Cohesion of nuclear matter

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The fused system formed in a heavy-ion collision breaks up immediately if the longitudinal energy per particle is greater than 2 MeV/A (c.m.). This result is derived using the classical theory of compression and rarefaction waves. The dependence of the result on the equation of state of nuclear matter is slight. There is some dependence on the range of the interaction.

[NUCLEAR REACTIONS Heavy ion reactions, nuclear matter, fragmentation.]

To obtain a qualitative understanding of heavy-ion collisions at higher bombarding energies, it is desirable to delineate the conditions for the fused system of nuclear matter to break up immediately, and the conditions for the system to remain fused. On the experimental side, the phenomenon of deep inelastic scattering shows that rapid fission is extremely probable when the mass of the system is large enough, even when the bombarding energy is relatively low. On the theoretical side, numerical studies of one-dimensional dynamics[1,3] show that nuclear matter is not very strong: when the energy exceeds 2 MeV/nucleon, the fused system quickly splits apart. At first sight this breakup threshold seems low, since the most obvious controlling parameter is the equilibrium binding energy which has a value of 16 MeV/A. In this note we will show how this result arises from continuum mechanics.

Before proceeding, it is necessary to justify the application of continuum mechanics and one-dimensional models. It is always possible to define for a many body system the local density $\rho(r)$, current $\vec{j}(r)$, and the stress tensor S_{ij}. The equation of continuity relating ρ and $\vec{j}$ can be derived from the quantum equation of motion.[4] An equation of motion for the $\partial \vec{j}/\partial t$ can also be derived, relating it to S_{ij}. Classically, a closed system of equations is obtained when S_{ij} is expressed in terms of local quantities such as ρ and the gradient $\vec{j}$. The simplest approximation, giving the Euler equation of hydrodynamics, is that S_{ij} depends on ρ alone:

$$S_{ij} = \delta_{ij} P(\rho).$$

In this situation the motions in the three dimensions are strongly coupled together. On the other hand, if the stress associated with a distortion $\epsilon_{ij} = \frac{1}{2}$ ($\nabla_i u_j + \nabla_j u_i$) is given by

$$S_{ij} = c \epsilon_{ij},$$

then the three directions are essentially uncoupled

in the equation of motion. In mean field theory, with parameters in a range considered reasonable for nuclear matter, there is in fact considerable decoupling. More compelling than the theoretical argument is the experimental evidence on the dynamics from the giant multipole vibrations. There is a well-established quadrupole mode at $63/A^{1/3}$ and recent evidence for a monopole at $80/A^{1/3}$ (Ref. 5). The near degeneracy of these two modes implies that distortional motion along the different coordinate axes is nearly uncoupled, and thus the motion is essentially one dimensional. Of course, we know that the fluid description also has a realm of validity. The liquid drop formula for nuclear binding describes the potential energy of the usual fission process quite well, at least before the scission point. The difference is the time scale. The stress tensor remains anisotropic on a time scale characteristic of the damping of the giant quadrupole state. The giant quadrupole has a width of about 3 MeV in ^{208}Pb, which implies a damping time for the stress tensor of about 60 fm/c. The usual fission of nuclei near their ground state takes a much longer time. An appropriate measure is perhaps the residual interaction between quasi-particles. Estimating this as 100 keV, the isotropic fluid description holds for times ~2000 fm/c.

We will consider two basic models for the stress tensor. The first is the Thomas-Fermi model, in which the particles are assumed to occupy a Fermi sphere of minimum radius. In that case the stress is isotropic and given by

$$S_{ii} = P = \frac{\partial E}{\partial V} = \rho^2 \frac{\partial E/A}{\partial \rho}. \tag{1}$$

This requires knowledge of the relationship between energy and density. We will follow Zamick[6] and parametrize the equations of state by an equation of the form

$$E/A = \frac{\hbar^2}{2m} \langle k^2 \rangle + A\rho + B\rho^{\sigma+1}. \qquad (2)$$

The first term represents the single-particle kinetic energy, and the last two terms represent effects of an ordinary nucleon potential and of a density- or velocity-dependent potential. The constants are adjusted to reproduce the equilibrium binding and density. The dynamic Thomas-Fermi model results when Eq. (2) is substituted in Eq. (1), where the explicit dependence of the kinetic energy on density is

$$\frac{\hbar^2}{2m} \langle k^2 \rangle = \frac{3\hbar^2}{10m} k_F^2 \left(\frac{\rho}{\rho_0}\right)^{2/3} \quad \text{(fluid)} \qquad (3)$$

with ρ_0 the equilibrium density and k_F the Fermi momentum at equilibrium.

The second model of the stress tensor, based on mean field theory, assumes that the Fermi surface of the nucleon distribution deforms as the system becomes distorted. In mean field dynamics, a compression in the x direction produces a change in the x component of the momenta only. Thus, the dependence of the kinetic energy on density will be

$$\frac{\hbar^2}{2m} \langle k^2 \rangle = \frac{\hbar^2}{2m} \langle k_y^2 + k_z^2 \rangle + \frac{\hbar^2}{2m} \langle k_x^2 \rangle$$

$$= \frac{1}{5} \frac{\hbar^2}{m} k_F^2 + \frac{1}{10} \frac{\hbar^2}{m} k_F^2 \left(\frac{\rho}{\rho_0}\right)^2 \quad \text{(elastic).} \quad (4)$$

In the two models, the stress in the x direction is given by

$$\frac{P}{\rho} = \frac{2}{5} \hbar^2 \frac{k_F^2}{2m} \left(\frac{\rho}{\rho_0}\right)^\alpha + A\rho + (\sigma+1)B\rho^{\sigma+1},$$

$$\alpha = \frac{2}{3} \quad \text{(fluid)}$$

$$= 2 \quad \text{(elastic)} \quad (5)$$

We now examine when a colliding system will break up. The most important continuum concept for discussion of breakup is the tensile strength of nuclear

matter. This is the negative maximum of the stress, and can be computed from

$$\frac{dP}{d\rho} = 0. \qquad (6)$$

This equation is equivalent to the classical condition for the sound velocity to go to zero, and for the system to become unstable against density fluctuations. When Eq. (2) is derived from Hartree-Fock theory, the same condition is obtained for the random phase approximation vibrations to go to zero frequency, and for the system to become unstable.

The critical density corresponding to Eq. (6) is given in Table I for a range of assumptions about the equation of state. It may be seen that the critical density falls in the range of 0.6–0.7 ρ_0. Note that it makes very little difference in the critical density, whether the fluid or elastic description is used. This point has also been noted by Holzwarth.[7] The energy needed to distort the system from equilibrium, to achieve the critical density, is also given in the table. The elastic model requires a greater energy to lower the density to critical. This energy is less than a quarter of the equilibrium energy. We also quote the value of the maximum negative stress. It is interesting to note that this is larger than the Coulomb stress that would be present in any system that could be made in the laboratory.

A first guess for the breakup threshold is that it equals the critical energy for maximum stress. In fact the breakup takes place at lower energies; to see why we need to trace the history of the collision in more detail. There are three important stages in the course of the initial collision:

(1) After the nuclei make contact, a step-function wave of compression propagates in both directions from the point of contact to the back sides. The density in the compressed region ρ' approximately satisfies

$$\frac{E}{A}(\rho') = E_0, \qquad (7)$$

TABLE I. Properties of nuclear matter at breakup threshold. The critical density is determined from Eqs. (5) and (6).

Eq. of state	σ	Critical density ρ_c	Max. pressure (MeV/fm^3)	$\frac{E}{A}(\rho_c) - \frac{E}{A}(\rho_0)$ (MeV)
Fluid	1	0.68	3.5	2.2
	$\frac{1}{3}$	0.64	2.4	1.8
Elastic	1	0.68	5.2	3.5
	$\frac{1}{3}$	0.64	4.2	3.2

where E_0 in the incoming energy of the nuclei in the
c.m. The leading edge of the wave is sharp in con-
tinuum mechanics, and may or may not be sharp in
mean field theory, depending on the specific equa-
tion of state.

(2) When the compression wave hits the back
sides of the nuclei, it is reflected as a decompres-
sion wave. The decompression is adiabatic and
the density on the decompressed side is ρ_0, the
normal density. In principle this wave does not
have a sharp edge; the thickness of the edge de-
pends on the difference in the sound velocities on
the two sides.

(3) When two decompression waves meet in the
middle, they essentially add and produce a second
decompression that propagates outward. This
rarefaction has a density below normal density. If
the density in the rarefaction is below critical den-
sity, the system will split apart.

An analogy to this wave behavior is the propaga-
tion of waves on a string. The boundary condition
on the surface of the nuclear slab, that it be free
and uncompressed, corresponds to a fixed boundary
condition for the displacement of the string. A
step-function wave propagating to the end is re-
versed upon reflection; the displacement behind the
wave is zero.

This picture is reasonable for the compressive
and first decompressive stages of the collision. It
suggests that the density change in the third stage
should be the same as the density change in the
second stage, but in fact when the density becomes
close to critical this linear analogy is inaccurate.
We need to consider in detail the behavior of rare-
faction waves. Analytic solutions can be obtained
for the shape of rarefaction waves, as well as re-
lations between velocity and density across the
wave.[8] For our purposes, it is only this latter re-
lation that is needed, which is

$$u = \int_{\rho_0}^{\rho''} \frac{c(\rho)d\rho}{\rho} . \tag{8}$$

Here u is the change in velocity of the medium

TABLE II. Dependence of critical velocity on equation
of state, from Eq. (8).

Eq. of state	σ	Critical velocity	Critical energy (MeV)
Fluid	1	0.048	1.1
	$\frac{1}{3}$	0.045	0.94
Elastic	1	0.062	1.8
	$\frac{1}{3}$	0.058	1.6

across the rarefaction, c is the sound velocity,
and ρ_0 and ρ'' are the initial and final densities. In
the third stage rarefaction of nuclear matter, the
velocity of the matter moving outwards changes
from some value slightly less than the incident
velocity, to zero. To find the critical velocity for
the second stage decompression, we solve Eq. (8)
with $\rho'' = \rho_c$. The integral may be roughly estimated
assuming a linear dependence of c on ρ. Since c
varies from c_0 to 0 when ρ changes from ρ_0 to ρ_c,
the critical velocity u_c is

$$u_c \simeq \tfrac{1}{2} c_0 \frac{\rho_0 - \rho_c}{\rho} \simeq \tfrac{1}{4} c_0 .$$

Thus the critical energy for the collision is

$$\frac{E}{A}\bigg|_{\text{crit}} = \tfrac{1}{2} m u_c^2 \simeq \frac{m c_0^2}{32} \tag{9}$$

if we assume that all of the initial energy is con-
verted to compressional energy in the first stage
and back to kinetic energy in the second stage.
Since c_0 is about $0.2c$, the critical energy is slight-
ly greater than 1 MeV$/A$. More accurate numeri-
cal values are given in Table II, along with the
implied initial energy. The dependence of the
threshold on the parameter σ in the equation of
state is slight, but there is a significant difference
between the elastic and fluid models. As mentioned
earlier, the elastic model, being based on mean
field theory, is more realistic.

We have also studied the breakup threshold

TABLE III. Breakup threshold from numerical hydrodynamics.

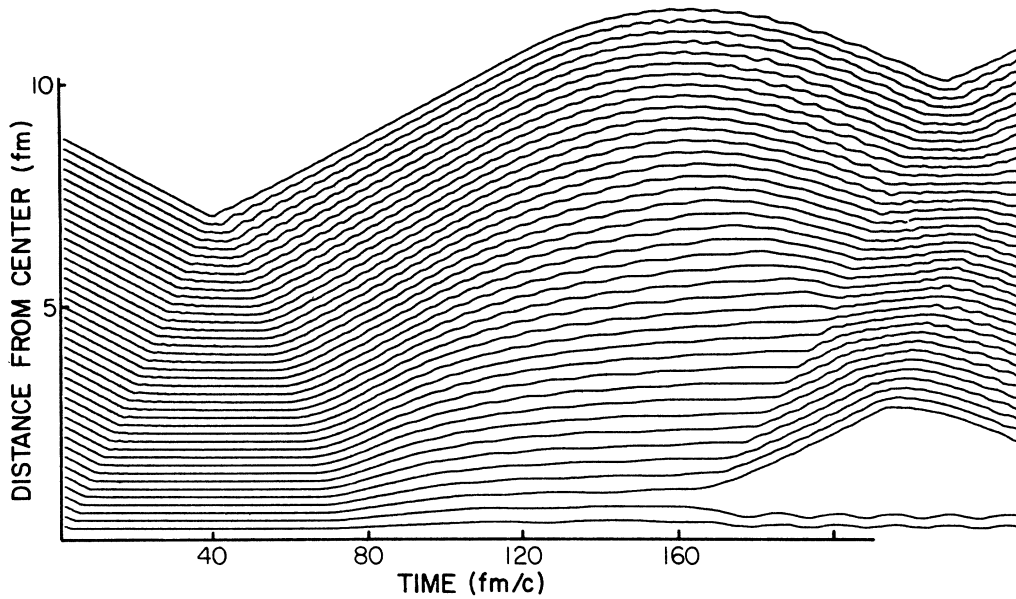| Fluid model σ	Breakup energy (MeV)	Static properties of model Range of interaction (fm)	Surface energy (MeV/fm^2)
1	1.2	0.61	0.31
$\frac{1}{3}$	1.1	0.8	0.31
1 with extended interactions	1.4	1.1	1.4
$\frac{1}{3}$ with extended interactions	1.5	2.0	2.2

FIG. 1. Slab collision at 1 MeV/A (c.m.). The position of mass elements as a function of time is shown for half of a symmetric collision of two slabs, each 9 fm thick. The slabs just make contact at $t = 0$; the equation of state is the fluid model with $\sigma = \frac{1}{3}$. Maximum compression occurs at 40 fm/c and the system snaps apart at ~160 fm/c.

numerically using a Lagrangian method to integrate the fluid dynamic equations. A typical collision is shown in Fig. 1, with time on the horizontal axis and position on the vertical axis. Elements of the fluid move with the lines. In Fig. 2 the behavior of the compression and rarefaction fronts are shown. There is essential agreement with the analytic arguments on the threshold energies and critical densities. In Table III, the numerical values of the threshold energy are given for several equations
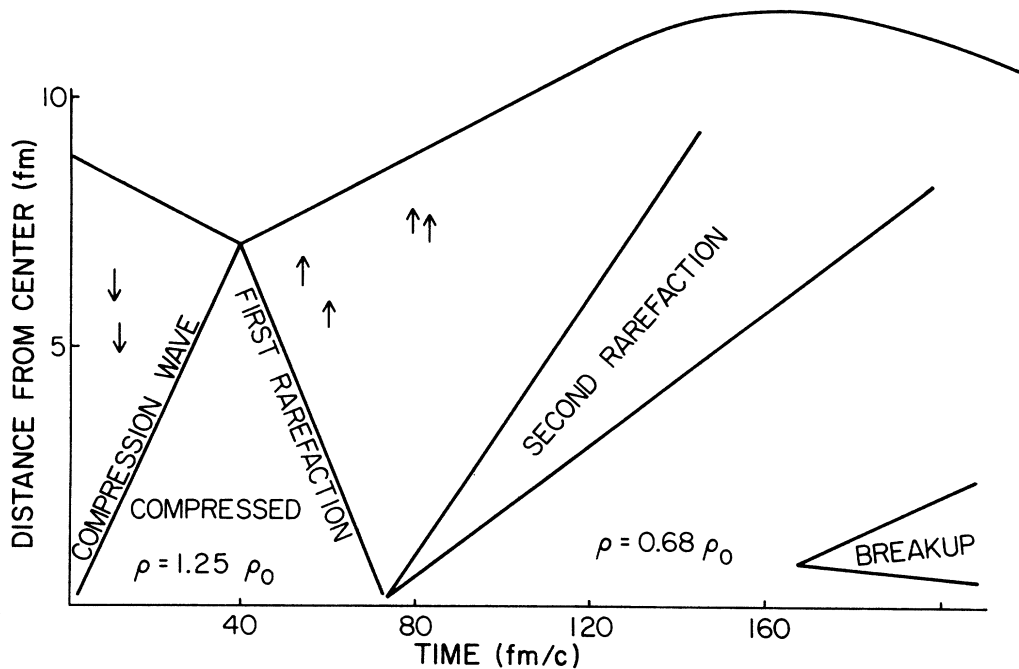

FIG. 2. Qualitative description of the slab collision in Fig. 1.

of state. The breakup threshold tends to be some-what higher in the numerical calculations than in the analytic treatment. There are two reasons for this. The nuclear matter can be below the critical density for a short time, and still not break apart, due to the finite time required for the instability to build up. Also, the finite range of the interaction can cause the system to pull together again after it has snapped apart. We introduce a finite range into the numerical model by allowing nonadjacent mass elements to interact, and also by varying the size of the mass elements. Collected results for the breakup threshold are shown in Table III. We de-fine the range of the interaction as the distance at which the potential energy between the slabs is half the value it has when the slabs are touching. The threshold for breakup is seen to depend on this quantity. We also compute the surface energy in these models, shown in the last column of the table.

The model with the greater surface energy has a higher breakup threshold, but we feel that this is just a consequence of the interaction having a longer range.

In conclusion, we have exhibited the dependence of the breakup threshold on the equation of state of nuclear matter. The threshold also is somewhat sensitive to the range of the interaction. There remain many important questions which we have not addressed. The dissipation in the first stage of the collision will increase the threshold energy, and it is important to develop a useful character-ization of the dissipation. We also need to under-stand how to modify the arguments in the presence of Coulomb and centrifugal fields. Finally, with these three-dimensional aspects of the collision considered in a minimal way, it should be possible to calculate cross sections in actual nuclei.

[1]P. Bonche, S. Koonin, and J. Negele, Phys. Rev. C 13, 1226 (1976).

[2]G. Holzwarth, Phys. Lett. 66B, 29 (1977).

[3]C. Y. Wong et al., Phys. Lett. 66B, 19 (1977).

[4]K. K. Kan and J. J. Griffin, Rev. Mod. Phys. 48, 467 (1976); E. Madelung, Z. Phys. 46, 322 (1926).

[5]D. H. Youngblood et al., Phys. Rev. Lett. 39, 1188

(1977); M. Sasao and Y. Torizuka, Phys. Rev. C 15, 217 (1977); N. Marty et al. (unpublished).

[6]L. Zamick, Phys. Lett. 45B, 313 (1973).

[7]G. Holzwarth, Phys. Rev. C 16, 885 (1977).

[8]L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Pergamon, New York, 1959), Eq. 92.6.