Comparison of two-term separable Λ-N potentials

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(Received 8 August 1977)

A comparison of two two-term separable Λ-N potentials which have been used in three-body calculations of the hypertriton binding energy is reported. The potentials compared are approximately on-shell T matrix equivalent, but yield quite different values for the hypertriton binding energy. It is shown that the radial wave function for one of the potentials has an extra node due to a spurious state. The comparison is based on a Fredholm determinant approach to the radial wave equation for a nonlocal potential.

[NUCLEAR STRUCTURE Two-term separable Λ-N potentials, Fredholm determinants, spurious states, extra nodes and separable potential three-body calculations.]

I. INTRODUCTION

Several years ago, Gibson and Lehman[1] studied the effect of variations of the Λ-N low-energy scattering parameters on the hypertriton binding energy. In this work, they employed attractive one-term separable nonlocal potentials as approximations to the Λ-N s-wave interactions. Their work was later criticized by Schick[2] for using approximate potentials which omitted the short-range repulsion of the Λ-N interactions. Specifically, using a two-term separable approximation to the potential, Schick showed that the hypertriton binding energy was significantly reduced in comparison with the binding energy obtained from a one-term approximation, and attributed the difference to the short-range repulsion of the second term. In reply, Gibson and Lehman[3] calculated the binding energy using a different two-term approximation and found a much smaller reduction of the binding energy in going from a one-term to a two-term approximation. The difference in the reduction of the hypertriton binding energy between these two calculations is not yet understood.[3]

The parameters of the two-term separable potentials used in each of the calculations were determined from Λ-N two-body information. This information, while not identical for the two calculations,[4] is sufficiently similar to warrant describing the two potentials as approximately on-shell T matrix equivalent. These circumstances suggest that the difference between the calculated hypertriton binding energies noted above reflects a substantial difference in the Λ-N wave function. Such a difference could be that one of the wave functions has an extra node while the other does not. Gibson and Lehman[3] suggest this possibility indirectly in referring to a paper by Arnold and MacKellar[5] which describes how a separable potential with short-range repulsion of the type used by Schick[2] can lead to a wave function with an extra node. In the present paper, this possibility is examined by treating it as an example of a systematic approach[6-8] to the properties of the radial equation for a nonlocal potential.

The analysis[6-8] used in this paper is based on a system of Fredholm determinants associated with certain integral equations. This system of Fredholm determinants is a compendium of the properties of the radial equation for a nonlocal potential in the same way that a single Fredholm determinant provides a complete description of the properties of the radial equation for a short-range local potential. The system reduces to the single Fredholm determinant appropriate for a local potential in the limit as the nonlocal potential becomes local. The Fredholm determinant approach is summarized in Sec. II of this paper. One of the determinants of the system, defined as $D(k)$ in Sec. II, is particularly important in understanding the difference between the calculated hypertriton binding energies. A comparison of the potentials used by Schick[2] and by Gibson and Lehman[3] in terms of this Fredholm determinant reveals a substantial difference between the two potentials. The determinant $D(k)$ for the potential used by Schick has a zero on the real k axis, while $D(k)$ for the potential used by Gibson and Lehman does not. Such a zero of $D(k)$ on the real k axis is called a spurious state. A spurious state does not affect the modulus π relative phase shift or the on-shell T matrix. However, it does affect the absolute phase shift by introducing an extra node in the radial wave function.

The Λ-N potentials used by Schick[2] and by Gibson and Lehman[3] in hypertriton binding energy calculations are compared in Sec. III. The radial wave function for the potential used by Schick is

shown to have an extra node due to the zero of $D(k)$ for this potential. The suggestion that the extra node is responsible for differences between the off-shell T matrix elements for the two potentials, which in turn affect the hypertriton binding energy calculations, is then discussed.

II. FREDHOLM DETERMINANT APPROACH

Two of the Fredholm determinants in the system are considered here.[9] They are $D^+(k)$, the Fredholm determinant associated with the integral equation

$$\psi^+(k,r)=\sin kr+\int_0^\infty\int_0^\infty G^+(k,r,r')\times V(r',s)\psi^+(k,s)dsdr' \tag{1}$$

for the physical solution $\psi^+(k,r)$, and $D(k)$, the Fredholm determinant associated with the integral equation

$$\varphi(k,r)=k^{-1}\sin kr+\int_0^r\int_0^\infty G(k,r,r')\times V(r',s)\varphi(k,s)dsdr' \tag{2}$$

for the regular solution $\varphi(k,r)$. The physical and regular solutions are related by

$$\psi^+(k,r)=k\varphi(k,r)/\mathfrak{L}^+(k), \tag{3}$$

where

$$\mathfrak{L}^+(k)=D^+(k)/D(k) \tag{4}$$

is the Jost function.

A phase shift $\delta(k)$ is defined by

$$\delta(k)=-\text{phase}[\mathfrak{L}^+(k)]. \tag{5}$$

With this definition,

$$\delta(0)=n\pi, \tag{6}$$

where n is the number of nodes of $\psi^+(0,r)$. Thus, $\delta(k)$ as given by Eq. (5) provides an absolute definition of the phase shift. It differs from another definition,

$$\delta_D(k)=-\text{phase}[D^+(k)] \tag{7}$$

by the factor $D(k)$ in the denominator of Eq. (4). The two definitions are equivalent only if $D(k)$ and $D^+(k)$ have no zeros on the real k axis. In any event, the two definitions are equivalent to within modulus π, and the scattering amplitude or on-shell T matrix is not affected by $D(k)$. The two definitions are also equivalent for a short-range local potential since $D(k)=1$ and $D^+(k)\neq 0$ for real $k\neq 0$ for this class of potentials. Thus, the distinction between Eqs. (5) and (7) can be important only for nonlocal potentials, in which case Eq. (5) is preferred because it is always related to the absolute phase of the wave function.[7]

Both $D(k)$ and $D^+(k)$ may have zeros on the real k axis for a nonlocal potential. A zero of only $D(k)$ on the real k axis is called a spurious state. A zero of $D^+(k)$ for real $k\neq 0$ is always accompanied by a zero of $D(k)$ in such a way that $\mathfrak{L}^+(k)=D^+(k)/D(k)\neq 0$ for real $k\neq 0$. This simultaneous occurrence of zeros of $D(k)$ and $D^+(k)$ on the real k axis is called a continuum bound state. Examples of spurious and continuum bound states are given in Refs. 5–7; the figures in Ref. 7 show the influence of these states on wave functions and phase shifts. Similar calculations for the Λ-N potentials used by Schick[2] and Gibson and Lehman[3] are given in the next section.

III. COMPARISON OF Λ-N POTENTIALS

The coordinate representation of the two-term potential used by Schick is

$$V(r,r')=-\lambda_1 e^{-\alpha_1(r+r')}+\lambda_2 e^{-\alpha_2(r+r')}, \tag{8}$$

while Gibson and Lehman use a two-term potential of the form

$$V(r,r')=-\lambda_1 e^{-\alpha_1(r+r')}+\lambda_2(1-\tfrac{1}{2}\alpha_2 r)\times(1-\tfrac{1}{2}\alpha_2 r')e^{-\alpha_2(r+r')}. \tag{9}$$

In each case the parameters λ_1, λ_2, α_1, and α_2 are determined by the following constraints: (1) a set of Λ-N effective range and scattering length parameters, and (2) estimates of the "repulsive" term in the potential which results in a zero of the s-wave scattering amplitude near $E_{\text{c.m.}}=160$ MeV. The effective range r and scattering length a used in determining the parameters of the Schick and Gibson-Lehman potentials are given in Table I. The Gibson-Lehman scattering length and effective range are average values, which they used

TABLE I. Scattering length a and effective range r associated with the Gibson-Lehman and Schick potentials, the binding energy B_Λ of the hypertriton in the one-term separable potential approximation, and the change δB_Λ in the binding energy due to the introduction of a two-term separable potential.

Potential	a (fm)	r (fm)	B_Λ (MeV)	δB_Λ[a] (MeV)	δB_Λ[b] (MeV)
Gibson-Lehman	−2.21	2.24	0.83	−0.135	−0.18
Schick singlet	−2.415	2.035			
			0.95	−0.32	−0.37
Schick triplet	−1.19	2.43			

[a] Two-term potential in the singlet state only. The value listed for the Gibson-Lehman potential is an estimate as described in the text.

[b] Two-term potential in both the singlet and triplet states.

to represent both singlet and triplet s states. Thus, the Gibson-Lehman two-term potential is assumed to be spin independent. Schick, on the other hand, determined values of the parameters of the potential given by Eq. (8) for the singlet and triplet states separately.

The third column of Table I gives the hypertriton binding energy B_Λ for the one-term approximate potential. The last two columns give the reduction while keeping a and r fixed. The first of the two columns shows the change found by Schick due to going from a one-term potential to a two-term potential for the singlet state only, while retaining the one-term form for the triplet potential. There is no corresponding calculation for the Gibson-Lehman case since they used a spin-independent potential. The value listed in Table I for this case is an estimate based on the relative singlet ($\frac{3}{4}$) and triplet ($\frac{1}{4}$) contributions to B_Λ for a spin-independent potential. The second of the two columns gives the reduction in the hypertriton binding in going from one-term to two-term potentials for both singlet and triplet states. As expected, the reduction of B_Λ for the two calculations is dominated by the singlet potential.[10] The reduction of B_Λ found by Schick is about twice[11] the reduction of B_Λ found by Gibson and Lehman. Since this reduction is dominated by the singlet potential in both calculations the following discussion of the two-term Schick potential in terms of Fredholm determinants is limited to the Schick singlet potential.

Figure 1 shows the Fredholm determinants $D(k)$ for the Gibson-Lehman and Schick potentials. The dashed line in this figure at $D(k)=1$ represents the values of $D(k)$ for a short-range local potential. $D(k)$ for the Gibson-Lehman potential is positive for all energies with a maximum value of 2.31 at

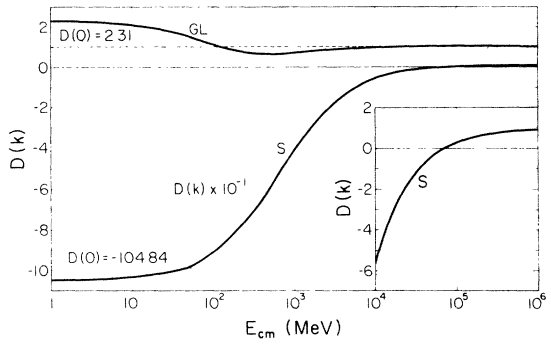

FIG. 1. Comparison of the Fredholm determinants $D(k)$ for the Gibson-Lehman (GL) and Schick (S) potentials. $D(k)$ for the Schick potential has a zero near $E_{c.m.}=70$ GeV which is called a spurious state. Note that $D(k)$ for the Schick potential has been scaled by a factor of 10^{-1}, except in the insert.

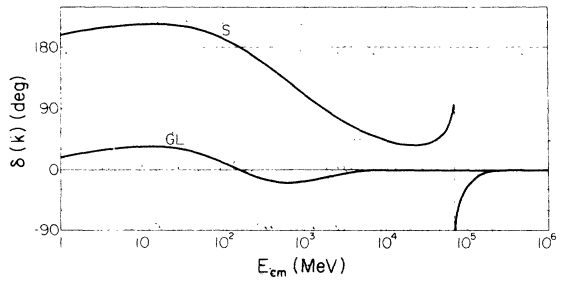

FIG. 2. Comparison of the phase shifts $\delta(k)$ defined by Eq. (5) for the Gibson-Lehman (GL) and Schick (S) potentials. $\delta(k)$ for the Schick potential has a π discontinuity near $E_{c.m.}=70$ GeV due to the spurious state at this energy. The π difference between the Gibson-Lehman and Schick phase shifts over the energy range from 1 to 200 MeV is due to the spurious state near $E_{c.m.}=70$ GeV.

zero energy, a minimum of 0.65 near 500 MeV, and an asymptotic value of 1 in the high-energy limit. $D(k)$ for the Schick potential also attains the asymptotic value of 1 in the high-energy limit, but approaches it more slowly. However, the important feature of $D(k)$ for the Schick potential is that it is negative at low energies with a zero of $D(k)$ near $E_{c.m.}=70$ GeV. Thus, an essential difference between the Gibson-Lehman and Schick potentials is the spurious state for the Schick potential near $E_{c.m.}=70$ GeV, with $D(k)<0$ in the energy range of practical interest.

An immediate consequence of the spurious state for the Schick potential near $E_{c.m.}=70$ GeV is that the phase shift $\delta(k)$ for this potential differs by π from the phase shift $\delta(k)$ for energies below the spurious state. The phase shift $\delta(k)$ has a π discontinuity at the spurious state.

Figure 2 shows the phase shifts $\delta(k)$ for the Gibson-Lehman and Schick potentials. Over the energy range from 0 to 200 MeV, which is the energy range of practical interest, the phase shifts $\delta(k)$ for the Gibson-Lehman and Schick potentials differ by about π. The phase shifts $\delta_D(k)$ are almost identical; the differences between the effective range and scattering length parameters for the Gibson-Lehman and Schick potentials given in Table I result in very slight differences between the phase shifts $\delta_D(k)$ for these potentials. While the phase shift $\delta(k)$ for the Schick potential at low energies looks like the phase shift for a local potential with one bound state, calculations show that the nonlocal Schick potential does not have a bound state.

Figure 3 shows the zero energy regular solutions for the Gibson-Lehman and Schick potentials. The solutions at $E_{c.m.}=500$ MeV are shown in Fig. 4.

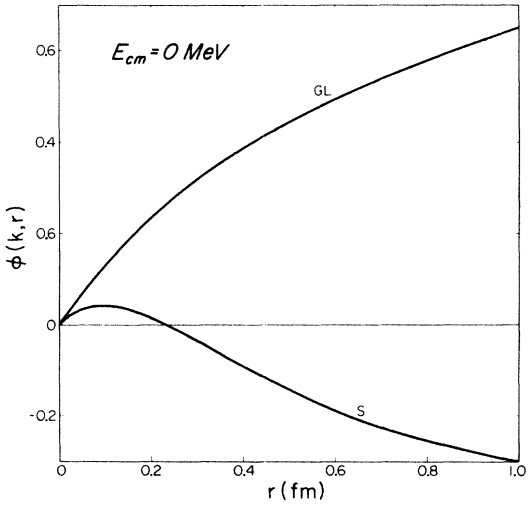

FIG. 3. Comparison of the zero-energy regular solutions $\varphi(k,r)$ defined by Eq. (2) for the Gibson-Lehman (GL) and Schick (S) potentials. The extra node in $\varphi(k,r)$ for the Schick potential is due to the spurious state near $E_{\text{c.m.}} = 70$ GeV.

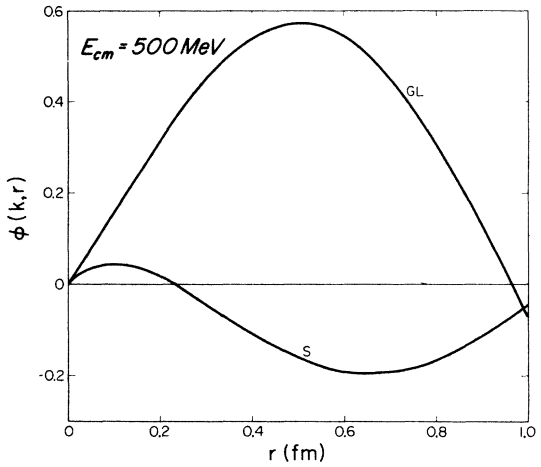

FIG. 4. Comparison of the regular solutions $\varphi(k,r)$ for the Gibson-Lehman (GL) and Schick (S) potentials at $E_{\text{c.m.}} = 500$ MeV. The extra node in $\varphi(k,r)$ for the Schick potential is due to the spurious state near $E_{\text{c.m.}} = 70$ GeV

These figures illustrate the extra node in the solution for the Schick potential due to the spurious state near $E_{\text{c.m.}} = 70$ GeV. The phase shift $\delta(k)$ defined by Eq. (5) takes the extra node into account, while the phase shift $\delta_D(k)$ defined by Eq. (7) does not.

The comparison of the Gibson-Lehman and Schick potentials given above reveals that these potentials are quite different even though both of them were introduced to take the short-range repulsion of the Λ-N interaction into account. In particular, this comparison emphasizes the importance of the Fredholm determinant $D(k)$ in discussing this difference. The spurious state near $E_{\text{c.m.}} = 70$ GeV for the Schick potential is responsible for the difference between the wave functions for the Gibson-Lehman and Schick potentials in the energy range of interest. Wave function differences involving an extra node are known to result in significant off-shell T matrix differences for N-N potentials that are important in descriptions of the structure of many-nucleon systems. The same should be true for Λ-N potentials and the structure of hypernuclei. Thus, it is reasonable to attribute the hypertriton binding energy differences of the last column in Table I to the extra node in the wave function for the Shick potential and therefore to the spurious state.

IV. SUMMARY

In this paper, a Fredholm determinant approach[6-8] to the properties of the radial equation for a nonlocal potential has been used to compare two Λ-N potentials that are approximately on-shell T matrix equivalent but which yield quite different results in three-body calculations of the hypertriton binding energy. It has been shown that one of the potentials has a spurious state. This spurious state results in an extra node in the radial wave function that is responsible for the differences in the calculated values of the hypertriton binding energy.

The Fredholm determinant approach used in this paper is a convenient way of classifying effects due to the nonlocality of a two-body potential which influence calculations of three-body observables. Such a classification of nonlocal effects does not by itself determine which of two two-body potentials is the best to use in a three-body calculation. Rather, the utility of the Fredholm determinant approach stems from the fact that it includes all relevant information about the two-body potential.

We are grateful to B. F. Gibson and D. R. Lehman for useful discussions about their calculations and for providing us with their potential parameters, and to L. H. Schick for helpful correspondence.

[1]B. F. Gibson and D. R. Lehman, Phys. Rev. C 10, 888 (1974); 14, 2346 (1976).

[2]L. H. Schick, Phys. Rev. C 11, 2089 (1975), and references therein to earlier work by Schick and collaborators.

[3]B. F. Gibson and D. R. Lehman, Phys. Rev. C 11, 2092 (1975).

[4]Gibson and Lehman use an average potential to represent both singlet and triplet s states, while Schick treats the singlet and triplet states separately. The net effect of this difference in the Λ-N two-body information used as input is that the Schick calculation is somewhat more sensitive to the singlet potential than the Gibson-Lehman calculation. As a result, part of the difference in the reduction of the hypertriton binding energy between the two calculations can be attributed to differences in the Λ-N two-body information used as input.

[5]L. G. Arnold and A. D. MacKellar, Phys. Rev. C 3, 1095 (1971).

[6]B. Mulligan, L. G. Arnold, B. Bagchi, and T. O. Krause, Phys. Rev. C 13, 2131 (1976).

[7]B. Bagchi, T. O. Krause, and B. Mulligan, Phys. Rev. C 15, 1623 (1977).

[8]L. G. Arnold, B. Bagchi, and B. Mulligan (unpublished).

[9]The equations and remarks given in this section hold for symmetric s-state nonlocal potentials of the type considered in Refs. 2 and 3.

[10]It is well known that B_Λ is dominated by the singlet potential. The fractional change in the singlet contribution to B_Λ, $(\delta B_\Lambda^s/B_\Lambda^s)$, in going from a one-term to a two-term potential is about −0.22 for the Gibson-Lehman calculation and −0.39 for the Schick calculation.

[11]It is probable that part of the difference between the Schick and Gibson-Lehman calculations for δB_Λ is due to differences in the Λ-N scattering lengths and effective ranges used as input. The effect of these input differences is small but not negligible. An estimate suggests that the input differences are accounted for by decreasing (increasing in magnitude) the value of δB_Λ found by Gibson and Lehman from −0.18 MeV to about −0.22 MeV.