calculation illustrates that the Pauli principle is not as effective in the collision process as in the stationary state. If $|v_1-v_2|$

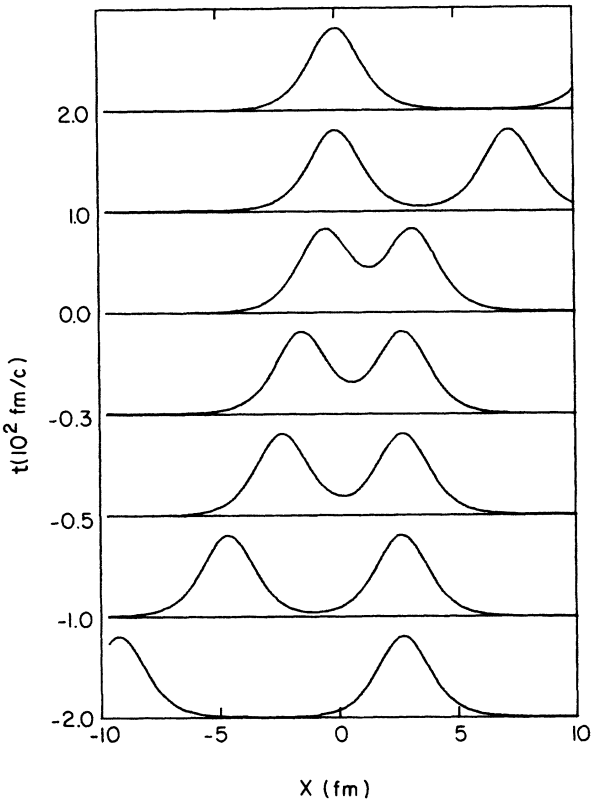

FIG. 1. The density $\rho(x,t)$ for "α-α scattering." The kinetic energy per nucleon $\frac{1}{2}mv^2$ of the incident nucleus is 1 MeV. The units are fm both for x anc ct, where c is the speed of light.

$\gg 2\hbar/m$, or $m|v_1-v_2|^2/2 \gg 2(\hbar\kappa)^2/m$, we find

$$|\psi_1(x,0)| \sim |\psi_2(x,0)| \sim \phi(x)\,. \tag{3.25}$$

In the limit of $|v_1-v_2| \to \infty$, we obtain $|\psi_1(x,0)| = |\psi_2(x,0)| = \phi(x)$, and the nucleon density in the α-α system at $t=0$ is exactly twice that in a single α particle. As we will illustrate below, the density reaches its maximum at $t=0$.

The density $\rho(x,t)$ for the "α-α scattering" with $v_1=0$ and $v_2=v>0$ is shown in Fig. 1. For the parameters of the model we take the nucleon mass $mc^2=938.9$ MeV, $\hbar^2/m=41.47$ MeV fm^2, and g such that the "deuteron" binding energy becomes 2.2 MeV, that is[12]

$$\frac{\hbar^2}{16m}\left(\frac{g}{3}\right)^2 = 2.2 \text{ MeV}\,. \tag{3.26}$$

With this g, $|E|$ of Eq. (3.6), the binding energy of the α particle becomes 13.2 MeV. For Fig. 1 we took $\frac{1}{2}mv^2$, the kinetic energy per particle of the incident nucleus, to be 1 MeV. The density builds up to its maximum at $t=0$.

The space-time trajectory of the incident nucleus is $x=vt$ (for $t\to-\infty$) and that of the emerging nucleus is $x=c+vt$ (for $t\to+\infty$). Therefore the time advance in this scattering process is given by

$$\Delta t = \frac{c}{v} = -\frac{1}{\kappa v}\ln|\xi| = \frac{1}{2\kappa v}\ln\left[1+\left(\frac{2\hbar\kappa}{mv}\right)^2\right]. \tag{3.27}$$

If κ is very small, Δt becomes

$$\Delta t \approx \frac{2\hbar^2\kappa}{m^2v^3} = \frac{\hbar^2\kappa}{(2m)^{1/2}\epsilon_k^{3/2}} \tag{3.28}$$

where $\epsilon_k=\frac{1}{2}mv^2$. This time advance could be approximately related to the scattering phase shift which in turn is related to χ defined by Eq. (3.19).

Dolan[13] discussed a similar (but different) problem with a one-component NLS equation. In her case, the NLS equation can be interpreted as the time-dependent Hartree equation for bosons. The effective interaction between two solitons, then, is attractive and she obtained a time delay rather than advance. In our case the effective interaction between the solitons is repulsive because of the Pauli principle, and hence the time advance.

IV. EXTENSION

The model presented in Sec. III allows only one (spin-isospin saturated) nucleus. This restriction can be removed by introducing another degree of freedom. We consider a system of nucleons in two dimensions, but they interact with each other through the same one-dimensional interaction (3.1) as that in Sec. III. For the single-particle wave function, we assume

$$\psi_{\alpha n}(x,y,t) = \psi_\alpha(x,t)h_n(y)\,, \tag{4.1}$$

where $h_n(y)$ with $n=1,2,\ldots$ are harmonic oscillator wave functions and are normalized by $\int_{-\infty}^{\infty}h_n^2(y)dy=1$. The oscillator constant is a fixed parameter. For the index α we use $\alpha=0,1,2,\ldots$.

First we consider a stationary state. For the mass number $A=4$, there is no difference between this model and the one considered in Sec. III. The simplest nontrivial generalization can be seen for $A=12$, which we now examine. For the ground state of this nucleus let us assume the configuration

$$(\alpha,n)=(0,0),\ (1,1),\ \text{and}\ (2,0)\,. \tag{4.2}$$

Each (α,n) orbital accommodates four nucleons. It is possible to choose the oscillator constant so that configuration (4.2) leads to an energy lower than any others, such as $(\alpha,n)=(0,0)$, (1, 1), and (2.2).

The total energy of the system consists of two parts, $E=E_x+E_y$, where E_x and E_y are due to the x and y degrees of freedom, respectively. E_y is

just the kinetic energy with respect to y. Let us denote the single-particle state attached to ψ_0, ψ_1, and ψ_2 by $|0\rangle$, $|1\rangle$, and $|2\rangle$, respectively. Denoting the Hamiltonian by $H=\sum_i t_i+\frac{1}{2}\sum_{ij}v_{ij}$, the energy is given by

$$E_x=4(\langle 0|t|0\rangle+\langle 1|t|1\rangle+\langle 2|t|2\rangle)+6(\langle 00|v|00\rangle+\langle 11|v|11\rangle+\langle 22|v|22\rangle)+16\langle 01|v|01\rangle+12\langle 02|v|02\rangle+16\langle 12|v|12\rangle, \tag{4.3}$$

$$E_y=5\hbar\omega_y. \tag{4.4}$$

Here ω_y is the harmonic oscillator constant with respect to y.

The HF equations are given by

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+g(\phi_0^2+\tfrac{4}{3}\phi_1^2+\phi_2^2)\right)\phi_\alpha=\epsilon_\alpha\phi_\alpha, \tag{4.5}$$

for $\alpha=0, 2$, and

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+g(\tfrac{4}{3}\phi_0^2+\phi_1^2+\tfrac{4}{3}\phi_2^2)\right)\phi_1=\epsilon_1\phi_1. \tag{4.6}$$

These equations are not of the form of Eq. (2.3) because the HF potentials in Eqs. (4.5) and (4.6) are different. However, if we rewrite the potential terms in the above equation as

$$\phi_0^2+\tfrac{4}{3}\phi_1^2+\tfrac{7}{6}\phi_2^2-\tfrac{1}{6}\phi_2^2, \tag{4.7}$$

$$\tfrac{4}{3}\phi_0^2+\phi_1^2+\tfrac{7}{6}\phi_2^2+\tfrac{1}{6}\phi_2^2, \tag{4.8}$$

and ignore the $\pm\frac{1}{6}\phi_2^2$ terms, we can put $\phi_0=\phi_1$ and $\epsilon_0=\epsilon_1$, and $\phi_1(=\phi_0)$ and ϕ_2 then satisfy

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{7g}{6}(2\phi_1^2+\phi_2^2)\right)\phi_\alpha=\epsilon_\alpha\phi_\alpha, \tag{4.9}$$

which can be identified with Eq. (2.3), with

$$\lambda_1=2g',\quad \lambda_2=g',\quad g'=\tfrac{7}{6}g. \tag{4.10}$$

Let us examine the solutions of this case as an approximation.[14] We will discuss the effect of the ignored terms later.

The single-particle wave functions ϕ_1 and ϕ_2 are determined by Eq. (2.2). Unlike the cases discussed in Sec. III, however, the choice of A_α seems to have a nontrivial consequence. If we put

$$A_\alpha=2\kappa_\alpha\gamma_\alpha, \tag{4.11}$$

we find

$$\phi_1=(A_1^{1/2}/\Delta)e^{\kappa_1 x}[1+(\gamma_2/\gamma)e^{2\kappa_2 x}], \tag{4.12}$$

$$\phi_2=(A_2^{1/2}/\Delta)e^{\kappa_2 x}[1-(\gamma_1/\gamma)e^{2\kappa_1 x}], \tag{4.13}$$

where

$$\Delta=1+\gamma_1 e^{2\kappa_1 x}+\gamma_2 e^{2\kappa_2 x}+(\gamma_1\gamma_2/\gamma^2)e^{2(\kappa_1+\kappa_2)x} \tag{4.14}$$

and

$$\gamma=(\kappa_1+\kappa_2)/(\kappa_1-\kappa_2). \tag{4.15}$$

The κ_α and ϵ_α are determined by Eqs. (2.4) and (4.10); hence $\gamma=3$. The occupation numbers for levels 1 and 2 are 8 and 4, respectively. Unlike the usual shell model, the lower level has larger occupation number. The nucleon density in the "carbon" nucleus is given by

$$\rho(x)=4(2\phi_1^2+\phi_2^2). \tag{4.16}$$

Now recall that the choice of A in Eq. (3.4) was simply related to that of the origin. Although we did not discuss it, the choice of the A's of Eq. (3.14) is related to that of the initial conditions for the colliding "nuclei." In all these cases the structure of the nucleon density in an isolated nucleus is not affected by the choice of the A's. For the two level system described by the above ϕ_1 and ϕ_2, however, the nucleon density distribution does depend on the A's (or γ_1 and γ_2). If we put

$$\gamma_1=\gamma_2=\gamma, \tag{4.17}$$

ϕ_1 and ϕ_2 become even and odd functions of x, respectively. For any other choice of γ_1 and γ_2, ϕ_1 and ϕ_2 will not have any definite parity. Figure 2 compares the functions ϕ_1 and ϕ_2 with γ_1 and γ_2 of Eq. (4.17) with those of

$$\gamma_1=\gamma,\quad \gamma_2=\tfrac{1}{10}\gamma. \tag{4.18}$$

In the latter case the center of mass of the nucleus is still at the origin (see Appendix A).

Scattering between two nuclei, with mass number 12 and 4 (carbon and α particle) can be examined in a manner very similar to the scattering in the model of Sec. III. The only difference is that we now have three single-particle wave

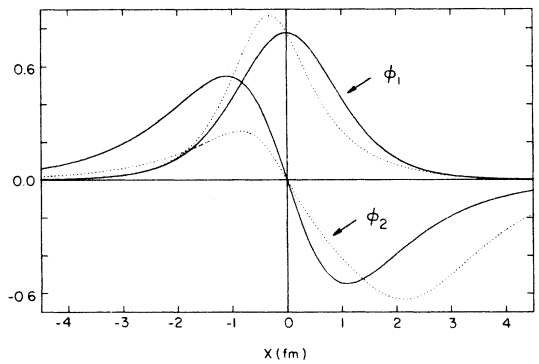

FIG. 2. The single-particle wave functions ϕ_1 and ϕ_2 given by Eqs. (4.12) and (4.13), respectively. The solid lines are those with γ_1 and γ_2 of Eq. (4.17), while the dotted lines are for Eq. (4.18).

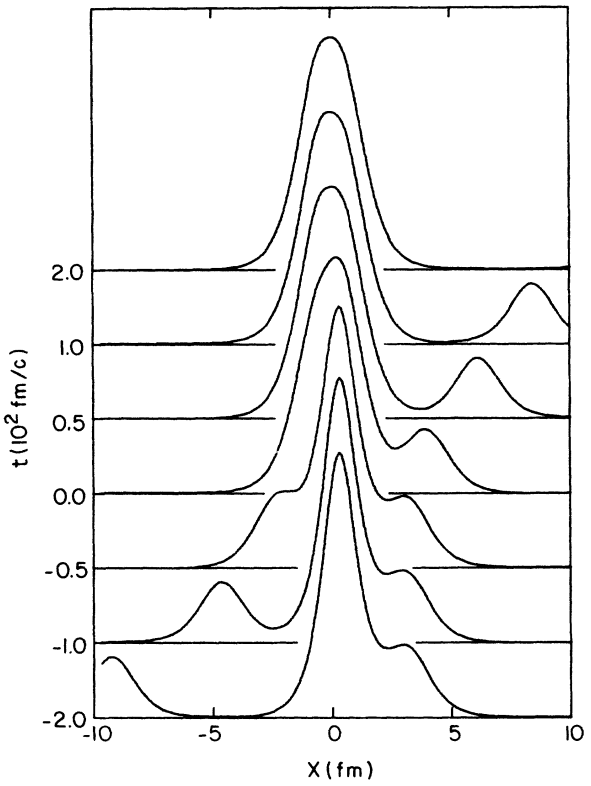

FIG. 3. The density $\rho(x,t)$ for "carbon-α" scattering. The kinetic energy per nucleon $\frac{1}{2}mv^2$ of the incident α particle is 1 MeV. The units are fm both for x and ct, where c is the speed of light. For $|t| \gtrsim 200$ fm/c, the two nuclei are well separated and their shapes no longer change.

functions ψ_1, ψ_2, and ψ_3. The first two are for nucleus 12, while ψ_3 is for nucleus 4. Choose κ's such that $\mathrm{Im}\kappa_1 = \mathrm{Im}\kappa_2 \neq \mathrm{Im}\kappa_3$. In this way, orbitals ψ_1 and ψ_2 are bound together, while orbital ψ_3 moves with a different velocity. Figure 3 shows $\rho(x,t)$ for the carbon-α scattering. Here we have used the same m and g as in Sec. III, and $\gamma_1 = \gamma_2 = \gamma$ and $\gamma_3 = 1$. Also $v_1 = v_2 = 0$, and the kinetic energy per particle of the incident α particle is $\frac{1}{2}mv_3^2 = 1$ MeV.

For $t \to \infty$, the carbon nucleus is symmetric. This corresponds to wave functions shown by solid lines in Fig. 2. However, the carbon nucleus for $t \to -\infty$ is asymmetric. This means that the symmetry of the density distribution in the carbon nucleus is not conserved throughout the scattering process.[15] The carbon-α scattering shown in Fig. 3 is different from a stereotyped soliton-soliton collision in which the solitons emerge from the collision having the same shapes with which they entered.

The equations which we have solved are approximate ones [Eq. (2.3) with Eq. (4.10)] rather than the true TDHF equations (4.5) and (4.6). A natural question is as follows: Will the solution which we obtained resemble the solution of the true equations? In order to be able to answer this question we have to know what the degeneracy which we found with respect to the shape change means. To this end we estimated the effect of the neglected terms in Eqs. (4.7) and (4.8) by perturbation theory. The single-particle energy ϵ_α is changed to $\epsilon_\alpha + \Delta\epsilon_\alpha$, where

$$\Delta\epsilon_0 = -\Delta\epsilon_1 = \frac{\hbar^2}{2m}\frac{g}{6}\int_{-\infty}^{\infty}(\phi_1\phi_2)^2 dx\,, \tag{4.19}$$

$$\Delta\epsilon_2 = \frac{\hbar^2}{2m}\frac{g}{6}\int_{-\infty}^{\infty}\phi_2{}^4 dx\,, \tag{4.20}$$

where ϕ_1 and ϕ_2 are those of Eqs. (4.12) and (4.13). The contribution to the total energy is

$$\Delta E = 2\sum_{\alpha=0}^{2}\Delta\epsilon_\alpha = 2\Delta\epsilon_2\,. \tag{4.21}$$

Without losing generality we put $\gamma_1 = \gamma = 3$; then $\Delta\epsilon_2$ is a function of γ_2. It can be shown that $\int\phi_2{}^4 dx = 9\kappa_2/35$ and $(d/d\gamma_2)\int\phi_2{}^4 dx = 0$ for $\gamma_2 = 3$. For $\gamma_2 \approx 3$, we obtain

$$\Delta\epsilon_2 = 1.98 + 0.025(\gamma_2 - 3)^2 + \cdots \text{ MeV}. \tag{4.22}$$

It is therefore very likely that the exact solution of the true equations (4.5) and (4.6) will yield a symmetric density distribution $\rho(x)$.

The dependence of the energy on the shape parameter γ_2 probably implies that the model nucleus has a vibrational spectrum. We speculate that the deformed state of the carbon nucleus for $t \to -\infty$ in Fig. 3 would correspond to a superposition of a number of vibrational states. Following the time backward in Fig. 3, the carbon nucleus is excited from the ground state into vibrational states due to the collision with an α particle.[15]

V. SLAB IN THREE DIMENSION

This is the same slab model in three dimensions as that adopted by Bonche *et al*,[1] except that we use a much simpler interaction

$$V_{ij} = -\frac{\hbar^2}{2m}\frac{g}{3}\delta(\vec{r}_i - \vec{r}_j)\,. \tag{5.1}$$

This is equivalent to taking only the first term of the effective nucleon-nucleon interaction which Bonche *et al*.[1] used. Many of the formulas that we need can be found in their paper. We will use essentially the same notations as theirs.

For a static slab, we start with the single-particle wave function

$$\psi_i(r) = \psi_{\alpha\vec{k}_\perp}(r) = \phi_\alpha(x)e^{i\vec{k}_\perp\cdot\vec{r}_\perp}/\Omega^{1/2}\,, \tag{5.2}$$

where $\vec{k}_\perp$ and $\vec{r}_\perp$ are transverse vectors, and Ω is

the area of the slab in the yz plane. The normalization is

$$\int|\psi_i(\vec{r})|^2 d\vec{r} = \int_{-\infty}^{\infty} \phi_\alpha^2(x)dx = 1 . \tag{5.3}$$

The single-particle energy is given by $(\hbar\vec{k}_1)^2/2m + \epsilon_\alpha$ where $(\hbar\vec{k}_1)^2/2m$ is the kinetic energy due to the yz degrees of freedom, while ϵ_α is determined by solving Eq. (2.3) with

$$\lambda_\beta = g\mathfrak{A}_\beta , \quad \mathfrak{A}_\beta = \frac{2m}{\hbar^2}\frac{\epsilon_F - \epsilon_\beta}{\pi} , \tag{5.4}$$

where ϵ_F is the Fermi energy, $\mathfrak{A}_\beta$ is the number of particles in state β per unit area. Incidentally, the density per unit area is given by

$$\rho(x) = 4\sum_i |\psi_i|^2/\Omega = \sum_\alpha \mathfrak{A}_\alpha \phi_\alpha^2(x) , \tag{5.5}$$

and the number of particles per unit area by

$$A/\Omega \equiv \mathfrak{A} = \sum_\alpha \mathfrak{A}_\alpha . \tag{5.6}$$

Equation (2.4) reads

$$g\mathfrak{A}_\alpha = 4\kappa_\alpha \text{ and } \epsilon_\alpha = -(\hbar\kappa_\alpha)^2/2m . \tag{5.7}$$

Hence,

$$\kappa_\alpha^2 - (4\pi/g)\kappa_\alpha + (2m/\hbar)\epsilon_F = 0 , \tag{5.8}$$

$$\kappa_\alpha = (2\pi/g) \pm [(2\pi/g)^2 - (2m/\hbar^2)\epsilon_F]^{1/2} . \tag{5.9}$$

If $\epsilon_F < 0$, there is only one acceptable (positive) root. In this case there is only one orbital and the model essentially reduces to the model of Sec. III.

If $(2m/\hbar^2)\epsilon_F > (2\pi/g)^2$, there is no solution.

If $(2m/\hbar^2)\epsilon_F < (2\pi/g)^2$, there are two acceptable roots, κ_1 and κ_2. However, it follows from $\kappa_1 + \kappa_2 = 4\pi/g$ and Eqs. (5.4) and (5.6) that

$$\mathfrak{A} = \frac{2m}{\hbar^2}\frac{1}{\pi}\sum_\alpha (\epsilon_F - \epsilon_\alpha) = \frac{4\pi}{g}\sum_\alpha \kappa_\alpha = (4\pi/g)^2 . \tag{5.10}$$

This means that this case could hold only for this specific density.

The energy E of the slab is given by

$$\frac{E}{\Omega} = \frac{\pi\hbar^2}{4m}\sum_\alpha \mathfrak{A}_\alpha^2 + \tfrac{1}{2}(T_x + \sum_\alpha \mathfrak{A}_\alpha\epsilon_\alpha) . \tag{5.11}$$

The first term on the right-hand side is the transverse kinetic energy, while

$$T_x = \frac{\hbar^2}{2m}\sum_\alpha \mathfrak{A}_\alpha \int_{-\infty}^{\infty} \left(\frac{d\phi_\alpha}{dx}\right)^2 dx \tag{5.12}$$

is the longitudinal kinetic energy. Using the virial theorem as in Sec. III we find $T_x = -\frac{1}{3}\sum_\alpha \mathfrak{A}_\alpha\epsilon_\alpha$, and hence, $\frac{1}{2}(T_x + \sum_\alpha \mathfrak{A}_\alpha\epsilon_\alpha) = \frac{1}{3}\sum_\alpha \mathfrak{A}_\alpha\epsilon_\alpha$. Equation (5.7) then enables us to write

$$\frac{E}{\Omega} = \frac{\hbar^2}{m}\frac{2}{3g}\sum_\alpha \left(\frac{12\pi}{g} - \kappa_\alpha\right)\kappa_\alpha^2 . \tag{5.13}$$

Now the question we ask is as follows: What is the ground state energy for given g and $\mathfrak{A}$? For this we have to determine κ's and ϵ_F. We can proceed as follows. (i) For the special value of $\mathfrak{A}$ given by Eq. (5.10) there are two κ's, κ_1 and κ_2, such that $\kappa_1 + \kappa_2 = 4\pi/g$. However, the minimization of E of Eq. (5.13) requires $\kappa_1 = \kappa_2$, and hence the model in this case essentially reduces to the model of Sec. III. (ii) For other values of $\mathfrak{A}$ there is only one value of κ at most, and κ is related to $\mathfrak{A}$ by $\mathfrak{A} = 4\kappa/g$. If $\kappa > 4\pi/g$, then $\epsilon_F < 0$, which corresponds to the first case discussed below Eq. (5.9), and the model reduces to the model of Sec. III. If $\kappa < 4\pi/g$, there is no acceptable root for κ.

VI. DISCUSSION

We have presented models of quantum mechanical many-body systems such that the HF and TDHF equations can be solved analytically. The solutions we obtained simulate an isolated nucleus or nucleus-nucleus collision. The models are all essentially one dimensional, and this is a severe restriction. Nevertheless, our models exhibit some interesting features. In Sec. III we illustrated that the Pauli principle is not as effective in the collision process as in the stationary state. This helps in building up a high density. In Sec. IV we showed that the solutions of the approximate equations exhibit degeneracy with respect to the shape change of the model nucleus. We conjectured that this degeneracy might be related to the vibration in the solutions of the correct equations. We discussed how the nucleus is deformed, or vibrationally excited, through the collision process.

In the model discussed in Sec. V, there is no saturation in the sense that $E \to -\infty$ as $\mathfrak{A} \to \infty$. This could be remedied by introducing a repulsive three-body force of the form of $\delta(\vec{r}_i - \vec{r}_j)\delta(\vec{r}_j - \vec{r}_k)$. With this additional force, the TDHF equation becomes

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \lambda\rho - \lambda'\rho^2\right)\psi_\alpha = i\hbar\frac{\partial\psi_\alpha}{\partial t} . \tag{6.1}$$

We have been able to solve this equation only for the one-component, stationary state case, i.e., $\psi(x,t) = \phi(x)\exp(-i\epsilon t/\hbar)$. We show this in Appendix B. It would be very interesting if Eq. (6.1) with many components could be solved.[16]

Apart from specific features of our models, we feel that the TDHF method is too deterministic. We mean by this that, for a given initial state of nucleus-nucleus scattering, the density distribution in the final state is uniquely determined. The TDHF method is, in this sense, almost a classical

description of nucleus-nucleus scattering. The TDHF method could be extended in this respect by quantizing solitons. Such an attempt has been done for bound states recently by Klein and Krejs.[17] Quantum effects would probably be much more important for scattering problems than for bound states, and we are hoping that our models would serve as a starting point for such an extension.

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APPENDIX A

We will show for our model of the carbon nucleus, Sec. IV, that, if $\gamma_1=\gamma=3$, the center of mass is at the origin, that is

$$\int_0^\infty (2\phi_1^2+\phi_2^2)dx=\int_{-\infty}^0 (2\phi_1^2+\phi_2^2)dx\,. \tag{A1}$$

Using Eq. (2.6) we obtain

$$I_\alpha^+\equiv\int_0^\infty \phi_\alpha{}^2dx=D_{\alpha\alpha}^{-1}(0)-D_{\alpha\alpha}^{-1}(\infty)=D_{\alpha\alpha}^{-1}(0)\,, \tag{A2}$$

$$I_\alpha^-\equiv\int_{-\infty}^0 \phi_\alpha{}^2dx=1-D_{\alpha\alpha}^{-1}(0)\,. \tag{A3}$$

From the definition of D, Eq. (2.1), follows

$$D_{\alpha\alpha}{}^{-1}(0)=\frac{1}{\Gamma}\times\begin{cases}1+\gamma_2\\ 1+\gamma_1\end{cases}\quad\text{for }\alpha=\begin{cases}1\,,\\ 2\,.\end{cases} \tag{A4}$$

where $\Gamma=1+\gamma_1+\gamma_2+(\gamma_1\gamma_2/\gamma^2)$. Equations (A2-A4), with $\gamma_1=\gamma$, yield

$$2(I_1^+-I_1^-)+(I_2^+-I_2^-)=\frac{(3-\gamma)(\gamma-\gamma_2)}{\gamma\Gamma}\,, \tag{A5}$$

which vanishes if $\gamma=3$, irrespective of the value of γ_2, hence Eq. (A1).

APPENDIX B

For the one-component, stationary case. Eq. (6.1), with

$$\psi(x,t)=\phi(x)\exp(i\hbar\kappa^2t/2m)\,,$$

is reduced to

$$\left(\frac{d^2}{dx^2}+\lambda\phi^2-\lambda'\phi^4\right)\phi=\kappa^2\phi\,. \tag{B1}$$

This equation can be integrated in the same manner as that for the case of $\lambda'=0$, yielding

$$\phi^2=\frac{4\kappa^2}{\lambda+[\lambda^2-(\frac{1}{3}16\lambda'\kappa^2)]^{1/2}\cosh(2\kappa x)}\,, \tag{B2}$$

Here it is understood that $\lambda>0$, but λ' can be positive or negative. Also we have chosen the origin such that $\phi(x)$ becomes an even function of x. If we require that ϕ be normalized by

$$\int_{-\infty}^\infty \phi^2dx=1\,, \tag{B3}$$

κ is determined to be

$$\kappa^2=\frac{3\lambda^2}{16|\lambda'|}\times\begin{cases}\tanh^2[(\lambda'/3)^{1/2}]\\ \tan^2[(-\lambda'/3)^{1/2}]\end{cases}\quad\text{for }\begin{cases}\lambda'>0\,,\\ \lambda'<0\,.\end{cases} \tag{B4}$$

For $\lambda'\to 0$, $\kappa\to\frac{1}{4}\lambda$, and Eq. (B2) is reduced to Eq. (3.4).

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[1] P. Bonche, S. Koonin, and J. W. Negele, Phys. Rev. C 13, 1226 (1976).

[2] R. Y. Cusson and J. Maruhn, Phys. Lett. 62B, 134 (1976); R. Y. Cusson, R. K. Smith, and J. A. Maruhn, Phys. Rev. Lett. 36, 1166 (1976).

[3] S. E. Koonin, K. T. R. Davies, V. Maruhn-Rezuani, H. Feldman, S. J. Krieger and J. W. Negele, Phys. Rev. C 15, 1359 (1977).

[4] Y. Nogami and C. S. Warke, Phys. Lett. 59A, 251 (1976).

[5] Equations (2.2) and (2.10) correspond to Eqs. (3) and (10) of Ref. 4, respectively, but the sign of the $\hat{e}$ and $\hat{u}$ terms has been reversed. The functions ϕ_α and ψ_α in this paper correspond to $-f_\alpha$ and $-g_\alpha$ of Ref. 4, respectively.

[6] The following simple proof was communicated to us by Dr. A. Hayashi.

[7] The factor of 3 in Eq. (3.1) is cancelled by another factor of 3 which arises due to spin and isospin.

[8] The equation then becomes the same as that considered by F. Calogero and A. Degasperis, Phys. Rev. A 11, 265 (1975).

[9] This A should not be confused with the mass number A. More generally one can put $A=2\kappa\exp(-2\kappa x_0)$. Then, x in Eq. (3.4) is replaced by $x-x_0$.

[10] L. L. Foldy, Am. J. Phys. 44, 1192 (1976). This

follows from the scaling property $\delta(x)=c\delta(cx)$.

[11]J. B. McGuire, J. Math. Phys. 5, 622 (1964).

[12]In this connection see H. T. Coelho, Y. Nogami, and M. Vallières, Can. J. Phys. 54, 376 (1976). For the accuracy of the HF method for McGuire's model, see Ref. 8.

[13]L. Dolan, Phys. Rev. D 13, 528 (1976).

[14]In this respect, the title of our paper is misleading.

[15]We can reverse the sense of time by $t\rightarrow -t$ and $\psi \rightarrow \psi^*$. Then the initial and final states are interchanged; the target nucleus is initially symmetric and becomes asymmetric after the collision.

[16]We conjecture that the time-independent equations $[d^2/dx^2 - V(\rho)+\epsilon_\alpha]\phi_\alpha=0$ have no solutions other than those in which all ϕ_α's are identical. We have no proof, but this is based on computer experiments done in collaboration with Dr. M. Vallières. If this conjecture is true, it would mean that the nonlocal term $\int \rho(x')\exp[-|x'-x|/a]dx$ is playing an essential role in the slab model of Bonche *et al.* (Ref. 1).

[17]A. Klein and F. Krejs, Phys. Rev. D 13, 3282 (1976). See also C. R. Nohl, Ann. Phys. (N.Y.) 96, 234 (1976).