Comparison of the unitary pole and Adhikari-Sloan expansions in the three-nucleon system

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The binding energy of ^{3}H, the percentage S-, S'-, and D-state probability, and the charge form factor of ^{3}He are calculated using the unitary pole and Adhikari-Sloan separable expansions to the Reid soft core potential. Comparison of the results for the two separable expansions show that the expansion of Adhikari and Sloan has the better convergence property, and the lowest rank expansion considered (equivalent to the unitary pole approximation) gives a good approximation to the binding energy of ^{3}H and the charge form factor of ^{3}He, even at large momentum transfer ($K^2 < 20$ fm^{-2}).

[NUCLEAR STRUCTURE ^{3}H binding energy, ^{3}He charge form factor, Faddeev approach, separable expansion to realistic N-N interactions.]

I. INTRODUCTION

Over the past few years it has become clear that we cannot reproduce the experimental binding energy and charge form factor of the trinucleon using any of the present realistic nucleon-nucleon potentials. In fact, a survey of the present theoretical results seems to indicate that as we reduce the discrepancy in the binding energy, we increase the disagreement in the charge form factor.[1-4] One possible solution to the above dilemma is to introduce three-body forces[5-9] and meson exchange effects.[9-13] Although in any final theory these effects may be intimately related to the basic choice of nucleon-nucleon interaction, the fact that they are small may justify the use of perturbation theory to calculate the contribution of three-body forces and meson exchange effects. For this program to have any success we need to solve the Faddeev equations for a "realistic" two-body interaction and obtain a simple, yet accurate, wave function for use in perturbation theory.

Furthermore, with the advent of the new meson facilities there is a growing interest in (p,π) reactions as a probe of short range correlations in nuclei, as the momentum transfer in these reactions is greater than 2 fm^{-1}. However, at present there are two schools of thought as to the reaction mechanism, and one way of resolving the ambiguity is to examine the reaction $d(p,\pi)t$ at low energies.[14-17] Here, we know both the initial and final nuclear wave function, and thus may gain some insight into the reaction mechanism. However, before we can calculate such cross sections we need the trinuclear wave function that has the correct behavior at momentum transfer of 2–3 fm^{-1}.

With the above need for a simple yet accurate trinucleon wave function, we have examined the three-nucleon problem (i.e., binding energy and charge form factor) using the recent separable expansion of Adhikari and Sloan.[18-20] These are the main features of the Adhikari-Sloan expansion (ASE) for the potential: (i) It converges in the operator sense while other expansions converge pointwise. This we hope will allow us to use less terms in the expansion for better accuracy. (ii) By proper choice of basis functions we can get the T matrix for the ASE to be exact, half off shell at selected energies. This second feature we use to minimize the number of terms needed in the expansion for optimum convergence in the three-nucleon bound state. For comparison we also examine the unitary pole expansion (UPE), which has been extensively used in three-nucleon calculations.[21,22] As we will show, the pole dominance idea which has led to the success of the UPE, is in fact incorporated in the Adhikari-Sloan expansion.

Since the main contribution to the binding energy of ^{3}H (E_T), and the charge form factor of ^{3}He (CFF) comes from the $^1S_0$ and $^3S_1$-$^3D_1$ nucleon-nucleon channels,[23] we will restrict our discussion of the trinucleon with only these channels included. Furthermore, this restriction will allow us to compare our results with the more exact solution of the Faddeev equations in coordinate[24] and momentum[25] space.

On comparing the trinucleon results using the Reid soft core (RSC) potential[26] for the two separable expansions we find that: (i) the binding energy of ^{3}H (E_T) converges as we increase the rank of the ASE without any resort to perturbation theory. This was not the case with the UPE[21] where it was found that one needs T-matrix perturbation theory to get the desired accuracy. (ii) Taking the ASE of rank equivalent to the unitary pole approx-

imation (UPA), we get a binding energy of 7.08 MeV which is better than the UPA result of 7.15 MeV[21] when compared with the exact result of 7.0 MeV. (iii) The resultant trinucleon wave function and charge form factor with the lowest order ASE are far superior to the UPE wave function, and are in good agreement with the exact results obtained by solving the two dimensional Faddeev equation with the exact RSC T matrix.[24,25]

From the above result, we see that the trinucleon wave function from the lowest rank ASE (the rank is equivalent to the rank of the UPA) can be used for calculating other processes. More important, this wave function is simpler to determine and easier to use.

In Sec. II we briefly present the results for the two separable expansions and discuss the important features of each. We then proceed in Sec. III to discuss the binding energy of ^{3}H. In particular we discuss the convergence of the two methods and the possible sources of numerical inaccuracy. As a test on the quality of the wave function we calculate the S, S', and D probabilities and the charge form factors of ^{3}He. These results are presented in Sec. IV, while Sec. V is devoted to some concluding remarks.

II. SEPARABLE EXPANSION FOR THE T MATRIX

The main advantage to the use of separable expansions in three-body calculation is the reduction of the Faddeev equations to a set of coupled one dimensional integral equations. The number of such equations is determined by the rank of the expansion and the number of two-body partial waves included. Since we will be restricting ourselves to the $^1S_0$ and $^3S_1$-$^3D_1$ partial waves of the RSC potential, the number of equations is completely determined by the rank of the expansion. Thus the vital criterion for the choice of expansion is to get optimum convergence with smallest rank. With this in mind we now proceed to a discussion of first the UPE which has been extensively used in three-nucleon calculations, and second the ASE which turns out to have better convergence properties.

Unitary pole expansion

The motivation behind the development of the UPE was the desire to construct a separable expansion for the T matrix which (a) satisfies two-body unitarity in all ranks, and (b) reproduces the position and residue of any bound state poles.[27] To satisfy both of these criteria, we write an expansion for the symmetrized kernel of the Lippmann-Schwinger (LS) equation in the $^3S_1$-$^3D_1$ channel

$$K_{ll'} = G_0^{1/2}(-B)V_{ll'}G_0^{1/2}(-B) = \sum_{mm'=1}^{\infty} |\phi_l^m\rangle C_{ll'}^{mm'} \langle\phi_{l'}^{m'}| . \quad (1)$$

Here, l and l' refer to the angular momenta in the coupled channels, $G_0(E) = (H_0 - E)^{-1}$, and B is the deuteron binding energy. The states $|\phi_l^m\rangle$ are solutions to the homogeneous LS equation written as an eigenvalue problem

$$|\phi_l^m\rangle = -\lambda_m \sum_{l'} K_{ll'} |\phi_{l'}^m\rangle , \quad (2)$$

with the normalization $\sum_l \langle\phi_l^m | \phi_l^{m'}\rangle = \delta_{mm'}$. Truncating the sum in Eq. (1) enables us to write an N term separable expansion for the potential

$$V_{ll'}^{\mathrm{UPE}}(N) = \sum_{mm'=1}^{N} |\chi_l^m\rangle C_{ll'}^{mm'}(N) \langle\chi_{l'}^{m'}| , \quad (3)$$

where the form factors $|\chi_l^m\rangle = G_0^{-1/2}(-B)|\phi_l^m\rangle$ and the strength matrix $\underline{C}_{ll'}(N)$ is given as

$$\underline{C}_{ll'}(N) = \underline{A}_l^{-1} \underline{B}_{ll'} \underline{A}_{l'}^{-1} , \quad (4)$$

in terms of the $N \times N$ matrices $\underline{A}_l$ and $\underline{B}_{ll'}$, where

$$[\underline{A}_l]_{mm'} = \langle\phi_l^m | \phi_l^{m'}\rangle \quad (5)$$

and

$$[\underline{B}_{ll'}]_{mm'} = \langle\phi_l^m | K_{ll'} | \phi_{l'}^{m'}\rangle . \quad (6)$$

Since the N-N interaction has both attraction and repulsion, the eigenvalues λ_m are either positive (attractive) or negative (repulsive). Thus, to construct the potential in Eq. (3) we take M_1 form factors that correspond to the smallest positive eigenvalues and M_2 form factors corresponding to the smallest in magnitude, but negative eigenvalues. The resultant potential is labeled $M_1 A + M_2 R$ and is of rank $M = M_1 + M_2$.

For central potentials (i.e., $^1S_0$) we have $V_{ll'} = \delta_{ll'} V_l$ and the strength matrix $\underline{C}_{ll'}(N)$ is diagonal. In this case due to the absence of a bound state we take $B = 0$ in Eqs. (1) and (2). This is justified by the fact that the $^1S_0$ antibound state pole is close to zero energy but on the second sheet.

The potential given in Eq. (3) leads to a two-body T matrix that satisfies unitarity and reproduces the deuteron bound state pole and its residue. The latter condition is only satisfied provided the form factor corresponding to $\lambda_m = 1$ is included in the definition of V^{UPE}. Since the Faddeev equations for the trinucleon require as input the T matrix at negative energies, it would not be sufficient for the UPE to get the T matrix only in a small neighborhood of the bound or antibound state pole. However, it has been shown[27] that increasing the rank of the UPE potential enlarges the domain of agreement between the UPE and exact T matrix. Thus

we achieve the convergence in three-body observables by increasing the rank M of the expansion.

Adhikari-Sloan expansion

While most separable expansions for the potential converge in a pointwise manner [i.e., $V^{(\text{rank } M)}(p,q) \to V(p,q)$ for M sufficiently large], the Adhikari-Sloan expansion is designed to converge to the exact potential in an operator sense to be illustrated shortly. This we hope will improve the convergence of the ASE as compared with other expansions. Another unique property of the ASE is its ability to reduce to other expansions (see Refs. 28 and 29). In particular, the expansion can be constructed to give the half-shell T matrix for selected energies and momenta.

In its most general form a rank M ASE for the potential V is given by

$$V^{\text{ASE}} = \sum_{n,m=1}^{M} V|f^n\rangle C^{nm}\langle g^m|V \tag{7}$$

with

$$[\underline{C}^{-1}]^{nm} = \langle g^n|V|f^m\rangle . \tag{8}$$

Here the states $|f^n\rangle$ and $|g^n\rangle$ are arbitrary and for the present we take them to be identical. Solving the Lippmann-Schwinger equation for V^{ASE} we get the T matrix to be of the form

$$T^{\text{ASE}}(E) = \sum_{n,m=1}^{M} V|f^n\rangle[\underline{\tau}^{-1}(E)]_{nm}\langle f^m|V \tag{9}$$

with

$$[\underline{\tau}(E)]_{nm} = \langle f^n|V + VG_0(E)V|f^m\rangle . \tag{10}$$

At first sight it may seem that the introduction of the exact potential V into the form factor in Eq. (9) has made the calculation of the ASE T matrix more difficult than the corresponding UPE T matrix. This, however, is not the case since the determination of the UPE form factor involves the use of Eq. (2) which includes the exact potential V.

It is easy to show from Eq. (7) that

$$V^{\text{ASE}}|f^n\rangle = V|f^n\rangle, \quad \langle f^n|V^{\text{ASE}} = \langle f^n|V, \quad n = 1,\ldots M . \tag{11}$$

The first of these equations means that V^{ASE} is exact when considered as an operator acting on the subspace of L^2 spanned by the states $|f^n\rangle$. The second means that for any state $|u\rangle$ in L^2, for which $V|u\rangle$ is defined, the projection of $V^{\text{ASE}}|u\rangle$ onto this subspace is exact. These properties become particularly important if the state $|f^n\rangle$ can be chosen to span a subspace which is in some sense especially important to the problem at hand. They also guarantee that as M is increased, and more independent basis states $|f^n\rangle$ are added, the ASE potential will become an increasingly accurate representation of the exact potential.

The basis states for coupled $^3S_1$-$^3D_1$ channels are assumed to have a partial wave expansion

$$|f^n\rangle = \sum_{l=0,2} |JSTl\rangle|f_l^n\rangle , \tag{12}$$

where, in momentum space, $f_l^n(p) = \langle p|f_l^n\rangle$ are real functions. Following Adhikari and Sloan[20] we chose the basis states $f_l^n(p)$ to be

$$f_0^n(p) = (p^2 + m^2\alpha^2)^{-1}\psi_m(p) , \qquad f_2^n(p) = 0 \tag{13}$$

for $n = 2m-1$, and

$$f_0^n(p) = 0, \qquad f_2^n(p) = p^2(p^2 + m^2\alpha^2)^{-1}\psi_m(p) \tag{14}$$

for $n = 2m$, where $n = 1,2,\ldots,M/2$, and $\psi_m(p)$ are the Sturmian functions of the S-wave Hulthén potential; the variable α is taken to be 0.5 fm^{-1}. This choice of basis gives the extra degree of freedom necessary for the potential to have the correct admixture of S and D components. However, the minimum value of M for which we get both S- and D-wave components to the potential is $M = 2$. This is to be compared with $M = 1$ for the UPE. We will see in the next section that $M = 1$ UPE and $M = 2$ ASE give the same number of coupled integral equations for the three-nucleon system. In this sense we will consider the two to be equivalent.

For the $^1S_0$ channel the basis functions are simply taken as

$$f_0^n(p) = (p^2 + n^2\alpha^2)^{-1}\psi_n(p) , \quad n = 1,\ldots,M \tag{15}$$

with $\alpha = 0.5$ fm^{-1}. In this case, a rank M UPE is equivalent to a rank M ASE. In fact if we choose the ASE basis functions such that $|\chi_0^n\rangle = V|f_0^n\rangle$ is the UPE form factor then the two expansions are identical. There is, however, no such simple equivalence in the $^3S_1$-$^3D_1$ channel.

One useful feature of the ASE, which we will make use of in the next section, is the possibility of choosing M' of the basis function such that

$$V|\bar{f}^n\rangle = T(E_n)|l_n p_n\rangle , \quad n = 1,2,\ldots M' \le M , \tag{16}$$

where $T(E_n)$ is the exact T matrix at energy E_n. With this new choice of basis function it is possible to show from Eqs. (9) and (10) that

$$\langle p'l'|T^{\text{ASE}}(E_n)|l_n p_n\rangle = \langle p'l'|T(E_n)|l_n p_n\rangle , \quad n = 1,\ldots M' . \tag{17}$$

In other words by proper choice of basis function the ASE T matrix is exact for all momenta p', angular momenta l' at M' set of energy momentum,

and angular momentum (E_n, p_n, l_n). In particular by taking E_n to be the energy of the bound state $({}^3S_1\text{-}{}^3D_1)$, or antibound state $({}^1S_0)$, we can ensure that the ASE T matrix gets the correct behavior at the pole, as in the case of the UPE. However, we note that while the UPE reproduces the deuteron pole and its residue with one term in the expansion, it takes the two term ASE to achieve the same result. This is due to our choice of basis function in Eqs. (13) and (14). Finally we note that taking $p_n^2\hbar^2/m = E_n$ and $M' = M$ the ASE becomes formally equivalent to the Ernst, Shakin, and Thaler expansion,[29] as implemented by Pieper.[28]

To implement the condition in Eq. (16) we make use of the fact that T^{ASE} for sufficiently large M is identical to the exact T matrix, in this case

$$\begin{aligned} V|\bar{f}^n\rangle &= T(E_n)|l_n p_n\rangle \\ &\cong \sum_{r,s=1}^{M} V|f^r\rangle \tau_{rs}(E_n)\langle f^s|V|l_n p_n\rangle \\ &= \sum_{r=1}^{M} V|f^r\rangle C_r(E_n l_n p_n)\,, \end{aligned} \tag{18}$$

where

$$C_r(E_n l_n p_n) = \sum_{s=1}^{M} \tau_{rs}(E_n)\langle f^s|V|l_n p_n\rangle\,. \tag{19}$$

In the present calculation we have found that $M = 20$ gives the desired accuracy. Since the optimum choice of energies, momenta, and angular momenta $(E_n p_n l_n)$ at which the half-shell T matrix is exact depends on the problem at hand, we will postpone the discussion as to their choice for the next section.

For a detailed comparison of the ASE and UPE with the corresponding exact RSC T matrix in both the ${}^1S_0$ and ${}^3S_1\text{-}{}^3D_1$ channels we refer the reader to Refs. 18–20, 22, 27, and 30. It suffices to say that both expansions have good convergence properties both at positive and negative energies. However, the ASE has the added advantage that it can be constructed to give the exact half-shell T matrix at selected energies and momenta.

III. TRINUCLEON BINDING ENERGY

The Faddeev equations for the trinucleon with separable two-body T matrices have been discussed extensively in the literature. However, since the relation between the number of coupled integral equations and the rank of the potential is different for the ASE and UPE, we will briefly derive the integral equations for the two separable expansions. This will be followed by a detailed discussion of the convergence of the binding energy of ${}^3\mathrm{H}$ (E_T) as a function of the rank of the expansions.

The difference between the structure of the two expansions is only present in the coupled ${}^3S_1\text{-}{}^3D_1$ channel for which the two-body T matrix is given by

$$t_{ll'}(p,p';E) = \sum_{n,n'=1}^{M} g_{nl}(p)\,\tau_{ll'}^{nn'}(E)\,g_{n'l'}(p')\,, \quad \text{for UPE} \tag{20a}$$

$$= \sum_{n,n'=1}^{M} g_{nl}(p)\,\bar{\tau}^{nn'}(E)\,g_{n'l'}(p')\,, \quad \text{for ASE}\,, \tag{20b}$$

where the form factors $g_{nl}(p)$ are

$$g_{nl}(p) = \langle p|\chi_l^n\rangle\,, \quad \text{for UPE} \tag{21a}$$

$$= \langle lp|V|\bar{f}^n\rangle\,, \quad \text{for ASE}\,. \tag{21b}$$

The difference in structure between the two expansions lies in the separability of $t_{ll'}$ in the angular momentum l. The propagator $\underline{\tau}(E)$ for the UPE is a $(2M \times 2M)$ matrix of the form

$$\underline{\tau}^{-1}(E) = \underline{C}^{-1} + \langle \underline{\chi}|G_0(E)|\underline{\chi}\rangle \tag{22}$$

with the matrix of strength $\underline{C}$ given in Eq. (4) and $\langle \underline{\chi}|$ a row matrix of length $2M$ and of the form $[\langle \chi_0^1|\cdots\langle \chi_0^M|\langle \chi_2^1|\cdots\langle \chi_2^M|]$. On the other hand the ASE propagator $\bar{\tau}(E)$ is an $(M \times M)$ matrix given in Eq. (10). As we will see, the independence of $\bar{\tau}^{nn'}(E)$ in Eq. (20b) of the orbital angular momentum will lead to a reduction in the number of coupled integral equations.

For three identical nucleons, the total wave function is given by[31]

$$|\Psi\rangle = \{e + (123) + (132)\}\,|\psi_\alpha\rangle\,, \tag{23}$$

where e, (123), and (132) are elements of the permutation group S_3, and $\alpha = 1$, 2, or 3. The requirement of antisymmetry of the total wave function is satisfied by the condition that

$$(\beta\gamma)\,|\psi_\alpha\rangle = -\,|\psi_\alpha\rangle\,, \tag{24}$$

where $(\beta\gamma)$ is an element of S_3. The Faddeev equations for $|\psi_\alpha\rangle$ are then given by[31]

$$\begin{aligned} |\psi_\alpha\rangle &= -2G_0(E)T_\alpha(E)(123)\,|\psi_\alpha\rangle \\ &= -2G_0(E)T_\alpha(E)\,|\psi_\beta\rangle\,, \end{aligned} \tag{25}$$

where $G_0(E) = (H_0 - E)^{-1}$ is the free three-particle Green's function, and E is the energy of the bound three-body system (i.e., $E < 0$). In Eq. (25) $T_\alpha(E)$ is the two-body T matrix for the pair $(\beta\gamma)$ in three-body Hilbert space.

To write Eq. (25) in partial wave form for the separable two-body T matrix, we use the J-J coupling scheme

$$\begin{aligned}
&\vec{s}_\alpha = \vec{\sigma}_\beta + \vec{\sigma}_\gamma\,, \quad \vec{j}_\alpha = \vec{l}_\alpha + \vec{s}_\alpha\,,\\
&\vec{I}_\alpha = \vec{L}_\alpha + \vec{\sigma}_\alpha\,, \quad \vec{J} = \vec{j}_\alpha + \vec{I}_\alpha\,,\\
&\vec{t}_\alpha = \vec{\tau}_\beta + \vec{\tau}_\gamma\,, \quad \vec{T} = \vec{t}_\alpha + \vec{\tau}_\alpha\,,
\end{aligned} \tag{26}$$

where l_α is the relative orbital angular momentum of the pair $(\beta\gamma)$ and L_α is the orbital angular momentum of particle α relative to the center of mass (c.m.) of $(\beta\gamma)$. In Eq. (26) τ_α and σ_α are the isospin and spin of particle α. The isospin, spin angular momentum eigenstates are then

$$|\Omega^{TJ}_{l_\alpha N_\alpha}\rangle = |(t_\alpha \tau_\alpha)T\rangle\,|[(l_\alpha s_\alpha)j_\alpha;\ (L_\alpha \sigma_\alpha)I_\alpha]J\rangle \tag{27}$$

with $N_\alpha = \{n_\alpha, L_\alpha, I_\alpha\}$ labeling the three-body channels and $n_\alpha = \{t_\alpha s_\alpha j_\alpha\}$ labeling the two-body channels.

With this coupling scheme the two-body T matrix in three-body Hilbert space is given for a rank M UPE potential, by[34]

$$\begin{aligned}
T_\alpha(E) &= \sum_{\substack{N_\alpha\\ l_\alpha l'_\alpha\\ JT}} \int_0^\infty dq_\alpha q_\alpha{}^2\, |\Omega^{TJ}_{l_\alpha N_\alpha}; q_\alpha\rangle\, t^{n_\alpha}_{l_\alpha l'_\alpha}\left(E - \frac{3\lambda}{4} q_\alpha{}^2\right) \langle \Omega^{TJ}_{l'_\alpha N_\alpha}; q_\alpha|\\
&= \sum_{\substack{N_\alpha\\ l_\alpha l'_\alpha\\ JT}} \int dq_\alpha q_\alpha{}^2\, |\underline{g}^{TJ}_{l_\alpha N_\alpha}; q_\alpha\rangle\, \underline{\tau}^{n_\alpha}_{l_\alpha l'_\alpha} \left(E - \frac{3\lambda}{4}\, q_\alpha{}^2\right) \langle \underline{g}^{TJ}_{l'_\alpha N_\alpha}; q_\alpha\,|
\end{aligned} \tag{28}$$

with q_α the momentum of particle α relative to the c.m. of $(\beta\gamma)$ and

$$|g^{TJ}_{l_\alpha N_\alpha}; q_\alpha\rangle = |\Omega^{TJ}_{l_\alpha N}; q_\alpha\rangle\, |\underline{g}_{n_\alpha l_\alpha}\rangle \tag{29}$$

with the row matrix $|\underline{g}_{n_\alpha l_\alpha}\rangle$ given by

$$|\underline{g}_{n_\alpha l_\alpha}\rangle = (|\chi^1_{n_\alpha l_\alpha}\rangle \cdots |\chi^M_{n_\alpha l_\alpha}\rangle)\,. \tag{30}$$

In Eq. (28) $\lambda = \hbar^2/m$ with the nucleon mass. For the ASE Eq. (28) still holds, provided we use the appropriate form factors and we recall that $\underline{\tau}^{n_\alpha}_{l_\alpha l'_\alpha} \to \overline{\tau}^{n_\alpha}$, which is independent of l_α and l'_α.

Making use of Eq. (28) we rewrite the Faddeev equations (25) as

$$|\psi_\alpha\rangle = -2G_0(E) \sum_{\substack{l_\alpha N_\alpha\\ JT}} \int_0^\infty dq_\alpha q_\alpha{}^2\, |\underline{g}^{TJ}_{l_\alpha N_\alpha}; q_\alpha\rangle\, \underline{X}^{TJ}_{l_\alpha N_\alpha}(q_\alpha) \tag{31}$$

with the column matrix of spectator wave function $\underline{X}^{TJ}_{l_\alpha N_\alpha}$ defined as

$$\underline{X}^{TJ}_{l_\alpha N_\alpha}(q_\alpha) = \sum_{l'_\alpha} \underline{\tau}^{n_\alpha}_{l_\alpha l'_\alpha}\left(E_0 - \frac{3\lambda}{4}\, q_\alpha{}^2\right) \langle \underline{g}^{TJ}_{l'_\alpha N_\alpha}; q_\alpha\,|\psi_\beta\rangle \tag{32}$$

and satisfying the coupled one dimensional integral equations

$$\underline{X}^{TJ}_{l_\alpha N_\alpha}(q_\alpha) = -2 \sum_{\substack{l'_\alpha l_\beta\\ N_\beta}} \underline{\tau}^{n_\alpha}_{l_\alpha l'_\alpha}\left(E_0 - \frac{3\lambda}{4}\, q_\alpha{}^2\right) \int_0^\infty dq_\beta\, q_\beta{}^2\, \underline{Z}^{TJ}_{l'_\alpha N_\alpha;\, l_\beta N_\beta}(q_\alpha, q_\beta; E_0)\, \underline{X}^{TJ}_{l_\beta N_\beta}(q_\beta)\,, \tag{33}$$

where

$$\underline{Z}^{TJ}_{l_\alpha N_\alpha;\, l_\beta N_\beta}(q_\alpha, q_\beta; E_0) = \langle \underline{g}^{TJ}_{l_\alpha N_\alpha}; q_\alpha\,|\, G_0(E_0)\,|\underline{g}^{TJ}_{l_\beta N_\beta}; q_\beta\rangle\,, \tag{34}$$

the explicit form of which is given in the Appendix. In the case of ASE, Eqs. (31)–(34) are still valid provided we do the substitution

$$\underline{\tau}^{n_\alpha}_{l_\alpha l_\beta} \to \overline{\tau}^{n_\alpha}$$

and

$$|\underline{g}^{TJ}_{l_\alpha N_\alpha}; q_\alpha\rangle \to |\underline{g}^{TJ}_{N_\alpha}; q_\alpha\rangle = \sum_{l_\alpha} |\underline{g}^{TJ}_{l_\alpha N_\alpha}; q_\alpha\rangle$$

and the corresponding change in the definition of the form factors. In doing this substitution we have effectively performed the sum over l'_α and l_β in Eq. (33), and thus reduced the number of coupled integral equations. For the present investigation we have restricted our two-body interaction to the ${}^1S_0$ and ${}^3S_1$-${}^3D_1$ channels. In this case the number of three-body channels (l_α, N_α) is five, and these are given in Table I. For the UPE with rank M_1

TABLE I. The three-body channels included in the solution to the Faddeev equation.

Channel	l_α	t_α	s_α	N_α j_α	L_α	I_α
1	0	1	0	0	0	$\frac{1}{2}$
2	0	0	1	1	0	$\frac{1}{2}$
3	2	0	1	1	0	$\frac{1}{2}$
4	0	0	1	1	2	$\frac{3}{2}$
5	2	0	1	1	2	$\frac{3}{2}$

TABLE II. The perturbed and unperturbed binding energy of ^{3}H ($E_T^0+\Delta E_T$) and (E_T^0) for different rank M unitary pole expansions.

M	$1A$	$1A+1R$	$1A+2R$	$2A+1R$
$E_T^0(M)$	7.148	6.991	6.947	7.037
$\Delta E_T(M)$	−0.181	+0.043	+0.083	−0.018
$E_T^0(M)+\Delta E_T(M)$	6.967	7.034	7.030	7.019

in $^1S_0$ and M_2 in the $^3S_1$-$^3D_1$ the number of coupled integral equations is M_1+4M_2, or $5M$ if $M_1=M_2=M$. On the other hand for the ASE, the number of coupled integral equations is M_1+2M_2 or $3M$ for $M_1=M_2=M$. Thus, we hope with ASE we can go to larger M and thus achieve convergence without any resort to perturbation theory which was necessary for the UPE.[21]

For the sake of comparison we briefly summarize the UPE results for the Reid soft core potential. Here, we first calculate the binding energy of ^{3}H by solving the Faddeev equation (33) for rank M potential. (For the sake of simplicity we have chosen $M_1=M_2=M$, in this way avoiding the optimization of both M_1 and M_2.) This gives us the unperturbed energy $E_T^0(M)$ and wave function. We then treat the difference between the exact and rank M UPE T matrix as a perturbation, with the resultant correction to the unperturbed energy $\Delta E(N)$ given by

$$\Delta E(M) = 12\langle \psi_\beta(M) \left| T^{\text{exact}} - T^{\text{UPE}}(M) \right| \psi_\beta(M)\rangle \cong 12\langle \psi_\beta(M) \left| T^{\text{UPE}}(N) - T^{\text{UPE}}(M) \right| \psi_\beta(M)\rangle \ , \tag{35}$$

where $N \gg M$ is chosen to be large enough so that $T^{\text{UPE}}(N) \cong T^{\text{exact}}$, and any further increase in N does not change $\Delta E(M)$. For the present investigation we have found $N=7A+6R$ to satisfy both of these conditions. In Table II we give the unperturbed and perturbed binding energies for $1A$, $1A+1R$, $1A+2R$, and $2A+1R$ potentials. A comparison of the unperturbed energy for the different rank potentials indicates that convergence has not been achieved, and that we may need to go to $2A+2R$ or higher rank potentials. With both $1A+2R$ and $2A+1R$ we have already 15 coupled integral equations, while with $2A+2R$ we go to 20 coupled equations with no guarantee of final convergence and an increase in numerical inaccuracy. However, we expect the correction to the energy and wave function due to higher terms in the UPE to be small. This is based on the fact that the $2A+1R$ T matrix is already a good approximation to the exact T matrix.[24] Furthermore, the first order correction to the energy in all cases except the $1A$ is less than 0.1 MeV, indicating that higher order corrections can be neglected. If we now examine the energy $E_T=E_T^0+\Delta E_T$ for different M, we observe that we have improved our convergence to the extent that $|E_T(M=3)-E_T(M=2)| \leq 0.015$ MeV as compared with $|E_T^0(3)-E_T^0(2)| \leq 0.05$ MeV. More important is the fact that $E_T(1A+2R) \simeq E_T(1A+1R)$ which is due to the fact that the second repulsive term can be treated with good accuracy in perturbation theory, while that is not the case with the second attractive term as is clear from Table II. This implies that $E_T(2A+1R)$ is a better approximation to the energy than $E_T^0(2A+2R)$ since it not only includes the second attractive but all higher terms in the UPE. Considering the fact that $|E_T(2A+1R)-E_T^0(2A+1R)| < 0.02$ MeV we would expect the error in neglecting higher order contributions in perturbation theory to be less than ± 0.01 MeV.

Let us now examine some of the possible sources of numerical inaccuracy in the above method.

(i) In Eq. (35), the replacement of the exact T matrix by a high rank UPE has introduced two possible sources of inaccuracy. One is the fact that the UPE T matrix is only an approximation to the exact one. More important is that high rank UPE involves eigenstates of the Lippmann-Schwinger kernel corresponding to large eigenvalues λ_n which are highly oscillatory. This may introduce inaccuracies in any integral involving such eigenstates. To test the above two sources of inaccuracy in the perturbation theory, we have repeated the calculation of E_T using the exact T matrix in Eq. (35) for $1A$ and $2A+1R$ potentials. The resultant binding energies are 6.968 and 7.019 MeV, respectively, which are in excellent agreement with the results in Table II. The advantage of replacing the exact T matrix by a high rank UPE is a major reduction in computing time.

(ii) To solve Eq. (33) we replace the integration by 16 point Gauss-Legendre quadratures mapped onto 0 to ∞. This replaces the set of $5N$ coupled one dimensional integral equations by $80N$ algebraic equations. To check this reduction we have repeated the $1A$ binding energy calculation using 24 points with a resultant change in energy of less than 0.005 MeV. For $2A+1R$ and $1A+2R$, rather than increasing the number of points (which would have produced a storage problem on the computer) we decreased them to 12 points, again producing a change of less than 0.005 in the binding energy. Thus the maximum possible numerical inaccuracy is ± 0.01 MeV. This gives us a binding energy of ^{3}H with the $^1S_0$ and $^3S_1$-$^3D_1$ Reid soft core potential of

$$7.02 \pm 0.02 \text{ MeV}.$$

TABLE III. The energies E_n and angular momenta l_n for which the ASE T matrix $\langle pl \mid T^{\mathrm{ASE}}(E_n) \mid l_n p_n \rangle$ $p_n^2 = -E_n/\varkappa$ is exact for all p and l.

	$^1S_0$		$^3S_1$-$^3D_1$
n	E_n (MeV)	l_n	E_n (MeV)
1	0.0	0	−7.013
2	−151.1	2	−269.3
3	−269.3	0	−151.1
4	−492.4	2	−492.4
5		0	−952.6
6		2	−952.6

Having established that the convergence in the binding energy of ^3H is slow as we increase the number of terms in the UPE, and the necessity of perturbation theory, we turn to a discussion of the use of ASE in three-nucleon calculations. One important feature of the ASE, which is not present in the UPE, is the possibility of choosing certain values of the energy E_n, momenta p_n, and angular momenta l_n, such that $\langle pl \mid T^{\mathrm{ASE}}(E_n) \mid l_n p_n \rangle$ is identical to the exact T matrix for all p and l [see Eq. (17)]. This feature which was also present in the separable expansion of Ernst, Shakin, and Thaler[29] as implemented by Pieper[28] can be used to improve the convergence of the expansion for the two-body T matrix. However, it is clear that the optimum choice of (E_n, p_n, l_n) will depend on the problem at hand. For the ASE, the Faddeev Eq. (33) reduces to the form

$$\underline{X}^{TJ}_{N_\alpha}(q_\alpha) = -2 \sum_{N_\beta} \underline{\tau}^{n_\alpha}\left(E_0 - \frac{3\varkappa}{4} q_\alpha^2\right) \times \int_0^\infty dq_\beta \, q_\beta^2 \underline{Z}^{TJ}_{N_\alpha N_\beta}(q_\alpha, q_\beta; E_0) \underline{X}^{TJ}_{N_\beta}(q_\beta) . \tag{36}$$

To solve these equations, we replace the integral by N_Q quadrature points $q_1 \ldots q_{N_Q}$. We then observe that there are N_Q energies at which the two-body T matrix is required, namely

$$E_n = E_0 - \frac{3\varkappa}{4} q_n^2 , \quad n = 1 \ldots N_Q . \tag{37}$$

It is then a fairly obvious choice to take M of the N_Q energies as the energies used in the change of basis for the rank M ASE T matrix. Of course E_0 is at this stage not known exactly, but we do have a rough idea of its value, and, since we hope that small changes in E_n will not change our results, this value of E_0 should be accurate enough to calculate the E_n.

It would now be most desirable to choose the momenta p_n for the change of basis to be those at which the T matrix in the Faddeev equation is needed after replacing the integrals by sums over quadrature points. Unfortunately, this is not easily implemented, since such a procedure would lead to a very high rank potential, and as we will find, convergence in the binding energy of ^3H is achieved with a relatively low rank potential. We have thus chosen $p_n^2 = -E_n/\varkappa$, which makes the half-off-shell T matrix exact at selected energies. In Table III, we present our choice of E_n and l_n in $^1S_0$ and $^3S_1$-$^3D_1$ channels, with the only exception to the above rule being $E_1 = 0$ for the $^1S_0$. In practice, as we increase the rank of the separable expansion used in the trinucleon calculation, the more values of (E_n, l_n) are taken in the change of basis. Thus for rank M_1 $^1S_0$ and M_2 $^3S_1$-$^3D_1$ we take the first M_1 energies E_n $(n = 1, 2, \ldots M_1)$ and M_2 energies and angular momenta $(E_n, l_n; n = 1, \ldots M_2)$. Such a potential will be referred to as ASE (M_1, M_2).

In Table IV we present our results for the binding energy of ^3H (E_T) with the RSC as a function of the rank of the Adhikari-Sloan expansion. We find that increasing the rank of the $^1S_0$ potential (M_1) from two to four changes E_T by as little as 0.002 MeV. Similarly, changing the rank of the $^3S_1$-$^3D_1$ potential (M_2) from 4 to 5 and 5 to 6 gives a change of 0.01 and 0.003 MeV, respectively. Thus with a rank (4, 6) ASE, we have achieved the convergence necessary without resorting to perturbation theory as was the case with the UPE. To test the sensitivity of our results to the choice of E_n, we have repeated the calculation of E_T with rank (1, 2) ASE taking E_1 the energy of deuteron (i.e., −2.2246 MeV), and $E_2 = -508.0$ MeV. This gives $E_T = 7.119$ MeV, a change of 0.037 MeV compared with the result in Table IV. We expect this sensitivity to the choice of energies E_n, at which the half-shell T matrix is exact, to decrease with increase in the rank of the expansion. In fact, similar tests with a rank (4, 6) ASE give a smaller change in E_T, and in all cases less than 0.01 MeV.

Combining the possible sources of error, i.e., finite rank expansion, choice of energies E_n, and finite quadrature points, we expect an error in E_T of less than 0.03 MeV. Thus the final result for the binding energy of ^3H using the ASE for the RSC

TABLE IV. Binding energy of ^3H using the Adhikari-Sloan expansion for the Reid soft core potential.

Rank of expansion (M_1, M_2)	E_T (MeV)
(1, 2)	7.082
(2, 4)	6.981
(4, 4)	6.983
(4, 5)	6.973
(4, 6)	6.970

potential in $^1S_0$ and $^3S_1$-$^3D_1$ is

6.97 ± 0.03 MeV.

In comparing this result with the corresponding UPE result of 7.02 ± 0.2 MeV we note: (i) With the ASE we have achieved convergence without perturbation theory, indicating that the trinucleon wave function for the ASE rank (4, 6) is a better approximation to the exact wave function than the $2A + 1R$ UPE wave function. This will be illustrated in the next section when we compare the charge form factor of ^{3}He for the two expansions. (ii) The Faddeev equation for the ASE with rank (4, 6) potentials consists of 16 coupled one dimensional integral equations, which are comparable to the 15 coupled equations we had for the $2A + 1R$ UPE potential.

IV. TRINUCLEON WAVE FUNCTION AND CHARGE FORM FACTOR OF ^{3}He

Having established the better convergence of the ASE as compared with the UPE in determining the binding energy of ^{3}H, we now examine the three-nucleon wave function for the two expansions. The aim of the present investigation is (i) to establish which of the two expansions gives a better approximation to the exact wave function; (ii) to see if we can get the short range behavior of the wave function with a low rank expansion such as the UPE (1A) (i.e., UPA) or ASE (1, 2). This second point might give us a simple yet accurate wave function for use in evaluating the contribution of three-body forces to the binding energy of ^{3}H, as a correction to the form factor due to meson exchange, and for use in such reactions as pion elastic scattering and absorption on the three-nucleon system.

Using again J-J coupling scheme [see Eqs. (26) and (27)] we can write the partial wave expansion for the total trinucleon wave function as

$$|\Psi\rangle = \sum_{l_\alpha N_\alpha} |\Omega^{TJ}_{l_\alpha N_\alpha}\rangle \, |U^{TJ}_{l_\alpha N_\alpha}\rangle \, , \tag{38}$$

where the angular part $|\Omega^{TJ}_{l_\alpha N_\alpha}\rangle$ is defined in Eq. (27), and the radial part $|U^{TJ}_{l_\alpha N_\alpha}\rangle$ can be written as

$$|U^{TJ}_{l_\alpha N_\alpha}\rangle = \sum_{r=1,2} |\eta^{TJr}_{l_\alpha N_\alpha}\rangle \tag{39}$$

with

$$\eta^{TJr=1}_{l_\alpha N_\alpha}(p_\alpha, q_\alpha) = \langle p_\alpha q_\alpha \, | \eta^{TJr=1}_{l_\alpha N_\alpha}\rangle = \langle p_\alpha q_\alpha ; \Omega^{TJ}_{l_\alpha N_\alpha} | \psi_\alpha \rangle \tag{40}$$

and

$$\eta^{TJr=2}_{l_\alpha N_\alpha}(p_\alpha, q_\alpha) = \langle p_\alpha q_\alpha ; \Omega^{TJ}_{l_\alpha N_\alpha} | \psi_\beta + \psi_\gamma \rangle = \tfrac{1}{2}[1 - (-)^{l_\alpha + s_\alpha + t_\alpha}] \sum_{l_\beta N_\beta} \int_{-1}^{+1} d\xi \, \Gamma^{TJ}_{l_\alpha N_\alpha ; l_\beta N_\beta}(q_\alpha, q'_\beta ; x) \eta^{TJr=1}_{l_\beta N_\beta}(p'_\beta, q'_\beta) \tag{41}$$

with

$$q'^2_\beta = p_\alpha^2 + \tfrac{1}{4} q_\alpha^2 + p_\alpha q_\alpha \xi \, , \quad p'^2_\beta = \tfrac{1}{4} p_\alpha^2 + \tfrac{9}{16} q_\alpha^2 - \tfrac{3}{4} p_\alpha q_\alpha \xi \, , \quad x = -\frac{1}{q'_\beta}\left(\tfrac{1}{2} q_\alpha + p_\alpha \xi\right) . \tag{42}$$

The function $\Gamma_{l_\alpha N_\alpha ; l_\beta N_\beta}$ is given explicitly in Eq. (A3) of the Appendix. The second line of Eq. (41) can be derived after some tedious angular momentum algebra (see Ref. 30).

From Eqs. (39)–(41) we see that the determination of the radial wave function $U^{TJ}_{l_\alpha N_\alpha}(p_\alpha, q_\alpha)$ has been reduced to obtaining an expression for $\eta^{TJr=1}_{l_\alpha N_\alpha}(p_\alpha, q_\alpha)$. The function of two variables (p_α, q_α) can be further simplified to almost a product of a function of p_α and one of q_α, if the two-body T matrix is separable. For the UPE, making use of Eq. (31) in Eq. (40), we get for $\eta^{TJr=1}_{l_\alpha N_\alpha}$ the result

$$\eta^{TJr=1}_{l_\alpha N_\alpha} = -2 G_0(p_\alpha, q_\alpha ; E_0) \langle p_\alpha \, | \underline{g}_{n_\alpha l_\alpha} \rangle \underline{X}^{TJ}_{l_\alpha N_\alpha}(q_\alpha) \tag{43}$$

with

$$G_0(p_\alpha, q_\alpha ; E_0) = [-E_0 + \tfrac{1}{4}(p_\alpha^2 + \tfrac{3}{4} q_\alpha^2)]^{-1} . \tag{44}$$

While this result is strictly correct for the UPE only, the equivalent expression for the ASE is obtained by dropping the subscript l_α from the spectator wave function (i.e., $\underline{X}^{TJ}_{l_\alpha N_\alpha} \to \underline{X}^{TJ}_{N_\alpha}$). In other words, the q_α dependence is the same for $l_\alpha = j_\alpha \pm 1$ for a given j_α when $s_\alpha = 1$. We also observe that $\eta^{TJr=1}_{l_\alpha N_\alpha}$ is nonzero only for the five three-body channels included in Table I. However, from Eq. (41) we see that $\eta^{TJr=2}_{l_\alpha N_\alpha}$ has a nonzero value for an infinite set of values of (l_α, N_α), and the sum in Eq. (38) is by no means finite, and we will have to cut it off at some reasonable value of (l_α, N_α). These higher partial waves in the total wave function are present due to the antisymmetry of the total wave function.

Using the definition of the total wave function (23), and the multiplication table for the permutation operators, we can write the normalization

TABLE V. Normalization of the total wave function for different separable expansions and number of partial waves.

Expansion	$\langle\Psi\mid\Psi\rangle$ Channels in Table I	$\langle\Psi\mid\Psi\rangle$ All channels with $l_\alpha+L_\alpha\leq 10$
UPA	0.9252	0.9966
$2A+1R$	0.9234	0.9965
ASE (1,2)	0.9218	0.9964
ASE (4,6)	0.9211	0.9963

of the total wave function as

$$\langle\Psi\mid\Psi\rangle = 3\langle\psi_\alpha\mid\psi_\alpha\rangle + 6\langle\psi_\alpha\mid\psi_\beta\rangle$$
$$= 3\sum_{l_\alpha N_\alpha}\left[\langle\eta^{TJr=1}_{l_\alpha N_\alpha}\mid\eta^{TJr=1}_{l_\alpha N_\alpha}\rangle + 2\langle\eta^{TJr=1}_{l_\alpha N_\alpha}\mid\eta^{TJr=2}_{l_\alpha N_\alpha}\rangle\right], \quad (45)$$

where the sum runs over the five three-body channels in Table I. Alternatively, we can write the normalization from Eq. (38) as

$$\langle\Psi\mid\Psi\rangle = \sum_{l_\alpha N_\alpha}\langle U^{TJ}_{l_\alpha N_\alpha}\mid U^{TJ}_{l_\alpha N_\alpha}\rangle, \quad (46)$$

where the number of channels in the sum is infinite. Our criterion for restricting the sum in Eqs. (38) and (46) to a finite number of channels is now determined by comparing the result of Eqs. (46) and (45) for the normalization. In this way we know what percentage of the wave function we have lost on truncating the sum in Eq. (38) to a finite number of partial waves.

To study this convergence we consider the UPA [i.e., UPE (1A)] with the wave function normalized according to Eq. (45). If we now restrict our sum in (46) to the five channels in Table I we get 0.9252, indicating the loss of 7.5% of the wave function. Increasing the sum in (46) to all three-body channels with $l_\alpha+L_\alpha\leq 10$ ($l_\alpha+L_\alpha\leq 20$), we increase the normalization from Eq. (46) to 0.9966 (0.9993). This indicates that a restriction to $l_\alpha+L_\alpha\leq 10$ with 42 terms in the sum in Eq. (46) is sufficient and the increase in accuracy by doubling the number of terms is not warranted. To show that this convergence is not unique to the UPA, we present in Table V similar results for higher rank UPE and ASE.

Having established that 42 terms (i.e., all channels with $l_\alpha+L_\alpha\leq 10$) in the sum in Eq. (38) are sufficient we now renormalize our wave function using Eq. (46), in this way guaranteeing a proper normalization of the total wave function.

As an initial test of the total trinucleon wave function resulting from the two separable expansions to the RSC potential, we evaluate the probability for the different components of the wave function in the Blatt-Derrick representation. In this way we can also compare our results with those obtained by solving the Faddeev equations for the exact RSC potential in coordinate space by Laverne and Gignoux (LG)[24] and in momentum space by Brandenburg, Kim, and Tubis (BKT).[25] In Table VI we give the percentage probability of the S- and D-wave contribution to the triton wave function obtained by using the UPE and ASE to the RSC potential. Since we have not included any P-wave nucleon-nucleon potential in our calculation, we have not tabulated the P-wave probabilities even though they are included in the normalization. We have included in Table VI all components with $l_\alpha+L_\alpha\leq 10$. In the table, L and S refer to the total orbital angular momentum and spin in L-S coupling. The column labeled P gives the symmetry under permutation of the spin isospin part of the wave function. Thus A denotes the totally antisymmetric component, and $\pm$ denotes the two mixed symmetry components. The totally symmetric components have zero probability for even total orbital angular momentum L, and are not shown. From the results in Table VI we observe that (i) there is good agreement between the low rank and high rank expansion of both types, this agreement being, in general, slightly better in the case of ASE, particularly in channel 1; (ii) there is good agreement between the two expansions.

To see how our results compare with those of other methods, we give in Table VII the results of LG and BKT along with our UPE ($2A+1R$) and ASE (4,6) results. At first sight it appears that there is very good agreement between the results of all four methods except in channel number one. This apparent discrepancy is in fact due to the different number of channels used in different calculations; for while BKT include only the nine channels in Table VII in their calculation, and thus have their wave function normalized to one with only these channels, the same nine account for only 99.5% of the LG wave function, 98.8% and 98.6% of the UPE ($2A+1R$) and the ASE (4,6) wave functions. We thus give in brackets the values obtained for the probability when the wave function is normalized including only the nine channels in Table VII. We see that to the accuracy given, only the probability in channel one is altered, bringing the four results into much closer agreement.

A further comparison of our wave functions for ^{3}H using separable expansion can be achieved by calculating the probability that a given pair of nucleons are in state with quantum numbers $(l_\alpha n_\alpha)$, i.e.

$$P_{l_\alpha n_\alpha} = \sum_{L_\alpha I_\alpha}\langle U^{TJ}_{l_\alpha N_\alpha}\mid U^{TJ}_{l_\alpha N_\alpha}\rangle. \quad (47)$$

TABLE VI. Percentage probabilities of the Blatt-Derrick components of the trinucleon wave function for the RSC potential.

Channel	L	S	l_α	L_α	P	UPA	$2A+1R$ UPE	ASE (1,2)	ASE (4,6)
1	0	$\frac{1}{2}$	0	0	A	87.74	87.55	87.16	87.16
2	0	$\frac{1}{2}$	0	0	−	0.74	0.77	0.78	0.80
3	0	$\frac{1}{2}$	1	1	+	0.74	0.77	0.79	0.81
4	0	$\frac{1}{2}$	2	2	A	1.18	1.30	1.24	1.35
5	0	$\frac{1}{2}$	2	2	−	0.057	0.059	0.062	0.063
6	0	$\frac{1}{2}$	3	3	+	0.047	0.048	0.051	0.050
7	0	$\frac{1}{2}$	4	4	A	0.26	0.28	0.28	0.29
8	0	$\frac{1}{2}$	4	4	−	0.0072	0.0070	0.0075	0.0073
9	0	$\frac{1}{2}$	5	5	+	0.013	0.012	0.013	0.012
10	2	$\frac{3}{2}$	0	2	−	1.08	1.06	1.10	1.07
11	2	$\frac{3}{2}$	1	1	+	2.64	2.56	2.69	2.60
12	2	$\frac{3}{2}$	1	3	+	1.05	1.09	1.15	1.14
13	2	$\frac{3}{2}$	2	0	−	3.07	3.08	3.22	3.17
14	2	$\frac{3}{2}$	2	2	−	0.18	0.18	0.19	0.19
15	2	$\frac{3}{2}$	3	1	+	0.36	0.39	0.39	0.40
16	2	$\frac{3}{2}$	2	4	−	0.15	0.16	0.16	0.16
17	2	$\frac{3}{2}$	3	3	+	0.14	0.14	0.14	0.14
18	2	$\frac{3}{2}$	3	5	+	0.21	0.22	0.22	0.22
19	2	$\frac{3}{2}$	4	2	−	0.059	0.062	0.064	0.064
20	2	$\frac{3}{2}$	4	4	−	0.020	0.020	0.019	0.020
21	2	$\frac{3}{2}$	4	6	−	0.034	0.035	0.036	0.036
22	2	$\frac{3}{2}$	5	5	+	0.028	0.028	0.028	0.029
23	2	$\frac{3}{2}$	5	3	+	0.091	0.095	0.097	0.099
24	2	$\frac{3}{2}$	6	4	−	0.016	0.017	0.017	0.017

We will refer to these probabilities by the spectroscopic notation $^{2S_\alpha+1}l_{\alpha j_\alpha}$. In Table VIII we compare our results for $P_{l_\alpha n_\alpha}$ using UPE ($2A+1R$) and ASE (4,6) with the results of LG who solved the Faddeev equation in coordinate space.[24] Since the states given in the table constitute 100% of the LG wave function, while they account for only 99% of our wave function, we also give in brackets the renormalized values of our probabilities (where they differ from the unrenormalized ones). Once again we see that this brings our results in close agreement with those of Laverne and Gignoux.[24] We also note that although the two-body interaction is only in the $^1S_0$ and $^3S_1$-$^3D_1$ channel, the total wave function has components in which the two nucleons are in other states.

Finally, in Table IX we give the total S, S', and D probability for the different separable expansions in conjunction with the results of Laverne and Gignoux[24] and Brandenburg, Kim, and Tubis.[25] In calculating the probabilities we have once again included all channels with $l_\alpha + L_\alpha \leq 10$, and for this reason we obtain a lower S-state probability than

TABLE VII. A comparison of the percentage probabilities of Table VI with those of LG (Ref. 24) and BKT (Ref. 25).

Channel	$2A+1R$ UPE	ASE (4,6)	LG	BKT
1	87.6(88.7)	87.2(88.4)	87.8(88.2)	88.9
2	0.8	0.8	0.8	0.8
3	0.8	0.8	0.8	0.9
4	1.3	1.4	1.4	1.3
10	1.1	1.1	1.1	1.0
11	2.6	2.6	2.7	2.5
12	1.1	1.1	1.2	1.1
13	3.1	3.2	3.3	3.1
15	0.4	0.4	0.4	0.4

TABLE VIII. Percentage probabilities of the two-body states in the triton for the RSC potential.

Channel \ Method	$2A+1R$ UPE	ASE (4,6)	LG
$^3S_1$	45.5(46.0)	45.3(45.7)	46.1
$^1S_0$	43.9(44.3)	43.7(44.1)	44.3
$^3D_1$	3.3	3.4	3.3
$^3P_1$	1.7	1.8	1.8
$^3P_2$	1.2	1.3	1.2
$^3P_0$	1.1	1.1	1.1
$^1D_2$	0.9	1.0	1.0
$^1P_1$	0.4	0.4	0.4
$^3F_2$	0.4	0.5	0.4
$^3D_2$	0.2	0.2	0.3
$^3D_3$	0.4	0.4	0.2

BKT. We get a higher D-state probability than BKT because in their calculations they have omitted a number of channels which contribute to our D-state probability; in particular they ignore the channels 14 and 18 of Table VI, which give large contributions. Since we have included the P-wave component of the wave function in calculating the normalization, our sum for the S, S', and D probability does not add to 100%. As a final check on the number of partial waves needed, we have extended our sum in Eqs. (38) and (46) from $l_\alpha + L_\alpha \leq 10$ to $l_\alpha + L_\alpha \leq 20$ with the resultant change in the S, S', and D probability from 89.17, 1.61, and 9.14% to 89.05, 1.62 and 9.26%. This further indicates that the restriction to $l_\alpha + L_\alpha \leq 10$ is a good approximation.

So far our results have shown that both separable expansions, even in lowest rank, give a good representation of the trinucleon wave function. However, if such wave functions are to be used in calculating reactions such as $d(p,\pi^+)^3\mathrm{H}$, the contribution from three-body forces and meson exchange (which involve high momentum transfer), we need to examine the short range behavior of the three-nucleon wave function. For this reason, and to see if one separable expansion is superior to the other and how high a rank we need, we have calculated the charge form factor of $^3\mathrm{H}$. The three-nucleon charge form factor is given as

$$F_{\mathrm{ch}}(k) = \frac{1}{2}\sum_{i=1}^{3} F_i(k)\,, \tag{48}$$

where

$$F_i(k) = \int d^3r\, d^3r_1\, d^3r_2\, d^3r_3\, e^{i\vec{k}\cdot\vec{r}}\Psi^*(\vec{r}_1,\vec{r}_2,\vec{r}_3)\rho_c(\vec{r},\vec{r}_i)\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3) \tag{49}$$

with $\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)$ the total three-nucleon wave function. The charge density $\rho_c(\vec{r},\vec{r}_i)$ is defined as

$$\rho_c(\vec{r},\vec{r}_i) = \tfrac{1}{2}(1+\tau_{iz})f^p_{\mathrm{ch}}(\vec{r}-\vec{r}_i) + \tfrac{1}{2}(1-\tau_{iz})f^n_{\mathrm{ch}}(\vec{r}-\vec{r}_i)\,, \tag{50}$$

where τ_{iz} is the Pauli isospin matrix, and f^p_{ch} and f^n_{ch} are, respectively, the Fourier transform of the proton and neutron charge form factors. Since we know the three-nucleon wave function in momentum space and in J-J coupling, we can write the above form factor in terms of $U^{TJ}_{l_\alpha N_\alpha}$ as[30]:

$$\begin{aligned} F^{^3\mathrm{He}}_{\mathrm{ch}}(k) &= \tfrac{1}{2}\{F^p_{\mathrm{ch}}(k)[3G_0(k)+G_1(k)] + 2F^n_{\mathrm{ch}}(k)G_1(k)\}\,, \\ F^{^3\mathrm{H}}_{\mathrm{ch}}(k) &= \{2F^p_{\mathrm{ch}}(k)G_1(k) + F^n_{\mathrm{ch}}(k)[3G_0(k)+G_1(k)]\}\,, \end{aligned} \tag{51}$$

where $G_{t_\alpha}(k)$ $t_\alpha = 0,1$ is given by

$$G_{t_\alpha}(k) = \sum_{I_\alpha \Lambda}\int_0^\infty dp\, p^2 \int_0^\infty dq\, q^2 \int_{-1}^{+1} dx \frac{1}{2}\left(-\frac{2k}{3}\right)^\Lambda \binom{L_\alpha}{\Lambda} |\vec{q}-\tfrac{2}{3}\vec{k}|^{-L_\alpha} q^{L_\alpha-\Lambda} P_\Lambda(x) U^{TJ}_{t_\alpha I_\alpha}(p,q) U^{TJ}_{t_\alpha I_\alpha}(p,|\vec{q}-\tfrac{2}{3}\vec{k}|) \tag{52}$$

with $x = \hat{k}\cdot\hat{q}$ and $I_\alpha = \{l_\alpha, s_\alpha, j_\alpha, L_\alpha, I_\alpha\}$. The functions F^p_{ch} and F^n_{ch} are the proton and neutron charge form factors, respectively. In our calculations we use the analytic expressions given by Janssens *et al.*[32] for these form factors. The radial wave function $U^{TJ}_{t_\alpha I_\alpha}$ is given by Eqs. (39)–(41) where it is written in terms of the spectator wave function.

TABLE IX. Percentage probabilities of the three-body states in the triton for the RSC potential.

Probability \ Method	UPA	UPE $2A+1R$	ASE (1,2)	ASE (4,6)	LG	BKT
$P(S)$	89.2	89.1	88.7	88.8	89.2	90.2
$P(S')$	1.6	1.7	1.7	1.7	1.8	1.7
$P(D)$	9.1	9.1	9.5	9.4	9.0	8.1

To test the effect of truncating the sum in Eq. (52), we have calculated the charge form factor at selected momenta K for the UPE $(2A+1R)$ using the five channels in Table I, all channels with $l_\alpha + L_\alpha \leq 10$, and all channels with $l_\alpha + L_\alpha \leq 20$. From Table X it is clear that for $K^2 = 7.0$ and 20.0 fm^{-2}, we have achieved convergence with $l_\alpha + L_\alpha \leq 10$. To make the equivalent statement for $K^2 = 14.0$ and 15.0 fm^{-2} we compare in Fig. 1 the charge form factor for the UPA with five channels and all 42 channels with $l_\alpha + L_\alpha \leq 10$. We observe that $K^2 = 14.0$ and 15.0 fm^{-2} is in the vicinity of the minimum in the charge form factor and thus the convergence, though not as good, is very reasonable, provided we take all 42 channels with $l_\alpha + L_\alpha \leq 10$. As a final test of convergence in the partial wave expansion we have calculated the charge rms radius for both the UPA and UPE

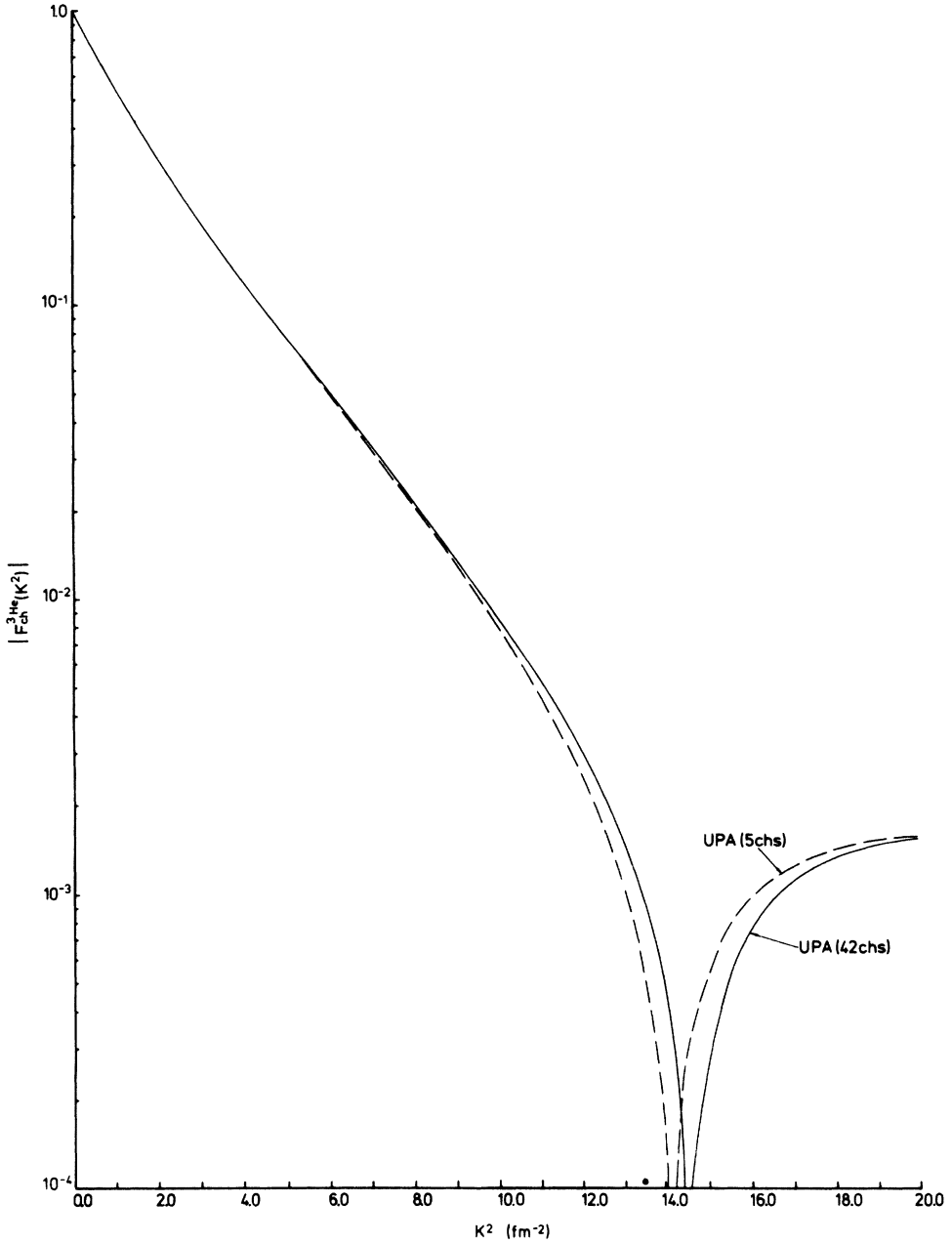

FIG. 1. Comparison of the RSC ^{3}He charge form factors calculated using all channels with $l_\alpha + L_\alpha \leq 10$ (42 channels) and the channels of Table I (5 channels).

TABLE X. The charge form factor for the $2A+1R$ UPE of the RSC potential.

K^2 (fm^{-2})	$F_{ch}^{^3He}(K^2)$ Channels included in calculation: Those in Table I	Those with $l_\alpha+L_\alpha \leq 10$	Those with $l_\alpha+L_\alpha \leq 20$
7.0	3.20×10^{-2}	3.33×10^{-2}	3.31×10^{-2}
14.0	2.72×10^{-4}	2.09×10^{-4}	1.80×10^{-4}
15.0	8.62×10^{-4}	4.77×10^{-4}	4.99×10^{-4}
20.0	1.75×10^{-3}	1.66×10^{-2}	1.66×10^{-2}

$(2A+1R)$. Again we see (Table XI) that with 42 terms in the partial wave expansion we get very good convergence.

Having established the number of terms we need in the sum in Eq. (52), we now present in Fig. 2 the charge form factor of ^{3}He using four different separable approximations to the RSC potential, and all three-body channels with $l_\alpha+L_\alpha \leq 10$. The experimental points in Fig. 2 are those of McCarthy *et al.*[33] The agreement between experiment and the calculated form factor is comparable to previous results using the exact RSC T matrix. This discrepancy between experiment and theory for $K \geq 10$ fm^{-1} can be partly reduced by including meson exchange effects and three-body forces.[9,13] More important from our point of view is the fact that the ASE gives a better convergence than the UPE to the extent that ASE (1,2) is closer to ASE (4,6) than is the case with the UPA and UPE $(2A+1R)$. Taking into consideration the fact that ASE converged without perturbation, which was not the case with the UPE, we expect the ASE (4,6) to give a more accurate wave function than the UPE $(2A+1R)$. This implies that the form factor with ASE (4,6) is closer to the exact result than the UPE $(2A+1R)$. This in fact is true if we compare our results with those of Laverne and Gignoux[24] and Brandenburg *et al.*[25] The surprising result is then the fact that the ASE (1,2) gives a better result than the UPE $(2A+1R)$. Thus we have achieved our aim in constructing a separable expansion of the same rank as the UPA and which gives a very accurate reproduction of the charge form factor even at large momentum transfer. Such a wave function for the trinucleon is simple to calculate yet gives the same results as the more complicated wave functions one gets by solving the Faddeev equation for the exact potential.

Finally in Table XII we give the rms charge radius, the position of the minimum in the charge form factor, the ratio R of the experimental over calculated form factor at $K^2=20$ fm^{-2}, and the binding energy of ^{3}H, for the different separable expansions and the LG[24] and BKT[25] results. We

TABLE XI. The charge radius of ^{3}He obtained using various numbers of channels in the UPA and $2A+1R$ calculations.

Channels included	Those in Table I	Those with $l_\alpha+L_\alpha \leq 10$	Those with $l_\alpha+L_\alpha \leq 20$
$\langle r_{ch}^2\rangle_{UPA}^{1/2}$ (fm)	2.121	2.055	2.054
$\langle r_{ch}^2\rangle_{2A+1R}^{1/2}$ (fm)	2.111	2.070	2.070

find that in all cases the agreement between the separable expansion results and the LG and BKT very good. Furthermore the ASE (1,2) is always better than the UPA and in some cases better than the UPE $(2A+1R)$, the only exception being the binding energy of ^{3}H. It is clear from a comparison of the ASE (1,2) with the results of Laverne and Gignoux and Brandenburg *et al.* that the trinucleon wave function for ASE (1,2) is much easier to calculate and as good as any of the other wave functions resulting from the solution of the Faddeev equation with the exact potential.

V. CONCLUSION

From the above detailed comparison of the trinucleon bound state results for the RSC potential using the separable expansions of Adhikari and Sloan and the UPE on the one hand, and the more exact solutions in coordinate space, and momentum space of the Faddeev equation on the other hand, we can draw the following conclusions: (i) The separable expansion gives a very good approximation not only to the binding energy of ^{3}H, but also the charge form factor of ^{3}He (even at large momentum transfer), the S-, S'-, and D-state probability of ^{3}H, and the rms radius. (ii) Of the two separable expansions the ASE has better convergence properties, to the extent that we need not resort to perturbation theory in calculating E_T, which we had to do for the UPE. (iii) The

TABLE XII. Comparison of trinucleon results for the two separable expansions of different rank, and the more exact solution of the Faddeev equation in coordinate space LG (Ref. 24) and momentum space BKT (Ref. 25).

Method	$\langle r_{ch}^2\rangle^{1/2}$ (fm)	K_{min}^2 (fm^{-2})	E_T (MeV)	R
UPA	2.05	14.4	7.15	3.61
$2A+1R$	2.07	14.2	7.04	3.32
ASE (1,2)	2.08	14.0	7.08	3.33
ASE (4,6)	2.09	13.9	6.97	3.22
BKT		13.9	6.98	3.5
LG		14.0	7.0	3.0

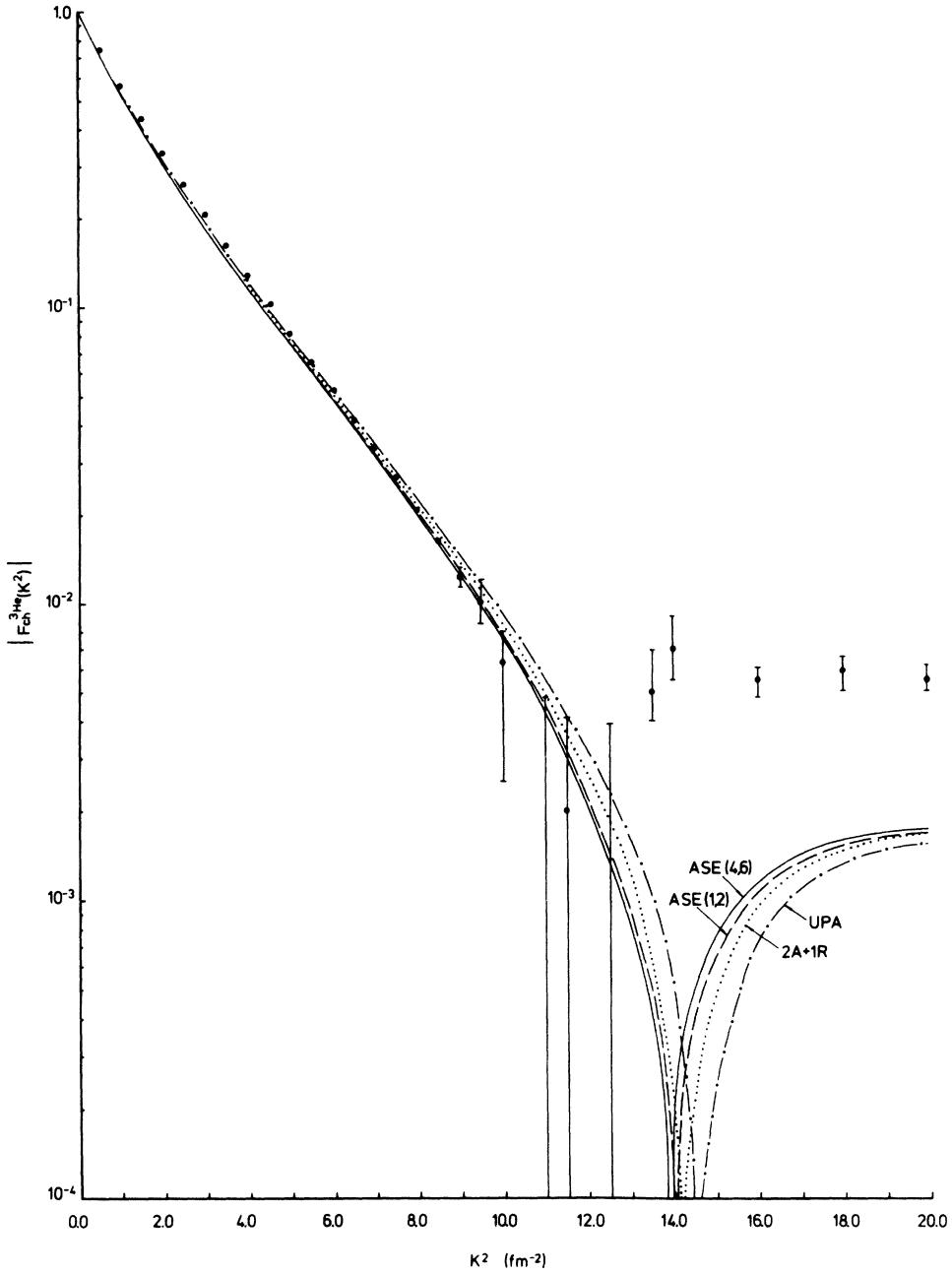

FIG. 2. Comparison of the RSC ^{3}He charge form factors using the four different expansions.

lowest rank ASE considered [i.e., ASE (1,2)] gives not only a good approximation to E_T, but reproduces the charge form factor of ^{3}He obtained using a more exact wave function. This means we can use the simpler wave function obtained from ASE (1,2) to calculate the contribution to the energy and form factor from three-body forces and exchange currents. Also the correction to the energy and charge form factor from the neglected higher nucleon-nucleon partial waves can now be calculated by perturbation theory. Furthermore, since the experimental form factor is reproduced for $K^2 \leq 12$ fm^{-2} with ASE (1,2) we may use such wave functions to study low energy pion production with the hope of learning more about the reaction mechanism.

APPENDIX

In the present Appendix we give an explicit expression for $Z^{TJ}_{l_\alpha N_\alpha;\, l_\beta N_\beta}$ in a form amenable to computation. Although we use J-J coupling for the present, the steps leading to this result are similar to those used in Ref. 34 to derive the equivalent expression in the channel spin coupling. We have

$$Z^{TJ}_{l_\alpha N_\alpha;\, l_\beta N_\beta}(q_\alpha \cdot q_\beta; E) = \langle \underline{g}^{TJ}_{l_\alpha N_\alpha}; q_\alpha | G_0(E) | \underline{g}^{TJ}_{l_\beta N_\beta}; q_\beta \rangle$$

$$= -\frac{1}{2}\int_{-1}^{+1} \frac{\langle \underline{g}_{n_\alpha l_\alpha} | p_\alpha \rangle \langle p_\beta | \underline{g}_{n_\beta l_\beta} \rangle}{E - \varkappa(q_\alpha{}^2 + q_\beta{}^2 + q_\alpha q_\beta x)} \Gamma^{TJ}_{l_\alpha N_\alpha;\, l_\beta N_\beta}(q_\alpha, q_\beta; x)\, dx \tag{A1}$$

with

$$\vec{p}_\alpha = -\vec{q}_\beta - \tfrac{1}{2}\vec{q}_\alpha\,, \quad \vec{p}_\beta = \vec{q}_\alpha + \tfrac{1}{2}\vec{q}_\beta\,, \quad x = \hat{q}_\alpha \cdot \hat{q}_\beta\,, \tag{A2}$$

and

$$\Gamma^{TJ}_{l_\alpha N_\alpha;\, l_\beta N_\beta}(q_\alpha, q_\beta; x) = \left(\frac{q_\alpha}{p_\alpha}\right)^{l_\alpha}\left(\frac{q_\beta}{p_\beta}\right)^{l_\beta} \sum_L P_L(x) \sum_{a=0}^{l_\alpha} \sum_{b=0}^{l_\beta} A^{L,a,b}_{\alpha,\beta}\left(\frac{q_\alpha}{q_\beta}\right)^{b-a}. \tag{A3}$$

In Eq. (A3) $P_L(x)$ is the Legendre polynomial of order L, and the coefficients $A^{L,a,b}_{\alpha,\beta}$ are given by

$$A^{L,a,b}_{\alpha,\beta} = (-)^R \hat{l}_\alpha \hat{l}_\beta \hat{s}_\alpha \hat{s}_\beta \hat{l}_\alpha \hat{l}_\beta \hat{L}_\alpha \hat{L}_\beta \hat{j}_\alpha \hat{j}_\beta \hat{I}_\alpha \hat{I}_\beta (2L+1) 2^{a+b-l_\alpha-l_\beta} \left\{(2l_\alpha - 2a + 1)\binom{2l_\alpha+1}{2a}(2l_\beta - 2b+1)\binom{2l_\beta+1}{2b}\right\}^{1/2} \begin{Bmatrix} \tau_\beta & \tau_\gamma & t_\alpha \\ \tau_\alpha & T & t_\beta \end{Bmatrix}$$

$$\times \sum_S (-)^{2S}(2S+1) \begin{Bmatrix} \sigma_\beta & \sigma_\gamma & s_\alpha \\ \sigma_\alpha & S & s_\beta \end{Bmatrix} \sum_{L'} (-)^{L'}(2L'+1) \begin{Bmatrix} l_\alpha & s_\alpha & j_\alpha \\ L_\alpha & \sigma_\alpha & I_\alpha \\ L' & S & J \end{Bmatrix} \begin{Bmatrix} l_\beta & s_\beta & j_\beta \\ L_\beta & \sigma_\beta & I_\beta \\ L' & S & J \end{Bmatrix}$$

$$\times \sum_{\Lambda\Lambda'} (2\Lambda+1)(2\Lambda'+1) \begin{pmatrix} l_\alpha - a & b & \Lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Lambda & L_\alpha & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & l_\beta - b & \Lambda' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Lambda' & L_\beta & L \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & \Lambda & b & l_\beta \\ L_\alpha & l_\alpha - a & l_\beta - b & L_\beta \\ L' & l_\alpha & a & \Lambda' \end{Bmatrix}, \tag{A4}$$

where $\hat{a} = (2a+1)^{1/2}$ and

$$R = \tau_\gamma + \tau_\alpha + \sigma_\gamma + \sigma_\alpha - t_\beta - s_\beta + l_\beta + 2T + L \tag{A5}$$

and the 12-J symbol is that defined by Ord-Smith.[35]

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