PHYSICAL REVIEW C VOLUME 16, NUMBER 2 AUGUST 1977

Validity of the particle-vibration model

J. M. Greben*
Kernfysisch Versneller Instituut, University of Groningen, Groningen, The Netherlands
(Received 9 August 1976)

The validity of various approximations in the particle-vibration model has been studied for the particle-vibration multiplet in ^{209}Bi. It is shown that the Pauli exclusion principle is taken into account by this model. Since the diagonal particle-vibration coupling, which is neglected in the particle-vibration model, has a sizable effect on the energy splittings, we have studied this interaction in detail. We show that its multipole structure is rather complex, so that it cannot easily be represented by a phenomenological model. The validity of certain symmetry relations in the particle-vibration model has been investigated. This study indicates that the particle-vibration model overestimates the influence of multiphonon intermediate states. Other multiplets in the lead region are briefly discussed, as well.

[NUCLEAR STRUCTURE Interactions between vibrational and fermion degrees of freedom. Approximations in the particle-vibration model. Septuplet in ^{209}Bi.]

I. INTRODUCTION

In the past decade the particle-vibration (PV) model has been used extensively in the description of multiplets in odd-nuclei neighboring doubly magic nuclei.[1,2] Since various assumptions and approximations are made in this model, the effects and adequacy of which are not completely understood, we have studied some of these approximations from a microscopic point of view. We have chosen the multiplet in ^{209}Bi for this analysis, since it is reasonably well understood theoretically and experimentally, contains a relatively large number of states, and therefore provides ample opportunities for testing the PV model.

The approximations in the PV model arise roughly speaking from two sources. First, the particle and vibrational degrees of freedom are treated as independent. In reality, these degrees of freedom are not independent and one may ask whether this treatment will lead to nonorthogonal wave functions, overcompleteness of the basis, and violation of the Pauli principle. Most of these questions have been considered by Bés, Broglia, and others[3,4] in the framework of nuclear field theory. They have proposed certain diagrammatic rules[3] which determine the graphs which should be included in the perturbation series to avoid the problems above. In some simple cases they have shown the validity of these rules by comparing the results of nuclear field theory with exact calculations or Feynman-Goldstone series.[4] Whether the Pauli principle is also correctly taken into account in the present case with many nondegenerate particle-hole states will be studied in the following. Apart from these basic questions about the Pauli principle one may ask whether certain symmetry relations between different particle-vibration graphs are generally valid, since these also originate from the assumption of independence of particle and vibrational degrees of freedom. We have investigated the deviations from these symmetry relations and discuss the consequence for the validity of the PV model.

A second source of approximations in the PV model is the phenomenological description of the vibration. In the Bohr-Mottelson model[5] the vibration is described as a harmonic surface oscillation of an incompressible fluid. Because of this assumption of harmonicity, the particle-vibration interaction which arises from the variation in the average nuclear potential is linear in the oscillator amplitude and therefore changes the number of phonons by one. The inclusion of phonon-conserving interactions between particle and phonon can therefore only be accomplished if one allows for anharmonicities in the vibrational motion. A phenomenological description of these so-called diagonal forces could be based on the quadrupole moment of the phonon, whose existence is also due to anharmonicities in the vibrational motion. In order to check this and other models for this force we have analyzed its multipole structure and its sensitivity to the basis and the residual interaction.

The phenomenological description of vibrational states is confined to those states which possess some degree of collectivity. In order to verify whether omitting noncollective states is justified, we have included such states in our microscopic calculation and have studied their importance.

In the next section we present the relevant graphs and formulas which we will need in our analysis in Sec. III. The conclusions are presented in Sec. IV. In the same section we briefly discuss some other

multiplets in the lead region. Finally, we consider the adequacy of the PV model in describing transition rates.

II. MICROSCOPIC DESCRIPTION OF PARTICLE-VIBRATION COUPLING

In a microscopic description of the coupling between particle and vibration, this coupling is expressed in terms of matrix elements of the residual interaction. In addition, the normalization of the particle-phonon states has to be taken into account. Since the residual force, together with the single-particle energies and wave functions, constitute the basic input for microscopic calculations, we will briefly discuss our choice for this force. The residual interaction, which is really an effective interaction, will depend on the configuration space which is used. For example, the inclusion of ground-state correlations in the description of the phonon will lead to a reduction of the strength of the residual force.[6] We have used a residual interaction which has been well tested in the lead region in the framework of the Tamm-Dancoff approximation.[7] Since we will only study those aspects of the PV model which are not sensitive to the inclusion of ground-state correlations, it is reasonable to neglect correlations in the lead core. Treatment of these correlations would, anyway, require further specific approximations like the random-phase approximation (RPA). The residual force contains a triplet-even and singlet-even part and has a Gaussian radial dependence. Since most energy levels in the lead region are rather insensitive to small changes in the residual force,[8] the form and force parameters are somewhat undetermined. On the other hand, this insensitivity implies that the precise form of this force is not so important, so that our conclusions will hardly be sensitive to the specific form of the interaction. We will explicitly investigate this point by using different interactions. True, Ma, and Pinkston[8] have also used a slightly different version of this residual force by including a P_2-type force. Since this modification does not affect the octupole state very much, we stick to the original force which is also used by Arita and Horie.[7]

The residual force is written in the following way:

$$V = -V_0(W + BP_B + HP_H + MP_M)\exp(-r^2/r_0^2), \qquad (2.1)$$

where P_B, P_H, and P_M are the Bartlett, Heisenberg, and Majorana projection operators. The coefficients used are $W = M = 0.417$, $B = H = 0.083$, and $r_0 = 1.86$ fm. For protons we also include the Coulomb interaction.

The vibrational states are written as a superposition of particle-hole states

$$|jm\rangle_i = \sum_{ab} x_{ab}^{(i)} C(j_a j_b j \,|\, m_a - m_b\, m)(-1)^{j_b - m_b} \times a^\dagger_{j_a m_a} a_{j_b - m_b} |0\rangle = \sum_{ab} x_{ab}^{(i)} |a\bar{b}(jm)\rangle , \qquad (2.2)$$

where j is the spin of the vibration and i labels the different excited states (called "phonons" in the following) with increasing energy. The operators $a^\dagger_{j_a m_a}$ and $(-1)^{j_b - m_b} a_{j_b - m_b}$ are creation operators for particles and holes. The phase is necessary to assure correct angular momentum transformation properties.[9] We have used harmonic oscillator single-particle wave functions with oscillator constant $b = 2.33$ fm. The single-particle energies are taken from experiment, where possible. The bare $i_{13/2}$ single-particle energy was determined by requiring that the lowest energy state in the $J^\pi = \frac{13}{2}^+$ basis (described at the end of this section) coincides with the experimental single-particle energy. The resulting shift was 460 keV. The single-particle energies in the continuum are taken from Ring and Speth[10] and Zawischa[11] ($k_{15/2}$). Phonon energies will be indicated by $E_j^{(i)}$.

The particle-vibration states are defined by

$$|j_i p(J)\rangle = \sum_{ab} x_{ab}^{(i)} |a\bar{b}(j) p(J)\rangle . \qquad (2.3)$$

In general, different particle-vibration states have a finite overlap which is given by

$$\langle j_{i'}' p'(J) \,|\, j_i p(J)\rangle = \delta_{jj'}\delta_{ii'}\delta_{pp'} + \hat{j}\hat{j}' \sum_b x_{pb}^{(i')} x_{p'b}^{(i)} \begin{Bmatrix} j_{p'} & j_b & j \\ j_p & J & j' \end{Bmatrix} , \qquad (2.4)$$

where $\hat{j} = (2j+1)^{1/2}$. The states in (2.3) can thus be normalized by multiplying with the factor $N_j^{(i)}$ which is given by

$$(N_j^{(i)})^{-2} = 1 + (2j+1)\sum_b (x_{pb}^{(i)})^2 \begin{Bmatrix} j_p & j_b & j \\ j_p & J & j \end{Bmatrix} . \qquad (2.5)$$

In Fig. 1 we have shown all two-body matrix elements which contribute to the energy of the particle-vibration states. For each graph, the exchange graph ($a \rightleftharpoons a'$) is also shown. In our calculation the exchange graphs are automatically included by using antisymmetrized shell-model matrix elements. The graphs have the same meaning as the ones discussed by Brown.[9] Broken lines represent the two-body interaction, whereas the wavy lines represent the phonon, and imply a summation over all particle-hole states. Contrary to the PV model, the particle-vibration vertex in these mi-

croscopic graphs does not represent a coupling mechanism.

Using a slightly different coupling scheme, Arita and Horie[7] have given expressions for the two particle-one hole matrix elements involved in these graphs. However, since one can perform the summation over the particle-hole states explicitly in the graphs (a), (b), and (c), one does not have to calculate the particle-hole matrix elements for these cases. The summation is performed using the fact that the amplitudes $x_{ab}^{(i)}$ satisfy the Tamm-Dancoff equation.[9] For example, in Fig. 1(a) (a') one performs the summation over the particle-hole states $|a'\bar{b}'\rangle$:

$$\hat{j}\hat{j}'\sum_{ba'b'} x_{pb}^{(i)}x_{a'b'}^{(i)}\begin{Bmatrix} j_{p'} & j_b & j \\ j_p & J & j' \end{Bmatrix} F_{j'}(p,b,a',b') = \hat{j}\hat{j}'\sum_{b} x_{pb}^{(i)}x_{pb}^{(i')}\begin{Bmatrix} j_{p'} & j_b & j \\ j_p & J & j' \end{Bmatrix}(E_{j'}^{(i')} - \epsilon_p + \epsilon_b)\,.$$

The matrix element $F_{j'}$ is the particle-hole matrix element. One can perform the same reduction for the graph in Figs. 1(b) and 1(c) and obtains after adding the single-particle terms

$$\langle j_i' p'(J)|H^{\text{n.d.}}|j_i p(J)\rangle = \langle j_i' p'(J)|j_i p(J)\rangle(E_j^{(i)} + \epsilon_p) + \hat{j}\hat{j}'\sum_{b} x_{pb}^{(i)}x_{pb}^{(i')}\begin{Bmatrix} j_{p'} & j_b & j \\ j_p & J & j' \end{Bmatrix}(E_{j'}^{(i')} - \epsilon_p + \epsilon_b)\,. \quad (2.6)$$

One easily checks that this expression is symmetric under the transformation $ip \rightleftarrows i'p'$. The superscript of the Hamiltonian is used to indicate that (2.6) does not include the diagonal contributions. The energy shift of a particle-phonon state from its unperturbed energy $E_j^{(i)} + \epsilon_p$ equals (set $i = i'$):

$$\Delta E_J = N_J{}^2\hat{j}^2\sum_{b} x_{pb}{}^2\begin{Bmatrix} j_p & j_b & j \\ j_p & J & j \end{Bmatrix}(E_j - \epsilon_p + \epsilon_b)\,. \quad (2.7)$$

In (2.7) we dropped the index of the phonon, since the lowest state is understood. If one expresses the particle-hole amplitudes x_{pb} in terms of the interaction Hamiltonian of the PV model by means of

$$x_{pb} = \frac{\langle j|H_{\text{int}}|p\bar{b}(j)\rangle}{E_j - \epsilon_p + \epsilon_b}\,, \quad (2.8)$$

then one easily identifies the energy shift (2.7) with the exchange graph in the PV model [Fig. 2(a)]. Obviously, the Pauli effect is taken into account in the PV model by means of this graph. Small deviations from the microscopic calculation will still occur, because the particle-hole amplitudes x_{pb} estimated by (2.8) will differ from those obtained by the Tamm-Dancoff equations, and the normalization factor $N_J{}^2$ is usually neglected in the PV model. This normalization can, however, also be calculated in terms of particle-vibration graphs,[12] so that an improvement in this direction seems to be rather easy. The foregoing discussion has also shown that the Pauli term is readily obtained in the microscopic formulation and that the PV model does not constitute a simplification in this respect. On the other hand, the diagonal terms which are not as easily obtained microscopically are not included in the PV model.

The diagonal particle-phonon force shown in Figs. 1(d) and 1(e) is given by

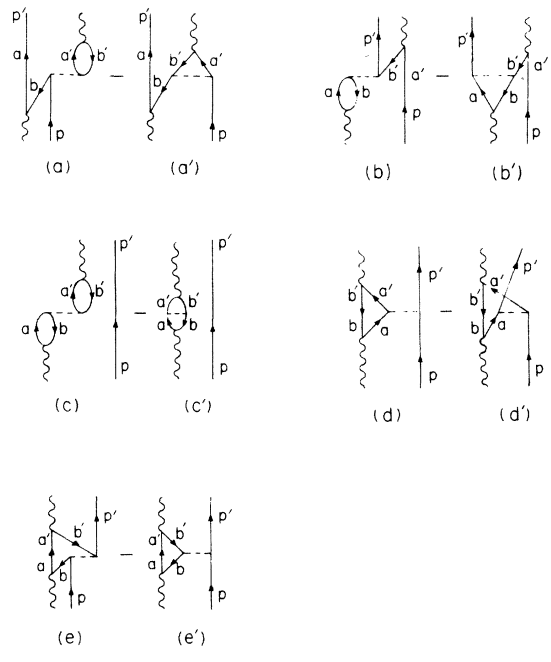

FIG. 1. Direct and exchange graphs (with prime) contributing to the expectation value of the Hamiltonian in the particle-phonon state.

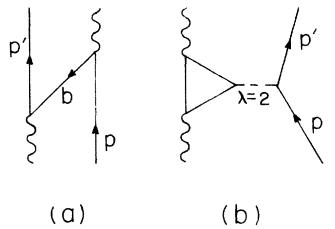

FIG. 2. (a) Exchange graph in the PV-model. (b) Quadrupole interaction between particle and phonon.

$$\langle j_i' p'(J)|H^{\mathrm{diag}}|j_i p(J)\rangle = \hat{j}\hat{j}' \sum_{acb} x_{ab}^{(i)} x_{cb}^{(i')} (-1)^{j_a - j_c + j - j'} \sum_{J'} (2J'+1) \begin{Bmatrix} j_a & j_b & j \\ J & j_p & J' \end{Bmatrix} \begin{Bmatrix} j_c & j_b & j' \\ J & j_{p'} & J' \end{Bmatrix} G_{J'}(c,p',a,p)$$

$$- \hat{j}\hat{j}' \sum_{abd} x_{ab}^{(i)} x_{ad}^{(i')} \sum_{J'} (2J'+1) \begin{Bmatrix} j_a & j_d & j' \\ j_b & J' & j_{p'} \\ j & j_p & J \end{Bmatrix} G_{J'}(p',b,p,d), \tag{2.9}$$

where $G_J(a,b,c,d)$ is the antisymmetrized shell-model matrix element. This matrix element does not contain the usual normalization factors $2^{-1/2}$ or 2^{-1} for $a=b$ and/or $c=d$, since these factors appear through the normalization of the states. The first term in (2.9) represents the attractive particle-particle component [Fig. 1(d)], the second term the repulsive particle-hole component [Fig. 1(e)]. This latter term can also be expressed in terms of particle-hole matrix elements. Bohr and Mottelson[5] discuss a possible phenomenological description of such a force. In this model the interaction is expressed in terms of the quadrupole moments of particle and phonon. Such an interaction can be represented by the graph in Fig. 2(b). In the next section we will analyze the adequacy of this model.

The matrix element between a particle and a particle-phonon state [Fig. 3(a)] is given by

$$\langle p'|H|j_i p(J)\rangle = \delta_{j_{p'}J} \sum_{ab} x_{ab}^{(i)} \hat{j} (\hat{j}_{p'})^{-1} (-1)^{j_a - j_b + j} \sum_{j'} (2j'+1) \begin{Bmatrix} j_a & j_b & j \\ j_{p'} & j_p & j' \end{Bmatrix} G_{j'}(b,p',p,a). \tag{2.10}$$

The graphs in Figs. 3(a) and 3(b) are analogous to the particle-vibration graphs in Figs. 4(a) and 4(b). The latter are related by a simple symmetry relation. Hence, we can check the validity of these relations by comparing the graphs in Figs. 3(a) and 3(b) numerically.

The correctness of the PV model clearly depends on the adequacy of the particle-phonon basis in describing the septuplet. This adequacy can be checked by determining the importance of other basis states, such as higher-lying particle-phonon states. We determined the importance of these states by performing calculations in basis with and without these states. The results are shown in Figs. 8(d)–8(f) (see next section), where it is to be understood that in calculating the effect of one specific state, all basis states mentioned to its left are already taken into account. Matrix elements between particle-phonon states are obtained from Eqs. (2.6) and (2.7), where the first term in (2.6) is due to the nonorthogonality of the states. Such nonorthogonality terms drop out if one performs a Hilbert-Schmidt orthonormalization of the basis. For future reference we picture the lowest order contribution of the single-particle term to the energy shift [Fig. 5(a)]. The effect of two-phonon one-particle states is treated in perturbation [cf., Fig. 8(g)]. The corresponding graph is shown in Fig. 5(b).

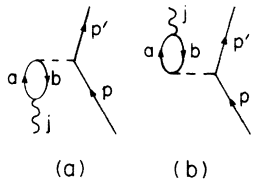

FIG. 3. Graphical representation of the matrix element (2.10). The exchange graph is omitted in this figure.

III. ANALYSIS OF THE PARTICLE-VIBRATION MODEL

A. Pauli term

In the previous section we have shown that the only difference between the PV model and the microscopic calculation lies in the particle-hole amplitudes x_{pb} and in the treatment of the normalization. The main contribution in the Pauli term [Eq. (2.7)] is due to the $h_{9/2}d_{3/2}^{-1}$ particle-hole state, which has the amplitude 0.394 in our calculation ($V_0 = 34.0$), 0.391 in the RPA calculation of Ring and Speth,[10] and 0.365 in the PV model.[1] This corresponds to a 15% difference in PV-model and microscopic estimates of the Pauli term. The inclusion of the normalization factor gives another correc-

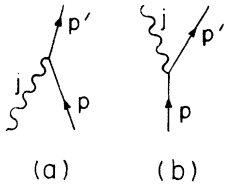

FIG. 4. Particle-vibration graphs representing the annihilation or creation of a phonon.

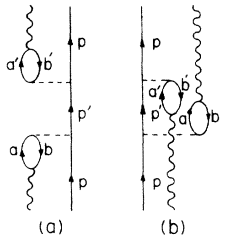

FIG. 5. Microscopic second order graphs contributing to the energy shifts of the particle-phonon states.

tion which is largest for the highest spin state ($N_{15/2}{}^2 = 1.13$) and much smaller in the other cases ($N_J{}^2 = 0.94$, 1.04, 0.97, 1.02, 0.99, and 1.00 for $J^\pi = \frac{13}{2}^+, \ldots, \frac{3}{2}^+$). The difference between the PV model and our calculation is therefore largest (≈30%) for the highest spin state. Since the Pauli term is very large for the high spin states it is very desirable to treat the normalization of the particle-phonon state also in the PV model. We have used two other residual interactions to test the model dependence of the $h_{9/2}d_{3/2}{}^{-1}$ particle-hole amplitude. A Soper mixture ($W = 0.3$, $B = 0.27$, $H = 0$, $M = 0.43$, and $V_0 = 38.5$) gives a value 0.397; an even force with relatively strong triplet interaction ($W = M = 0.375$, $H = B = 0.125$, and $V_0 = 37.8$) gives a value 0.403. These values also underline the small sensitivity of the Pauli term to the residual interaction. The sensitivity of the Pauli term to the particle-hole basis used has been investigated as well. We have performed calculations in particle-hole basis with maximal excitation energy 7, 7.43, 7.67, 8, 11, 13, and 15 MeV. For each of these calculations V_0 is adjusted to the experimental vibrational energy; $V_0 = 49.8$, 45.4, 38.8, 36.1, 34.5, 34.0, and 33.8. The results are presented in Fig. 6. On the left-hand side of the figure all $h_{9/2}$ p-h states are already included, except for the $h_{9/2}g_{7/2}{}^{-1}$ state. The latter is introduced at a dimension of 28, and has a sizable effect on the Pauli term. The other changes in the Pauli term, reflecting the modifications of the coefficients of the $h_{9/2}$ p-h states with increasing dimension, are very small. Comparison with estimates based on RPA calculations with 90 particle-hole states[10] shows that the Pauli term is not strongly affected by the inclusion of ground-state correlations, either. This analysis clearly shows that the magnitude of this Pauli term is not due to the cutoff in configuration space. Furthermore, the validity of the microscopic calculation is underlined by the insensitivity to both residual force and basis (provided at least 30 particle-hole states are included).

B. Diagonal particle-vibration coupling

Both the particle-particle and particle-hole components in the diagonal interaction are large in magnitude, as one can see in Fig. 8. The cancellation between these components is, however, so strong that they have a smaller effect on the energy

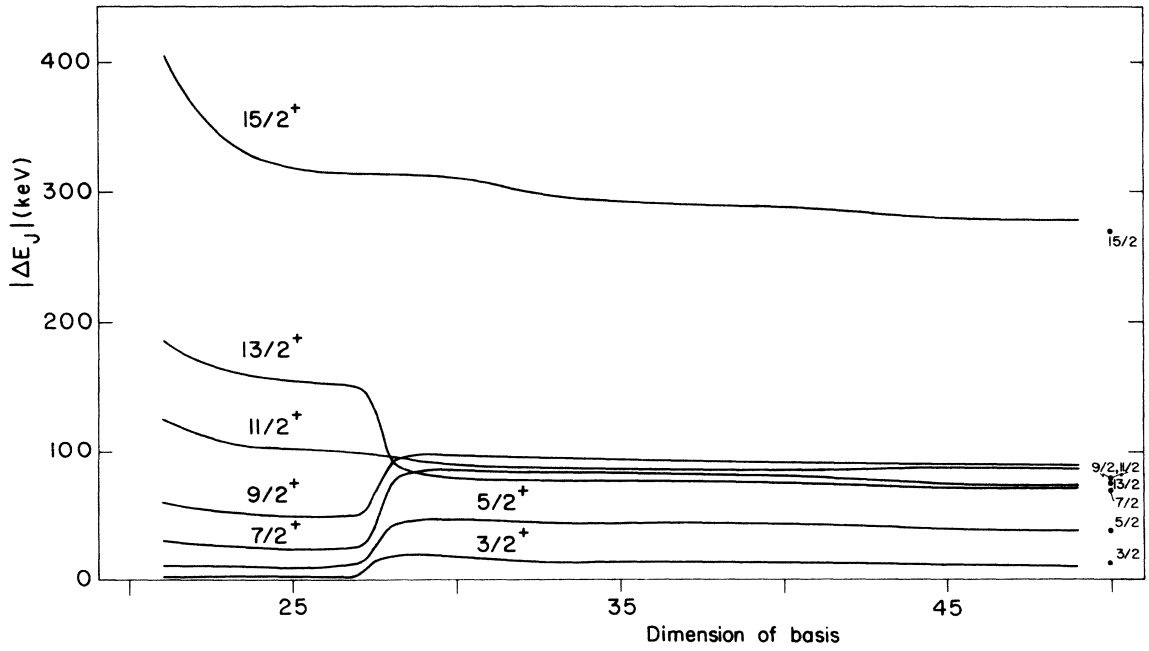

FIG. 6. Behavior of the Pauli terms with increasing basis. The points on the right are based on RPA calculations of Ring and Speth (Ref. 10).

splittings than the Pauli term. Since the cancellation will not occur for pairing phonons (which are made up either of particles or holes), one expects the diagonal interaction to be especially important for these degrees of freedom. In order to study the validity of the phenomenological model for the quadrupole component of this interaction, we will expand the interaction in multipoles. First we write the corresponding interaction Hamiltonian in second-quantized form:

$$H^{\text{diag}} = \sum_J \hat{J} \Delta E_J [(\beta_3^\dagger a_{h_{9/2}}^\dagger)_J (\tilde{\beta}_3 \tilde{a}_{h_{9/2}})_J]_0 , \quad (3.1)$$

where $\tilde{a}_{jm} = (-1)^{j-m} a_{j-m}$, and $\beta_3^\dagger$ is the creation operator for the 3^- phonon. The (unnormalized) energy shifts ΔE_J are given by Eq. (2.9) with $i = i'$. After changing the coupling scheme in (3.1) one obtains the multipole expansion

$$H^{\text{diag}} = \sum_l v_l [(\beta_3^\dagger \tilde{\beta}_3)_l (a_{h_{9/2}}^\dagger \tilde{a}_{h_{9/2}})_l]_0 . \quad (3.2)$$

The results for v_l are given in Table I. Although the quadrupole component is the largest component, it would give completely erroneous results if it were to be the only multipole to be used in the computation of the energy shift. Other multipoles, especially the dipole and hexadecapole, contribute substantially, too. We can compare our estimate of v_2 with the estimate of Bohr and Mottelson (Ref. 5, p. 576):

$$v_2 = 0.626 Q_{\text{phonon}} Q_{h_{9/2}} \text{ keV fm}^{-4} . \quad (3.3)$$

Using the experimental values for the quadrupole moment of the phonon[13] (-100 fm^2) and the $h_{9/2}$ particle (40 fm^2), one obtains $v_2 = -2504$ keV. The discrepancy with our result should probably be explained by the uncertainty in the experimental value of the quadrupole moment of the phonon. Theoretical estimates of this quadrupole moment (-6.1 fm^2 in Ref. 7 and -10 fm^2 in Ref. 5) lead to values of v_2 which are in much closer agreement with our value for v_2. Another possibility for deriving the diagonal force in a macroscopic way consists of the expansion of the particle-vibration Hamiltonian to higher order in the oscillator amplitude. This is a dangerous procedure, since it will also bring into play vibrations of other multipolarity (like quadrupoles). The resulting interaction Hamiltonian will contain a part analogous to (3.2); however, only even multipoles will be present. So we must conclude that in both phenomenological models important multipoles are neglected, which is very dangerous in actual calculations because of the strong cancellations between different multipole contributions. Once again we have studied the sensitivity of the calculation to the choice for the residual interaction by using the Soper mixture and the strong triplet force discussed before. The multipole coefficients change only very little as one one can see from Table I. In Table II we show multipole coefficients for the coupling with other particles and holes. These results and additional results for particles and holes which are not shown, underline the importance of the dipole, quadrupole, and monopole components in the diagonal interaction (of course the monopole does not contribute to the splitting of the levels). We have also studied the sensitivity of the diagonal interaction to the particle-hole basis (Fig. 7). Unlike the case of the Pauli term, where the only contributions came from the $h_{9/2}$ p-h states, all p-h states contribute to the diagonal interaction. The variation in the interaction with dimension, therefore, reflects both the change in the p-h coefficients and the inclusion of additional states. Apparently, the 25th and 26th p-h states $\nu(i_{11/2} h_{9/2}^{-1})$ and $\pi(i_{13/2} h_{11/2}^{-1})$ have a considerable effect on the diagonal interaction. Increasing the dimension above 30 hardly affects the interaction any more.

TABLE I. The first two columns contain the multipole coefficients of the particle-particle (p-p) and the particle-hole (p-h) component of the diagonal particle-phonon coupling in keV. The third column contains the full multipole coefficient. In the last two columns results for the Soper and the strong triplet force are given.

	p-p	p-h	Sum	Soper mixture	Triplet force
v_0	−758	666	−92	−93	−100
v_1	213	28	240	198	252
v_2	−518	228	−290	−297	−300
v_3	73	−1	72	62	82
v_4	−198	144	−54	−36	−57
v_5	60	−25	35	30	40
v_6	−173	22	−151	−144	−150

TABLE II. Multipole coefficients of the full diagonal interaction for coupling to other particles or holes (see table heading).

	$\pi 2f_{5/2}$	$\nu 2g_{9/2}$	$\nu 2f_{5/2}^{-1}$	$\nu 3p_{1/2}^{-1}$	$\pi 3s_{1/2}^{-1}$
v_0	234	114	−238	314	−142
v_1	161	130	147	8	−217
v_2	−153	−211	167		
v_3	35	46	42		
v_4	−31	−87	58		
v_5	13	24	18		
v_6		−161			

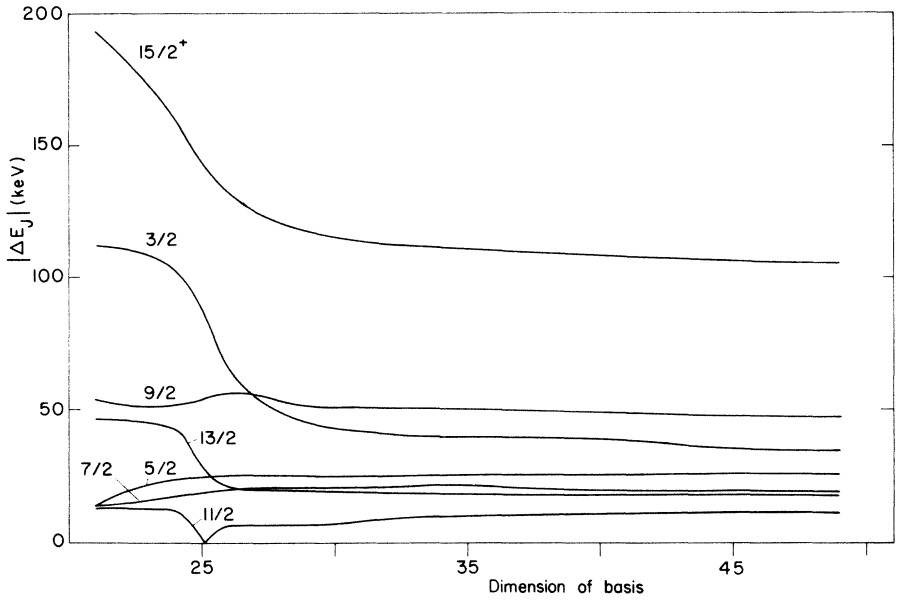

FIG. 7. Behavior of the diagonal particle-phonon coupling with increasing basis.

C. Symmetry relations

The particle-vibration graphs in Fig. 4 are connected by the following symmetry relation (cf. Ref. 5, p. 418):

$$h_j(p,p') = h_j(p',p)\,, \tag{3.4}$$

where $h_j(p,p')$ is defined by

$$h_j(p,p') = (-1)^{j_{p'}-1/2}\,\hat{j}_{p'}\langle p \,|\, H_{\rm int} \,|\, jp'(j_{p'})\rangle\,. \tag{3.5}$$

In Eq. (3.5) j represents the collective phonon (vibration) and $H_{\rm int}$ is the particle-vibration Hamiltonian.

The equality (3.4) is no longer valid if one replaces the particle-vibration matrix elements by their microscopic expression, Eq. (2.10). For the case of a two-body δ-function force,[14] explicit symmetry relations of the matrix element (2.10) can be derived, because various summations (in particular over j') can be performed. The resulting expression contains two parts; one which satisfies the symmetry relations exactly, and one which only satisfies them for unnatural parity states and gives a symmetry relation with opposite sign for the natural parity states. The latter part is not expected to be very large for isoscalar modes of excitation. Since the collective 3^- state is mainly isoscalar, we expect that Eq. (3.4) is reasonably well satisfied in the present case, although further deviations will occur if one uses forces of finite range. On the other hand, for isovector modes the part which violates the symmetry relation (3.4) is large and we do not expect (3.4) to be valid at all.

We have checked these symmetry relations quantitatively for the residual force discussed in Sec. II. The results are given in the first row of Table III. Note that the particle-phonon states used in

TABLE III. Matrix elements $h_3(p,p')$ between particle states p and particle phonon states $|3p'\rangle$, one of the particles being an $1h_{9/2}$ particle. Symmetry relations can be tested by comparing entries in the same squares. The middle row (Soper interaction) and last row (strong triplet force) are shown to illustrate the sensitivity to the residual force used.

p	$3d_{3/2}$	$3d_{5/2}$	$2g_{7/2}$	$2g_{9/2}$	$1i_{11/2}$	$1j_{13/2}$	$1k_{15/2}$
$h_3(p,1h_{9/2})$	400	−139	807	−328	2438	−847	−2811
$h_3(1h_{9/2},p)$	350	−138	689	−243	2188	−548	−2330
$h_3(p,1h_{9/2})$	333	−109	767	−292	2627	−881	−2999
$h_3(1h_{9/2},p)$	330	−130	686	−257	2368	−627	−2590
$h_3(p,1h_{9/2})$	409	−145	809	−326	2491	−818	−2850
$h_3(1h_{9/2},p)$	363	−146	704	−260	2190	−554	−2334

TABLE IV. Same as Table III for particles (holes) different from $1h_{9/2}$. The particle (hole) states p are determined by the requirement that they are the single-particle states closest to the Fermi surface with the appropriate quantum numbers.

j_p	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$
$h_3(p,\pi 2f_{7/2})$	524	930	849	1957	516	2486
$h_3(\pi 2f_{7/2},p)$	504	820	780	1650	495	1808
$h_3(p,\nu 1j_{15/2})$				−1538	−587	
$h_3(\nu 1j_{15/2},p)$				−1898	−626	
$h_3(p^{-1},\pi 1h_{11/2}{}^{-1})$		2465	898	2478		
$h_3(\pi 1h_{11/2}{}^{-1},p^{-1})$		1780	595	2086		

determining the matrix elements are not normalized. For higher spin values the matrix elements $h_3(p,h_{9/2})$ are considerably larger than the matrix elements $h_3(h_{9/2},p)$. Expressed in terms of energy splittings this means that the contribution of the graph in Fig. 5(b) is reduced considerably with respect to the contribution of the graph in Fig. 5(a). By converting this argument one can argue that the PV model overestimates the contribution of graphs which involve multiphonon intermediate states. In the next two rows we show the same comparison for the Soper force and the strong triplet interaction. For the latter force the results are practically the same, for the first the symmetry relation seems to be satisfied slightly better. In Table IV we give the results for another proton single-particle state ($\pi f_{7/2}$), a neutron state ($\nu j_{15/2}$), and a proton hole state ($\pi h_{11/2}{}^{-1}$). From the systematics of these and additional calculations we conclude that the matrix elements $h_3(p,p')$ with a proton (neutron) single-particle or hole state p' of negative parity are on the average larger (smaller) by a factor of 1.2 than their counterparts $h_3(p',p)$.

D. Higher-lying particle-phonon states

From Table V we see that the higher-lying particle-phonon states admix very little in the main $|3_1 h_{9/2}\rangle$ state. The most important admixture, the $|3_1 f_{7/2}\rangle$ configuration in the $\frac{13}{2}^+$ state, can also be treated in the PV model. The 5^- state in ^{208}Pb is not very collective and therefore does not admit a treatment within the PV model. The energy shifts arising from 5^- admixtures into the septuplet are given in Fig. 8(f). Relative to the total energy shift their effect is rather small. The admixtures of other noncollective states (3_2^-, 3_3^-, and 5_2^-) are even less important. It therefore seems well justified to omit these noncollective degrees of freedom in the PV model. The admixture of $|3_i h_{9/2}\rangle$ configurations ($i>1$) in the septuplet can be interpreted as a change of the phonon structure in the presence of the extra particle. For the $\frac{15}{2}^+$ state, where the Pauli principle is most effective, the superposition of three phonon states ($i=1,2,3$) with the appropriate coefficients leads to a reduction of $x_{h_{9/2}d_{3/2}}$ from 0.394 to 0.374. The changes in other coefficients are even smaller, so that we conclude that the phonon retains its original structure very well in the presence of the extra particle.

In Fig. 8 we have summarized all the contributions to the energy splitting. Also shown are experimental and PV-model results. The misfit for the $\frac{3}{2}^+$ state is usually attributed to the neglect of configurations arising from the coupling of pairing phonons in ^{210}Po to an $s_{1/2}$ hole. In the present paper we did not consider these degrees of freedom; however, they may be an interesting topic for further study. The importance of the single-particle terms [Fig. 8(d)] depends very much on the specific character of these states (energy, principal quantum number). The two-phonon one-particle term [Fig. 8(g)] gives a rather large contribution for all the members of the septuplet. As stated earlier in this section, this term is overestimated in the PV model.

IV. DISCUSSION AND CONCLUSIONS

The analysis of various approximations in the PV model has shown that, as far as the energies of the particle-phonon states are concerned, only the ne-

TABLE V. Coefficients of the particle-phonon admixtures into the multiplet. The excitation energies in the second column are given with respect to the unperturbed energy of the multiplet.

State	Exc. energy (keV)	$\frac{15}{2}$	$\frac{13}{2}$	$\frac{11}{2}$	$\frac{9}{2}$	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
$3_1\times f_{7/2}$	892	0	−0.141	0.025	−0.031	0.034	−0.023	0.002
$3_2\times f_{7/2}$	2436	0	−0.005	0.001	−0.002			0.000
$3_1\times h_{9/2}$	0	1.045	0.942	1.004	0.982	0.995	0.987	0.987
$3_2\times h_{9/2}$	1544	−0.007	0.003	0.011	0.007	−0.001	−0.001	0.016
$3_3\times h_{9/2}$	1707	−0.031	0.007	−0.011	0.004	0.001	0.012	0.007
$5_1\times f_{7/2}$	4260	−0.015	0.025	0.002	0.004	−0.020	0.028	0.024
$5_2\times f_{7/2}$	4665	−0.017	0.025	−0.003	0.007	−0.017	0.018	−0.011
$5_1\times h_{9/2}$	3368	0.117	0.051	0.035	0.032	0.087	−0.001	0.013
$5_2\times h_{9/2}$	3773	0.064	0.019	0.028		0.038	−0.011	0.010

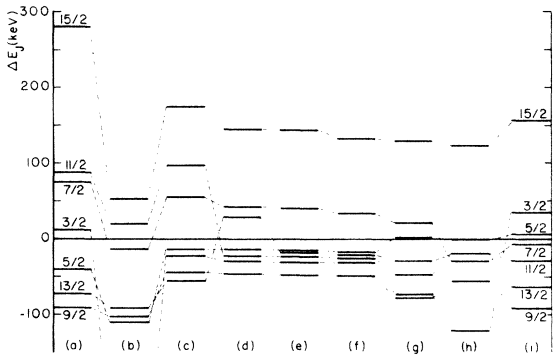

FIG. 8. Contributions to the energy shifts of the septuplet and comparison with experiment and particle-vibration model: (a) Pauli term; (b) diagonal particle-hole force; (c) diagonal particle-particle force; (d) single-particle contribution; (e) effect of higher-lying $|3^-p\rangle$ configurations; (f) effect of $|5^-p\rangle$ configurations; (g) effect of the graph in Fig. 5(b); (h) experimental spectrum; (i) particle-vibration model. The diagonal particle-hole force brings the $\frac{9}{2}^+$ level down to -191 keV in Fig. 8(b).

glect of the diagonal interaction leads to a significant difference between the PV model and the microscopic calculations. More specifically, we arrived at the following conclusions:

(a) The Pauli exclusion principle can be taken into account by one simple exchange graph both in the microscopic case and in the PV model. Although this graph is calculated in a different way in the PV model, the agreement with the microscopic calculation is rather good. We have shown that for higher spin values of the members of the multiplet, the normalization should also be taken into account in the PV model.

(b) The diagonal particle-phonon force is next most important for the energy shifts. The average diagonal contribution is 37 keV (the corresponding number for the single-particle contribution is 36 keV), whereas the total average energy shift is 54 keV. Since most energy shifts would change by as much as a factor of 2 if the diagonal contribution is not included, it is essential to include this contribution if one wants to compare the energy shifts with experiment (which is the main objective of the particle-vibration model). We showed that the diagonal force has a complex multipole structure, so that it cannot be represented by a simple quadrupole model,[5] nor by a simple expansion of the single-particle Hamiltonian in the vibrational parameter. Since the calculation of this term is very time consuming, it would be of much interest to develop other phenomenological models for this interaction. The present analysis constitutes a good quantitative basis for such an investigation.

(c) We have shown that there is a small but systematic deviation from the symmetry relations between different particle-vibration graphs. The creation of particle-phonon states of positive parity is favored (suppressed) over the creation of negative parity states in the case of proton (neutron) single-particle states. For the application of the PV model in ^{209}Bi this implies that graphs with two phonons in the intermediate state are enhanced over graphs with no phonons in the intermediate state. We have given a qualitative argument that for isovector ($T=1$) modes the symmetry relations will be strongly violated.

(d) Noncollective states in the lead core hardly play a role in the multiplet of ^{209}Bi. The impossibility of treating these states in the PV model, therefore, does not seriously affect the validity of the PV model.

The sensitivity of the microscopic calculations to the residual force has been estimated by using three different forces. We found an average variation in the particle particle-phonon matrix elements of 5%, which is quite small considering the significant changes in the residual force. The corresponding change in the Pauli term is roughly 10%, which preserves the qualitative agreement between the microscopic and macroscopic Pauli term. The average variation of the diagonal force is about 8%. The multipole distribution of this force is hardly affected by these changes. Since the sensitivity of the diagonal force to the dimension of the particle-hole basis is even smaller (provided the number of states is ≥ 30), the present calculations give a good quantitative determination of the diagonal interaction.

Other multiplets in the lead region do not lend themselves so well to analysis of this type. The 5^- multiplet in ^{209}Bi, starting at 2.987 MeV ($\frac{13}{2}^+$) and going up to 3.212 MeV ($\frac{9}{2}^+$ or $\frac{1}{2}^+ + \frac{7}{2}^+$) in excitation, can not be described very well in the PV model since the 5^- state is hardly collective (the dominant state is $\nu 2g_{9/2}3p_{1/2}^{-1}$). Since the diagonal and the single-particle term are, respectively, first and second order in the residual interaction, the diagonal interaction will be even more important in the present case (in the collective case, where the residual interaction strength is not the appropriate expansion parameter,[3] the previous argument does not hold true). There is not much experimental information about the $|3_1 2g_{9/2}\rangle$ septuplet in ^{209}Pb. Pairing modes in ^{209}Pb coupled to hole states may well destroy the weak-coupling nature of such a multiplet.[1] Again, members of this multiplet would be strongly affected by the Pauli effect, through the large $2g_{9/2}3p_{3/2}^{-1}$ particle-hole amplitude in the octupole phonon. The distribution of the multipole strength of the diagonal interaction is similar to the one in ^{209}Bi, as can be verified in Table II. The

average strength of the diagonal interaction (35 keV) is also similar. The only significant single-particle term is due to the low-lying single-particle state $1j_{15/2}$. The doublets in ^{207}Pb and ^{207}Tl, which are formed by coupling the octupole phonon to a $3p_{1/2}$ and a $3s_{1/2}$ hole, respectively, will not be shifted as strongly by the Pauli effect as the $A = 209$ states. The multipole distribution of the diagonal force for these doublets is shown in Table II. For both doublets, the diagonal force diminishes the gap between experiment and PV calculations[1]; however, the remaining gap is still of the order of 40 keV.

Experimental data on transition rates can also be used for comparing the PV model with the microscopic description. In the extreme weak-coupling case, $E3$ matrix elements for the transition between septuplet and $h_{9/2}$ ground state are identical to the octupole phonon matrix elements. Deviations from this behavior are due — and therefore sensitive — to the particle-phonon coupling. A rigorous comparison between PV-model and microscopic calculation is not possible here, since in our microscopic description ground-state correlations are neglected. A comparison with experiment is also ambiguous, because of the uncertainty in the effective proton and neutron charge. Nevertheless, we can look at the relative importance of the different contributions to the change in the transition matrix element. One would expect that the PV model gives slightly better results for the transition rates than for the energies, since it takes into account certain ground-state correlations, whereas the diagonal interaction does not give direct contribution to the transition matrix element. This expectation is not borne out by the results.[1] The calculated $B(E3)$ values are on the average 20% too high. The main corrections to the PV-model calculations come from the renormalization of the main particle-phonon state and the inclusion of higher-lying particle-phonon states. We found that these two effects contribute on the average 28% to the total shift in the transition matrix element. The Pauli exchange graph [see Fig. 8(d) in Ref. 1] contributes 37%, whereas two other particle-vibration graphs [Fig. 8(a) and 8(c) in Ref. 1] contribute 15 and 12%. The RPA-type graph which occurs in the PV model contributes about 7%, if averaged over all states. We conclude that for a good description of the transition rates, the PV model should also account for the normalization of the particle-phonon state and for particle-phonon states at higher excitation energy. The first could be done along the lines of nuclear field theory,[12] the latter can only be accomplished if these states can be approximated by (other) collective states (vibrations, pairing phonons) and single-particle states.

ACKNOWLEDGMENTS

I am grateful to Professor F. Iachello for suggesting this study and for many stimulating discussions. I also acknowledge discussions with Dr. M. Nomura. This investigation was performed as part of the research program of the Stichting voor Fundamenteel Onderzoek der Materie (FOM) with financial support from the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (ZWO).

*Present address: Physics Department, Brown University, Providence, Rhode Island 02912.

[1] I. Hamamoto, Nucl. Phys. A126, 545 (1969); A141, 1 (1970); Phys. Rep. C 10, 63 (1974).
[2] I. Hamamoto and E. Osnes, Phys. Lett. 53B, 129 (1974).
[3] D. R. Bés, G. G. Dussel, R. A. Broglia, R. Liotta, and B. R. Mottelson, Phys. Lett. 52B, 253 (1974).
[4] D. R. Bés and R. A. Broglia, in *Proceedings of the Conference on Problems of Vibrational Nuclei, Zagreb, 1974*, edited by V. Alaga (North-Holland, Amsterdam, 1975), p. 1.
[5] A. Bohr and B. R. Mottelson, Nuclear Structure (Benjamin, New York, 1975), Vol. II.
[6] V. Gillet, A. M. Green, and E. A. Sanderson, Nucl. Phys. 88, 321 (1966).
[7] J. C. Carter, W. T. Pinkston, and W. W. True, Phys. Rev. 120, 504 (1960); K. Arita and H. Horie, Nucl. Phys. A173, 97 (1971).
[8] W. W. True, C. W. Ma, and W. T. Pinkston, Phys. Rev. C 3, 2421 (1971); W. W. True, Phys. Rev. 168, 1388 (1968).
[9] G. E. Brown, *Unified Theory of Nuclear Modes and Forces* (North-Holland, Amsterdam, 1971).
[10] P. Ring and J. Speth, Nucl. Phys. A235, 315 (1974).
[11] D. Zawischa, Z. Phys. 266, 117 (1974).
[12] P. F. Bortignon, R. A. Broglia, R. Liotta, D. R. Bés, and V. Paar, Copenhagen, 1975 (unpublished).
[13] A. R. Barrett and W. R. Phillips, Phys. Rev. 186, 1205 (1969).
[14] H. Feshbach and F. Iachello, Ann. Phys. (N.Y.) 84, 211 (1974). The quantities R_J and P_J, defined in this reference, have an incorrect phase.